Black Hole Entropy in the presence of Chern-Simons terms

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Dec 25, 2006, Yukawa Inst.

based on hep-th/0611141,
to appear in Class. Quant. Grav.

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1. **Introduction**

2. Black Hole Spacetime

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Black Holes Thermodynamics

- **Classical General Relativity leads to**

$$\frac{\kappa}{2\pi}\frac{\delta A}{4G_N} = \delta m - \Omega\delta J$$

- **Semiclassical analysis identifies**

$$T_H = \frac{\kappa}{2\pi}.$$

- **Very natural to identify**

$$S = \frac{A}{4G_N}$$

and look for statistical explanation.

String theory

- **Extremal charged black hole $\sim$ D-branes**
- **Excitations can be counted, account for**

$$S = \frac{A}{4G_N}$$

[Strominger-Vafa], [Maldacena-Strominger-Witten]

String theory

- **Extremal charged black hole $\sim$ D-branes**
- **Excitations can be counted, account for**

$$S = \frac{A}{4G_N} + \cdots$$

[Strominger-Vafa], [Maldacena-Strominger-Witten]

- **Also predicts subleading corrections.**

Corrections to the area law

- **Einstein-Hilbert**

$$\mathcal{L} = \sqrt{-g}\ \frac{R}{16\pi G_N}$$

- **Area law**

$$S = \frac{A}{4G_N}$$

Corrections to the area law

- **Einstein-Hilbert corrected:**

$$\mathcal{L} = \sqrt{-g}\left(\frac{R}{16\pi G_N} + \frac{c}{2}R^2 + \cdots\right)$$

- **Area law accordingly modified :**

$$S = \frac{A}{4G_N} + 8\pi c \int_{\text{hor}} R_{rtrt} + \cdots$$

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$$S = \frac{A}{4G_N} + 8\pi c \int_{\text{hor}} R_{rtrt} + \cdots$$

- **Wald's formula:**

$$S = -2\pi \int_{\text{hor}} \sqrt{-g}\frac{\delta\mathcal{L}}{\delta R_{abcd}}\epsilon_{ab}\epsilon_{cd}$$

String theory works

- Two way to calculate corrections:

Microscopic	Macroscopic
d.o.f. on the brane	Wald's formula

- Completely agrees !

[de Wit et al.][Ooguri-Strominger-Vafa]

String theory works

- Two way to calculate corrections:

Microscopic	Macroscopic
d.o.f. on the brane	Wald's formula

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[de Wit et al.][Ooguri-Strominger-Vafa]

- In four dimensions.

Odd dimensions

- **Black rings in 5d.**
- **Entropy correction:**

Microscopic	**Macroscopic**
done	**not yet**

- **Why ?**

Odd dimensions

- **Black rings in 5d.**
- **Entropy correction:**

Microscopic	Macroscopic
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- **Why ? Wald's formula isn't applicable to gravitational Chern-Simons:**

$$\int F \wedge \mathrm{tr}\, \Gamma \wedge R$$

Odd dimensions

- Black rings in 5d.
- Entropy correction:

Microscopic	Macroscopic
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- Why ? Wald's formula isn't applicable to gravitational Chern-Simons:

$$\int F \wedge \mathrm{tr}\, \Gamma \wedge R$$

Our aim today

Entropy correction from grav. CS.

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2. **Black Hole Spacetime**

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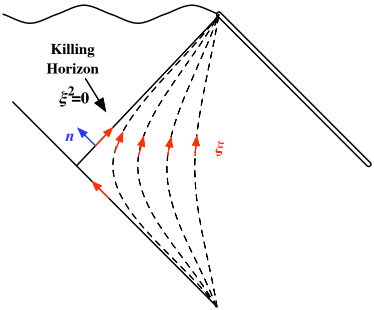
Terminology



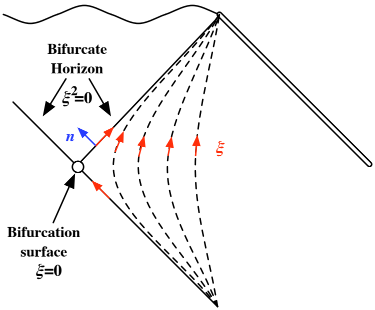
- **Killing Horizon : $\xi^2 = 0$.**
- **Binormal**

$$\epsilon_{ab} = \xi_a n_b - \xi_b n_a$$

- **Surface gravity**

$$\nabla_a \xi_b = \kappa \epsilon_{ab} + \cdots$$

Terminology

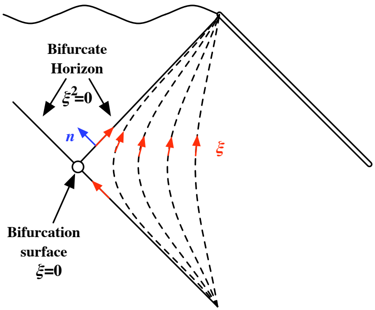


- **Bifurcation Surface**

$$\xi = 0.$$

- **Bifurcate horizon :**
  A pair of horizons which pass BS.

Hawking radiation

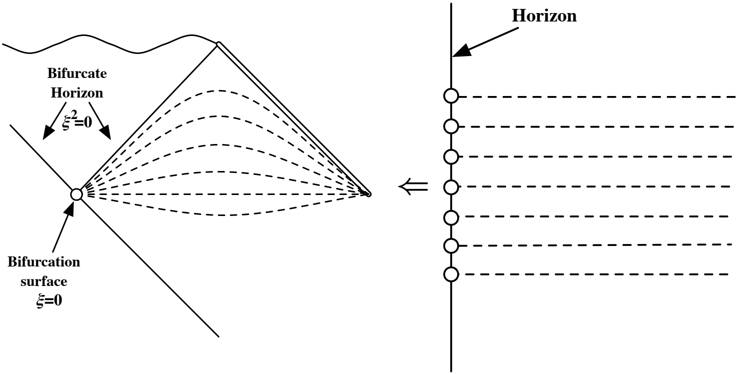


- **Free field radiates at**

$$T_H = \kappa/2\pi.$$

- **Unruh effect near B.S.**
- **Depends only on bg metric,**
- **Not quantum gravity per se.**

Bifurcation surface



- **Horizon cross section at finite t is the b.s.**

Holonomy at B.S.

- At the B.S.,

$$\nabla_c \epsilon_{ab} = \kappa^{-1} \nabla_c \nabla_a \xi_b = \kappa^{-1} R_{abcd} \xi^d = 0.$$

→ holonomy reduces to

$$SO(D-1,1) \supset SO(1,1)_N \times SO(D-2)$$

Black Hole Thermodynamics

Zeroth law

κ constant on the horizon.

First law

$$\frac{\kappa}{2\pi}\frac{\delta A}{4G_N} = \delta m - \Omega\delta J$$

Second law

A increases with time.

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Higher derivative corrections

- RG perspective
  → Planck suppressed terms R^2, R^4 etc.
- Coefficients calculable in string compactifications
- Affects black hole solutions.

Conceptual problems

- Null dir. of $g_{ab} \neq$ maximal propagation speed.
- e.g. Field redefinition

$$g_{ab} \to g_{ab} + cR_{ab} + c'Rg_{ab} + \cdots$$

changes the 'light cone'

- Physics should be invariant !

Bifurcate Horizon in Higher derivative gravity

- Define the horizon using the light cone of g_{ab}.
- Suppose it has Killing, bifurcate horizon :
  under $g_{ab} \to g_{ab} + h_{ab}$,

  Bifurcation surface

  invariant, defined by $\xi^a = 0$

  Bifurcate horizon

  $\xi^a \xi_a \to g_{ab}\xi^a\xi^b + h_{ab}\xi^a\xi^b$.
  2nd term invariant under ξ
  $\longrightarrow$ zero by evaluating at B.S.

  Hawking temperature

  Evaluate $\kappa^2 = |\nabla_a \xi^b|^2$ at B.S.
  $\longrightarrow$ Christoffel drops off because $\xi^a = 0$.

Black Hole Thermo. in Higher Derivative Gravity

Zeroth law

κ constant on the horizon : assumption

First law

$$\frac{\kappa}{2\pi} S = \delta m - \Omega \delta J,$$ S given by Wald's formula

Second law

Difficult to establish

Wald's formula

Black Hole Entropy

$$S = -2\pi \int_{\text{hor}} \sqrt{-g}\,\frac{\delta\mathcal{L}}{\delta R_{abcd}}\epsilon_{ab}\epsilon_{cd}$$

for $\quad \mathcal{L} = \mathcal{L}(g_{ab}, R_{abcd}\,, \nabla_e R_{abcd}\,, \cdots ; \phi, \cdots)$

- It does not include gravitational Chern-Simons

$$*\delta\mathcal{L} = \text{tr}\,\Gamma \wedge R^{2n-1}$$

- or Green-Schwarz type coupling

$$\delta\mathcal{L} = \frac{1}{2}|H|^2 \quad \text{with} \quad H = dB + \text{tr}\,\Gamma \wedge R.$$

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Notation

- D space-time dimension
- $L(\phi)$ Lagrangian density as D-form
- ϕ collective symbol for the fields. $g_{\mu\nu}$, A_μ, …
- $\mathcal{L}_\xi$ Lie derivative by a vector field ξ,

$$\mathcal{L}_\xi \omega = (d\iota_\xi + \iota_\xi d)\omega.$$

- δ_ξ Variation under diffeo. e.g.

$$\delta_\xi \Gamma^a_b = \mathcal{L}_\xi \Gamma^a_b + d(U_\xi)^a_b$$

where $(U_\xi)^a_b = \partial_b \xi^a$.

Diffeomorphism Invariance

- **Assumption by Wald:**

$$\delta_\xi L(\phi) = \pounds_\xi L(\phi).$$

Cannot incorporate the CS.

- **Today:**

$$\delta_\xi L(\phi) = \pounds_\xi L(\phi) + d\Xi_\xi.$$

- **Almost verbatim transcript of Wald's original.**

Covariant Hamiltonian Method

- EOM E_ϕ and symplectic potential Θ via

$$\delta L = E_\phi \delta\phi + d\Theta(\phi, \delta\phi)$$

- Θ pairs of 'coordinate' and 'momenta'

$$L(\phi) = \frac{1}{2} * d\phi \wedge d\phi \longrightarrow \Theta = *d\phi \wedge \delta\phi$$

- symplectic form given by

$$\Omega(\phi, \delta_1\phi, \delta_2\phi) = \delta_1\Theta(\phi, \delta_2\phi) - \delta_2\Theta(\phi, \delta_1\phi).$$

Lemma

- j : a form constructed from
    - fields ϕ
    - an external field ξ
- suppose j is closed on-shell for any ξ.
- It is then exact on-shell. i.e.

$$dj_\xi \simeq 0 \longrightarrow j_\xi \simeq dQ_\xi.$$

- $\simeq$: equality on-shell.

Noether's theorem

- **Symmetry leads to conserved current :**

$$j_\xi = \Theta(\phi, \delta_\xi \phi) - \iota_\xi L - \Xi_\xi$$

satisfies

$$dj_\xi \simeq 0$$

- **The lemma implies**

$$j_\xi \simeq dQ_\xi, \quad \longrightarrow \quad \int_C j_\xi \simeq \int_{\partial C} Q_\xi.$$

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i.e.

Global symmetry	→	Conserved charges
Gauge symmetry	→	Gauss law

Un-illuminating main part

- **Define Π_ξ via**

$$\delta_\xi \Theta = \pounds_\xi \Theta + \Pi_\xi.$$

- **Recall $\delta L = E_\phi \delta\phi + d\Theta, \quad \delta_\xi = \pounds_\xi L + d\Xi_\xi.$**

Un-illuminating main part

- Define Π_ξ via
$$\delta_\xi \Theta = \pounds_\xi \Theta + \Pi_\xi.$$
- Recall $\delta L = E_\phi \delta\phi + d\Theta, \quad \delta_\xi = \pounds_\xi L + d\Xi_\xi.$
- Calculating $\delta\delta_\xi L$ in two ways:
$$d\Pi_\xi \simeq \delta d\,\Xi_\xi \longrightarrow \Pi_\xi - \delta\,\Xi_\xi \simeq d\Sigma_\xi.$$

$$\longrightarrow \quad \begin{aligned} \delta j_\xi &= \delta\Theta(\phi, \delta_\xi\phi) - \iota_\xi \delta L - \delta\,\Xi_\xi \\ &\simeq \delta\Theta(\phi, \delta_\xi\phi) - \delta_\xi\Theta(\phi, \delta\phi) + d\iota_\xi\Theta + \Pi_\xi - \delta\,\Xi_\xi \\ &\simeq \Omega(\phi, \delta\phi, \delta_\xi\phi) + d(\iota_\xi\Theta + \Sigma_\xi). \end{aligned}$$

Un-illuminating main part

- Define Π_ξ via

$$\delta_\xi \Theta = \pounds_\xi \Theta + \Pi_\xi .$$

- Recall $\delta L = E_\phi \delta\phi + d\Theta, \quad \delta_\xi = \pounds_\xi L + d\Xi_\xi .$
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- Thus, for C_ξ with $\delta C_\xi = \iota_\xi\Theta + \Sigma_\xi$,

$$\delta d Q'_\xi \simeq \Omega(\phi, \delta\phi, \delta_\xi\phi)$$

where

$$Q'_\xi = Q_\xi - C_\xi .$$

Recap.

- $\delta dQ'_\xi \simeq \Omega(\phi, \delta\phi, \delta_\xi\phi)$ means

$$\int dQ'_\xi \text{ is the Hamiltonian generating } \xi.$$

- Let $\xi = t + \Omega\phi$ where

  ξ Horizon generating Killing
  t global time translation
  ϕ angular rotation

  Then

$$\delta \int_{\text{hor}} Q'_\xi \simeq \delta \int_\infty Q'_t + \Omega\, \delta \int_\infty Q'_\phi.$$

$$\delta \int_{\text{hor}} Q'_\xi \simeq \delta \int_\infty Q'_t + \Omega\, \delta \int_\infty Q'_\phi.$$

- $E = \int_\infty Q'_t$, $J = \delta \int_\infty Q'_\phi$.
- Taking the horizon at the bifurcation surface,

$$\int_{\text{hor}} Q'_\xi = \kappa \int_{\text{hor}} Q'_\xi \,\big|_{\xi \to 0, \nabla_a \xi_b \to \epsilon_{ab}}$$

- Then, define

$$S = 2\pi \int_{\text{hor}} Q'_\xi \,\big|_{\xi \to 0, \nabla_a \xi_b \to \epsilon_{ab}}$$

Result.

$$\frac{\kappa}{2\pi} \delta S = \delta E + \Omega \delta J.$$

3d Chern-Simons

- Start from $L_{CS} = \beta\,\mathrm{tr}(\Gamma R - \frac{1}{3}\Gamma^3)$

  $\longrightarrow$ $\delta_\xi L_{CS} = \pounds_\xi L_{CS} - d(\beta\,\mathrm{tr}\,dU_\xi \Gamma)$ $\longrightarrow$ $\Xi_\xi = -\beta\,\mathrm{tr}\,dU_\xi\Gamma$

- Non-covariant part in Θ is $-\beta\,\mathrm{tr}\,\Gamma\delta\Gamma$

  $\longrightarrow$ $\Pi_\xi = -\beta\,\mathrm{tr}\,dU_\xi\delta\Gamma$

  $\longrightarrow$ $\Pi_\xi - \delta\Xi_\xi \simeq 0$ $\longrightarrow$ $\Sigma_\xi = 0$

  $\longrightarrow$ $j_\xi = \Theta(\phi, \delta_\xi\phi) - \Xi_\xi + \cdots = 2\beta\,\mathrm{tr}\,dU_\xi\Gamma + \cdots$

  $\longrightarrow$ $Q'_\xi = 2\beta\,\mathrm{tr}\,U_\xi\Gamma + \cdots$.

- $(U_\xi)^a_b = \partial_b\xi^a$.

3d Chern-Simons

- $L_{CS} = \beta\,\mathrm{tr}(\Gamma R - \frac{1}{3}\Gamma^3) \longrightarrow S_{CS} = 8\pi\beta \int_{\mathrm{hor}} \Gamma_N$

  where $\Gamma_N = -\epsilon^{\nu}{}_{\mu}\Gamma^{\mu}{}_{\nu\rho}dx^{\rho}/2$.

- agrees with known corrections to BTZ.
- Naive application of Wald $\longrightarrow$ $4\pi\beta \int_{\mathrm{hor}} \Gamma_N$.

Chern-Simons in General Dimensions

- $L_{CS} = \beta\,\mathrm{tr}(\Gamma R^{2m-1} + \cdots)$

  $\longrightarrow \quad S_{CS} = 8\pi m\beta \int_{\mathrm{hor}} \Gamma_N R_N^{2m-2}$

- Recall

  Holonomy at the bifurcation surface

  $$= SO(1,1)_N \times SO(D-2)$$

- Entropy correction = CS of the normal bundle !

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Summary

- **Spacetime with black holes reviewed.**
- **Entropy correction from the grav. CS.**

Summary

- Spacetime with black holes reviewed.
- Entropy correction from the grav. CS.

Outlook – Need application !

- Black rings. – working on it with friends.
- 7d black holes. – idea wanted.

$$\mathrm{tr}\,\Gamma \wedge R^3 \longrightarrow \int_{\mathrm{hor}} \Gamma_N \wedge R_N^2$$