Surface operators and W-algebras

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based on various papers
with the help of various people
Most recently with H. Kanno, [1105.0357]

May 2011

4d instantons

$$\Updownarrow$$

W-algebras.

What is the phenomenon?

(no string theory necessary)

Why does it happen?

(string theory provides the explanation)

How should we generalize?

(W-algebras come back with vengeance.)

Contents

Virasoro coherent state

- Consider the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}\delta_{m,-n}(m^3 - m)$$

- and the Verma module

$$|\Delta\rangle, \quad L_{-1}|\Delta\rangle, \quad L_{-1}^2|\Delta\rangle, \quad L_{-2}|\Delta\rangle, \quad L_{-1}^3|\Delta\rangle, \quad \ldots$$

- $|\Delta\rangle$ annihilated by all annihilation operators:

$$L_0|\Delta\rangle = \Delta|\Delta\rangle, \qquad L_n|\Delta\rangle = 0, \qquad n > 0$$

- We'd like to consider a nice coherent state of the Virasoro algebra:

$$L_1|q,\Delta\rangle = q|q,\Delta\rangle, \quad L_{n\geq 2}|q,\Delta\rangle = 0,$$

normalized to have

$$|q,\Delta\rangle = |\Delta\rangle + \text{descendants}.$$

- It's not $e^{qL_{-1}}|\Delta\rangle$, but can be determined order by order:

$$|q,\Delta\rangle = |\Delta\rangle + \frac{q}{2\Delta}L_{-1}|\Delta\rangle + \cdots$$

with the norm

$$\langle q,\Delta|q,\Delta\rangle = 1 + \frac{q^2}{2\Delta} + \frac{q^4(c+8\Delta)}{4\Delta((1+2\Delta)c - 10\Delta + 16\Delta^2)} + \cdots$$

Instantons

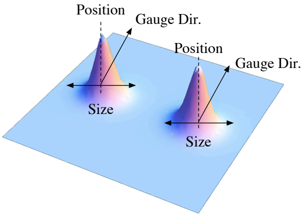
- Consider an instanton

$$F = dA + A \wedge A, \quad F = - \star F$$

where A is an $\mathbf{SU(2)}$ gauge field on $\mathbb{R}^4$ with

$$k = \frac{1}{8\pi^2} \int \mathbf{tr}\, F \wedge F \ : \ \text{positive integer}$$

- The space of parameters : $M_{2,k}$, $\quad \mathbf{dim}_{\mathbb{R}} = 8k$



- Consider a toy statistical mechanical model of instantons:

$$Z = \sum_k \Lambda^{4k} \int_{M_{2,k}} \mathrm{dvol}$$

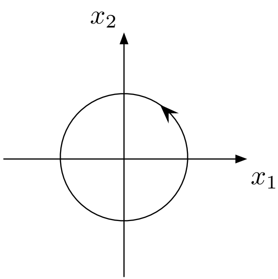
- Of course it diverges due to the volume of $\mathbb{R}^4$.
  Need to put the system into a "box".

- Consider a very stupid integral

$$\text{Area of } \mathbb{R}^2 = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2.$$

- Regularize it via

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \; e^{-\pi\epsilon(x_1^2+x_2^2)} = \frac{1}{\epsilon}.$$



- Let $\{x_1, x_2\}_{\text{P.B.}} = 1$.
- $H = x_1^2 + x_2^2$ is the rotation generator
- ϵ is the chemical potential.

- On $\mathbb{R}^4$, let $\{x_1, x_2\}_{\text{P.B.}} = \{x_3, x_4\}_{\text{P.B.}} = 1$.
- $M_{2,k}$ inherits the Poisson bracket.
- Introduce chemical potentials / Hamiltonians

ϵ_1	H_1	for the rotation of x_1-x_2 plane,
ϵ_2	H_2	for the rotation of x_3-x_4 plane,
a	J	for the σ^3 of the global $\mathbf{SU}(2)$ rotation.

- Our toy model is then

$$Z = \sum_k \Lambda^{4k} Z_k \quad \text{where} \quad Z_k = \int_{M_{2,k}} e^{-\pi(\epsilon_1 H_1 + \epsilon_2 H_2 + aJ)} \text{dvol}.$$

- It's just an ensemble of instantons
  with chemical potentials for all the conserved quantities.
- Exactly calculable. [Moore-Nekrasov-Shatashvili]

Statistical mechanical toy model of 4d instantons

We consider

$$Z(\Lambda, \epsilon_1, \epsilon_2, a) = \sum_k \Lambda^{4k} \int_{M_{2,k}} e^{-\pi(\epsilon_1 H_1 + \epsilon_2 H_2 + aJ)} \text{dvol}$$

Explicitly, we have

$$Z = 1 + \Lambda^4 \frac{2}{\epsilon_1\epsilon_2((\epsilon_1+\epsilon_2)^2 - 4a^2)} + \\ + \Lambda^8 \frac{((\epsilon_1+\epsilon_2)^2 + \epsilon_1\epsilon_2/8 - a^2)/(8\epsilon_1^2\epsilon_2^2)}{((\frac{\epsilon_1+\epsilon_2}{2})^2 - a^2)((\epsilon_1 + \frac{\epsilon_2}{2})^2 - a^2)((\frac{\epsilon_1}{2} + \epsilon_2)^2 - a^2)} + \cdots$$

Norm of Virasoro coherent state

We take

$$L_1|q,\Delta\rangle = q|q,\Delta\rangle, \quad L_{n\geq 2}|q,\Delta\rangle = 0,$$

normalized to have

$$|q,\Delta\rangle = |\Delta\rangle + \text{descendants}.$$

The norm was

$${}_c\langle q,\Delta|q,\Delta\rangle_c = 1 + \frac{q^2}{2\Delta} + \frac{q^4(c+8\Delta)}{4\Delta((1+2\Delta)c - 10\Delta + 16\Delta^2)} + \cdots$$

There's an equality [Alday-Gaiotto-YT], [Gaiotto]

$$ {}_c\langle q, \Delta | q, \Delta \rangle_c = Z(\Lambda, \epsilon_1, \epsilon_2, a) $$

under the identification

$$ \begin{aligned} c &= 1 + 6\frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2}, \\ \Delta &= \frac{(\epsilon_1 + \epsilon_2)^2}{4\epsilon_1 \epsilon_2} - \frac{a^2}{\epsilon_1 \epsilon_2}, \\ q &= \frac{\Lambda^2}{\epsilon_1 \epsilon_2}. \end{aligned} $$

- Note that it's not that strange that $Z = \langle\psi|\psi\rangle$:

$$Z = \sum_k \Lambda^{4k} \int_{M_{2,k}} e^{-\pi(\epsilon_1 H_1 + \epsilon_2 H_2 + aJ)} \mathrm{dvol}$$
$$= 1 + \Lambda^4 \langle\psi_1|\psi_1\rangle + \Lambda^8 \langle\psi_2|\psi_2\rangle + \cdots$$

where

$$\psi_k(X_1, \ldots, X_{8k}) = \exp^{-\frac{\pi}{2}(\epsilon_1 H_1(X_1,\ldots,X_{8k}) + \cdots)}$$

is a wavefunction in $\mathcal{H}(M_{2,k})$: the Hilbert sp. of QM on $M_{2,k}$.

- Therefore,

$$Z = \langle\psi|\psi\rangle \quad \text{where} \quad |\psi\rangle = |\psi_0\rangle \oplus \Lambda^2 |\psi_1\rangle \oplus \cdots$$

- So, to have

$$\langle q, \Delta | q, \Delta \rangle = Z(\Lambda, \epsilon_1, \epsilon_2, a) = \langle \psi | \psi \rangle,$$

we just need

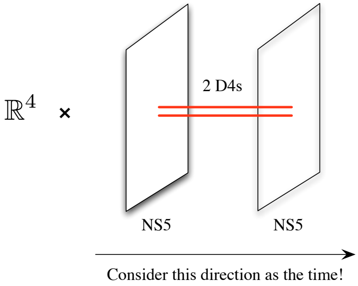
$$|q, \Delta\rangle = |\psi\rangle.$$

- It's just like Heisenberg's matrix mechanics $\leftrightarrow$ Schrödinger's wave mechanics.
- We need Dirac's transformation theory.

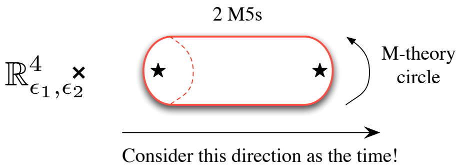
Contents

IIA realization and M-theory lift

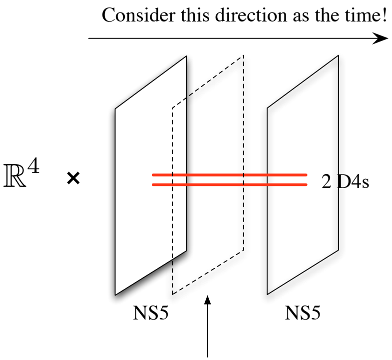
- Witten's IIA construction.
- 4d $\mathcal{N} = 2$ $\mathbf{SU(2)}$ theory.
- At constant 'time',
  the **BPS config.** is **instantons**.
  A **BPS state** is in $\oplus_k H^*_G(M_{2,k})$.
- NS5-branes at the start/end of 'time'
  prepare the same state $|\psi\rangle$.
- So $Z = \langle\psi|\psi\rangle$.



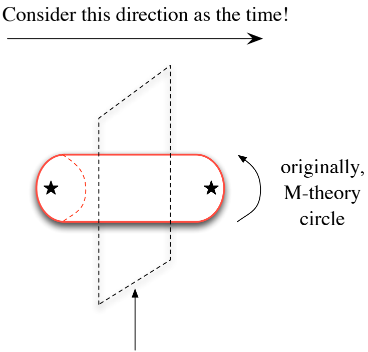
- Lift the system to M-theory.



- We get some 2d theory. This has Virasoro symmetry.
- That's **why** $\oplus_k H^*_G(M_{2,k})$ becomes **Virasoro Verma module**.
- $L_0 \sim$ KK momenta along the M-theory circle.
- which are D0s in the IIA description
- which are instantons on D4s!



At "constant time", we have instantons



At "constant time", we have the Virasoro Verma module

- Why do we have coherent states at the tips?
- M5-branes are wrapped on the Seiberg-Witten curve, given by

$$\lambda^2 - \phi_2(z) = 0, \qquad \phi_2(z) = (\Lambda^2 z + u + \frac{\Lambda^2}{z})\frac{dz^2}{z^2}.$$

- The general rule is that $\epsilon_1\epsilon_2\langle T(z)\rangle dz^2 \to \phi_2(z)$.
- Recall

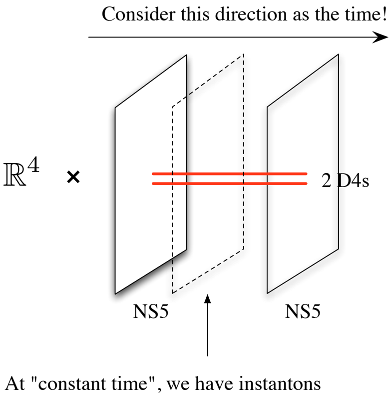
$$T(z) \sim \cdots + \frac{L_0}{z^2} + \frac{L_1}{z^3} + \cdots .$$

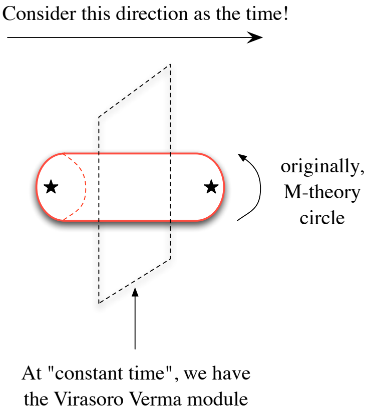
Then we see

$$L_0|\psi\rangle \sim \frac{u}{\epsilon_1\epsilon_2}|\psi\rangle, \qquad L_1|\psi\rangle \sim \frac{\Lambda^2}{\epsilon_1\epsilon_2}|\psi\rangle.$$

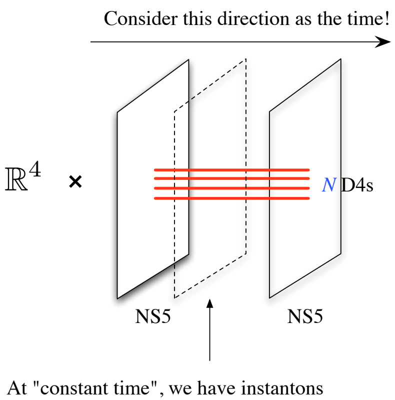
Contents

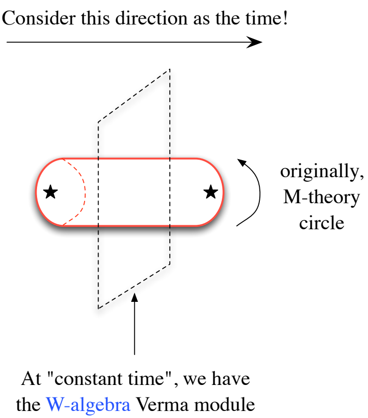
SU(2)


Consider this direction as the time!
$\mathbb{R}^4$ ×
2 D4s
NS5
NS5
At "constant time", we have instantons


Consider this direction as the time!
originally,
M-theory
circle
At "constant time", we have
the Virasoro Verma module

SU(N)


Consider this direction as the time!
$\mathbb{R}^4$ ×
N D4s
NS5
NS5
At "constant time", we have instantons


Consider this direction as the time!
originally,
M-theory
circle
At "constant time", we have
the W-algebra Verma module

- For $\mathbf{SU}(N)$, the statistical-mechanical model is

$$Z = \sum_k \Lambda^{2Nk} \int_{M_{N,k}} e^{-\pi(\epsilon_1 H_1 + \epsilon_2 H_2 + \sum a_i J_i)} \mathrm{dvol}$$

  where

$M_{N,k}$: the moduli space of k instantons of $\mathbf{SU}(N)$
(a_i) : chemical potentials for $\mathbf{SU}(N)$
(J_i) : Hamiltonians for $\mathbf{SU}(N)$

- Virasoro algebra = $W_2 = W(\hat{SU}(2))$ had

$$T(z)$$

corresponding to the Seiberg-Witten curve

$$\lambda^2 + \phi_2(z) = 0$$

- $W_N = W(\hat{SU}(N))$ algebra has

$$W_2(z), W_3(z), \ldots, W_N(z)$$

corresponding to the Seiberg-Witten curve

$$\lambda^N + \phi_2(z)\lambda^{N-2} + \cdots + \phi_N(z) = 0$$

- The instanton partition function

$$Z = \sum_k \Lambda^{2Nk} \int_{M_{N,k}} e^{-\pi(\epsilon_1 H_1 + \epsilon_2 H_2 + \sum a_i J_i)} \mathrm{dvol}$$
$$= \langle \psi | \psi \rangle$$

is the norm of the coherent state $|\psi\rangle$:

$$W_{N,1} |\psi\rangle \sim \frac{\Lambda^N}{(\epsilon_1 \epsilon_2)^{N/2}} |\psi\rangle$$

- For $N = 3$, checked by [Taki].
  For $N \geq 4$, not done yet. I don't know why...
- Rumored that [Maulik-Okounkov] have a geometric proof.

More general W-algebras

There are more general W-algebras [de Boer-Tjin '93]

$$W(\hat{\mathbf{SU}}(N), [n_1, \ldots, n_M]) \qquad \text{where} \quad n_1 + \cdots + n_M = N$$

with the special cases

$$W_N = W(\hat{\mathbf{SU}}(N), [N]),$$
$$\hat{\mathbf{SU}}(N) = W(\hat{\mathbf{SU}}(N), [1, \ldots, 1])$$

- E.g. $W(\hat{\mathbf{SU}}(N), [\overbrace{\mathbf{2},\ldots,\mathbf{2}}^{m}, \overbrace{\mathbf{1},\ldots,\mathbf{1}}^{n}])$ has

spin one	$J^a_b(z)$, $j^i_j(z)$,
spin 3/2	$G^a_i(z)$, $\tilde{G}^i_a(z)$
spin 2	$S^a_b(z)$

where $a, b = 1, \ldots, m$ and $i, j = 1, \ldots, n$.

- $W([\mathbf{2}, \mathbf{1}])$ is called Bershadsky-Polyakov algebra;
$W([\mathbf{2}, \mathbf{1}, \ldots, \mathbf{1}])$ is Romans' quasisuperconformal algebra.
(both 1991)

- Do they appear in 4d gauge theory?

- The answer is Yes! [Braverman-Feigin-Finkelberg-Rybnikov], [Wyllard], [Kanno-YT]
- Consider N D4-branes on

$$\mathbb{R}_t \times \mathbb{C}_z \times (\mathbb{C}_w/\mathbb{Z}_M)$$

where $\mathbb{Z}_M$ acts on the Chan-Paton indices as

$$(\underbrace{\omega, \ldots, \omega}_{n_1}, \underbrace{\omega^2, \ldots, \omega^2}_{n_2}, \ldots, \underbrace{\omega^M, \ldots, \omega^M}_{n_M})$$

- Bulk D0s + fractional D0s bound at the surface operator $w = 0$

- As before, the BPS part of the dynamics reduces to susy QM on the (fractional) instantons, with the Hilbert space

$$\mathcal{H} = H^*(\text{moduli space of instantons on } \mathbb{C}_z \times (\mathbb{C}_w/\mathbb{Z}_M))$$

- This equals the Verma module of the W-algebra

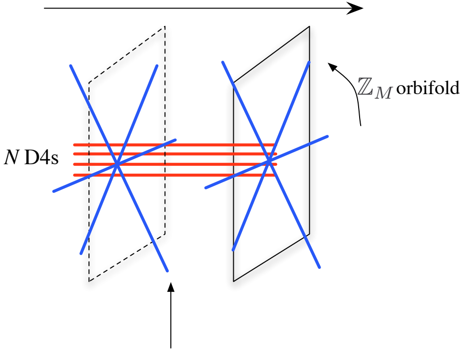
$$W(\hat{\mathbf{SU}}(N), [n_1, \ldots, n_M])$$

- Moreover, **the instanton partition function**
  = **the norm of the coherent state**

- (only explicitly checked for $W([2, \ldots, 2, 1, \ldots, 1])$, though)

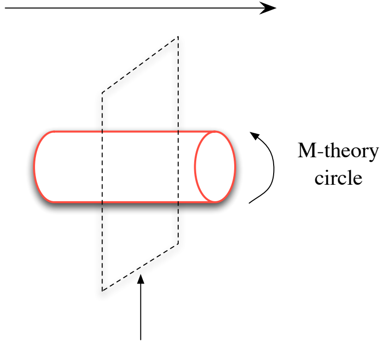
Summary

Consider this direction as the time!



At "constant time", we have instantons

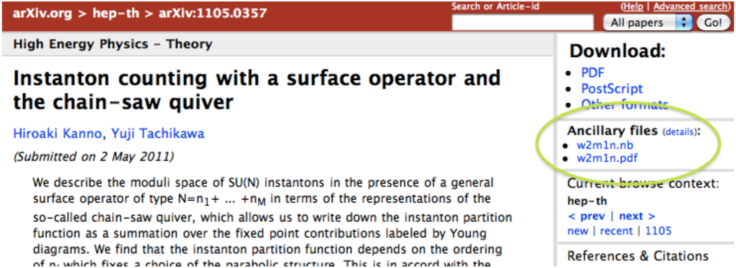
Consider this direction as the time!



At "constant time", we have
the general W-algebra Verma module

- W_N-algebra reappeared also in the context of higher-spin AdS_3/CFT_2 last year. [Gaberdiel,Gopakumar,Hartman,...]
- Might be a good time to study W-algebras again, after 17-year hiatus.
- One good thing is that PC is much faster now. We can do more complicated W-algebra calculation in short time, on this laptop.
- Due to programming complexity, detailed checks were done only for $W([2, \ldots, 2, 1, \ldots, 1])$. Anyone wants to help me?

- Mathematica code to calculate the **instanton partition function** and the **norm of the W-algebra coherent state** (at least the cases implemented) is available on the preprint webpage. **Enjoy**!



- You should also post your Mathematica / Maple code to the arXiv!