

Surface operators and W -algebras

Yuji Tachikawa (IPMU&IAS)

based on various papers
with the help of various people
Most recently with H. Kanno, [\[1105.0357\]](#)

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4d instantons



W-algebras.

What is the phenomenon?

(no string theory necessary)

Why does it happen?

(string theory provides the explanation)

How should we generalize?

(W-algebras come back with vengeance.)

Contents

- 1. What is the phenomenon?**
2. Why does it happen?
3. How should we generalize?

Virasoro coherent state

- Consider the Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}\delta_{m,-n}(m^3 - m)$$

- and the Verma module

$$|\Delta\rangle, \quad L_{-1}|\Delta\rangle, \quad L_{-1}^2|\Delta\rangle, \quad L_{-2}|\Delta\rangle, \quad L_{-1}^3|\Delta\rangle, \quad \dots$$

- $|\Delta\rangle$ annihilated by all annihilation operators:

$$L_0|\Delta\rangle = \Delta|\Delta\rangle, \quad L_n|\Delta\rangle = 0, \quad n > 0$$

- We'd like to consider a nice coherent state of the Virasoro algebra:

$$L_1|q, \Delta\rangle = q|q, \Delta\rangle, \quad L_{n \geq 2}|q, \Delta\rangle = 0,$$

normalized to have

$$|q, \Delta\rangle = |\Delta\rangle + \text{descendants.}$$

- It's not $e^{qL_{-1}}|\Delta\rangle$, but can be determined order by order:

$$|q, \Delta\rangle = |\Delta\rangle + \frac{q}{2\Delta}L_{-1}|\Delta\rangle + \dots$$

with the norm

$$\langle q, \Delta | q, \Delta \rangle = 1 + \frac{q^2}{2\Delta} + \frac{q^4(c + 8\Delta)}{4\Delta((1 + 2\Delta)c - 10\Delta + 16\Delta^2)} + \dots$$

Instantons

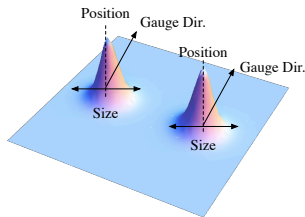
- Consider an instanton

$$F = dA + A \wedge A, \quad F = -\star F$$

where A is an $\mathbf{SU}(2)$ gauge field on \mathbb{R}^4 with

$$k = \frac{1}{8\pi^2} \int \text{tr } F \wedge F : \text{ positive integer}$$

- The space of parameters : $M_{2,k}$, $\dim_{\mathbb{R}} = 8k$



- Consider a toy statistical mechanical model of instantons:

$$Z = \sum_k \Lambda^{4k} \int_{M_{2,k}} \text{dvol}$$

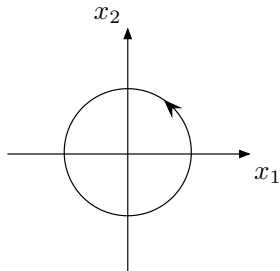
- Of course it diverges due to the volume of \mathbb{R}^4 .
Need to put the system into a “box”.

- Consider a very stupid integral

$$\text{Area of } \mathbb{R}^2 = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2.$$

- Regularize it via

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 e^{-\pi\epsilon(x_1^2+x_2^2)} = \frac{1}{\epsilon}.$$



- Let $\{x_1, x_2\}_{\text{P.B.}} = 1$.
- $H = x_1^2 + x_2^2$ is the rotation generator
- ϵ is the chemical potential.

- On \mathbb{R}^4 , let $\{x_1, x_2\}_{\text{P.B.}} = \{x_3, x_4\}_{\text{P.B.}} = 1$.
- $M_{2,k}$ inherits the Poisson bracket.
- Introduce chemical potentials / Hamiltonians
 - $\epsilon_1 \quad H_1$ for the rotation of x_1 - x_2 plane,
 - $\epsilon_2 \quad H_2$ for the rotation of x_3 - x_4 plane,
 - $a \quad J$ for the σ^3 of the global $\mathbf{SU}(2)$ rotation.
- Our toy model is then

$$Z = \sum_k \Lambda^{4k} Z_k \quad \text{where} \quad Z_k = \int_{M_{2,k}} e^{-\pi(\epsilon_1 H_1 + \epsilon_2 H_2 + aJ)} \text{dvol.}$$

- It's just an ensemble of instantons with chemical potentials for all the conserved quantities.
- Exactly calculable. [Moore-Nekrasov-Shatashvili]

Statistical mechanical toy model of 4d instantons

We consider

$$Z(\Lambda, \epsilon_1, \epsilon_2, a) = \sum_k \Lambda^{4k} \int_{M_{2,k}} e^{-\pi(\epsilon_1 H_1 + \epsilon_2 H_2 + aJ)} \mathrm{dvol}$$

Explicitly, we have

$$\begin{aligned} Z = & 1 + \Lambda^4 \frac{2}{\epsilon_1 \epsilon_2 ((\epsilon_1 + \epsilon_2)^2 - 4a^2)} + \\ & + \Lambda^8 \frac{((\epsilon_1 + \epsilon_2)^2 + \epsilon_1 \epsilon_2 / 8 - a^2) / (8\epsilon_1^2 \epsilon_2^2)}{((\frac{\epsilon_1 + \epsilon_2}{2})^2 - a^2)((\epsilon_1 + \frac{\epsilon_2}{2})^2 - a^2)((\frac{\epsilon_1}{2} + \epsilon_2)^2 - a^2)} + \dots \end{aligned}$$

Norm of Virasoro coherent state

We take

$$L_1|q, \Delta\rangle = q|q, \Delta\rangle, \quad L_{n \geq 2}|q, \Delta\rangle = 0,$$

normalized to have

$$|q, \Delta\rangle = |\Delta\rangle + \text{descendants}.$$

The norm was

$${}_c\langle q, \Delta | q, \Delta \rangle_c = 1 + \frac{q^2}{2\Delta} + \frac{q^4(c + 8\Delta)}{4\Delta((1 + 2\Delta)c - 10\Delta + 16\Delta^2)} + \dots$$

There's an equality [Alday-Gaiotto-YT], [Gaiotto]

$${}_c\langle q, \Delta | q, \Delta \rangle_c = Z(\Lambda, \epsilon_1, \epsilon_2, a)$$

under the identification

$$\begin{aligned} c &= 1 + 6 \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2}, \\ \Delta &= \frac{(\epsilon_1 + \epsilon_2)^2}{4\epsilon_1 \epsilon_2} - \frac{a^2}{\epsilon_1 \epsilon_2}, \\ q &= \frac{\Lambda^2}{\epsilon_1 \epsilon_2}. \end{aligned}$$

- Note that it's not that strange that $Z = \langle \psi | \psi \rangle$:

$$\begin{aligned} Z &= \sum_k \Lambda^{4k} \int_{M_{2,k}} e^{-\pi(\epsilon_1 H_1 + \epsilon_2 H_2 + aJ)} \text{dvol} \\ &= 1 + \Lambda^4 \langle \psi_1 | \psi_1 \rangle + \Lambda^8 \langle \psi_2 | \psi_2 \rangle + \dots \end{aligned}$$

where

$$\psi_k(X_1, \dots, X_{8k}) = \mathbf{exp}^{-\frac{\pi}{2}(\epsilon_1 H_1(X_1, \dots, X_{8k}) + \dots)}$$

is a wavefunction in $\mathcal{H}(M_{2,k})$: the Hilbert sp. of QM on $M_{2,k}$.

- Therefore,

$$Z = \langle \psi | \psi \rangle \quad \text{where} \quad |\psi\rangle = |\psi_0\rangle \oplus \Lambda^2 |\psi_1\rangle \oplus \dots$$

- So, to have

$$\langle q, \Delta | q, \Delta \rangle = Z(\Lambda, \epsilon_1, \epsilon_2, a) = \langle \psi | \psi \rangle,$$

we just need

$$|q, \Delta\rangle = |\psi\rangle.$$

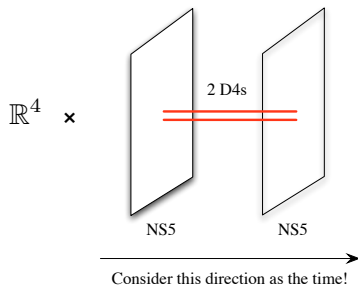
- It's just like Heisenberg's matrix mechanics \leftrightarrow Schrödinger's wave mechanics.
- We need Dirac's transformation theory.

Contents

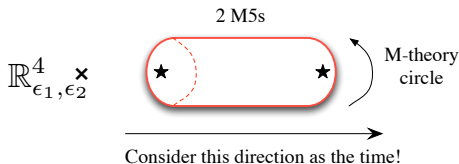
1. What is the phenomenon?
2. Why does it happen?
3. How should we generalize?

IIA realization and M-theory lift

- Witten's IIA construction.
- 4d $\mathcal{N} = 2$ $SU(2)$ theory.
- At constant 'time',
the **BPS config.** is **instantons**.
A **BPS state** is in $\oplus_k H_G^*(M_{2,k})$.
- NS5-branes at the start/end of 'time'
prepare the same state $|\psi\rangle$.
- So $Z = \langle \psi | \psi \rangle$.

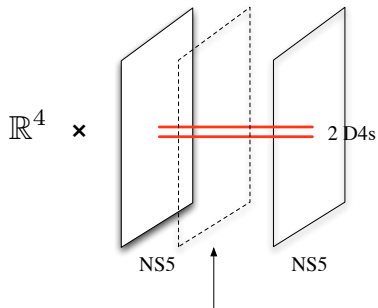


- Lift the system to M-theory.



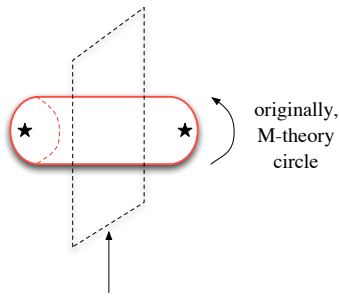
- We get some 2d theory. This has Virasoro symmetry.
- That's **why** $\oplus_k H_G^*(M_{2,k})$ becomes **Virasoro Verma module**.
- $L_0 \sim$ KK momenta along the M-theory circle.
- which are D0s in the IIA description
- which are instantons on D4s!

Consider this direction as the time!



At "constant time", we have instantons

Consider this direction as the time!



At "constant time", we have the Virasoro Verma module

- Why do we have coherent states at the tips?
- M5-branes are wrapped on the Seiberg-Witten curve, given by

$$\lambda^2 - \phi_2(z) = 0, \quad \phi_2(z) = (\Lambda^2 z + u + \frac{\Lambda^2}{z}) \frac{dz^2}{z^2}.$$

- The general rule is that $\epsilon_1 \epsilon_2 \langle T(z) \rangle dz^2 \rightarrow \phi_2(z)$.
- Recall

$$T(z) \sim \dots + \frac{L_0}{z^2} + \frac{L_1}{z^3} + \dots.$$

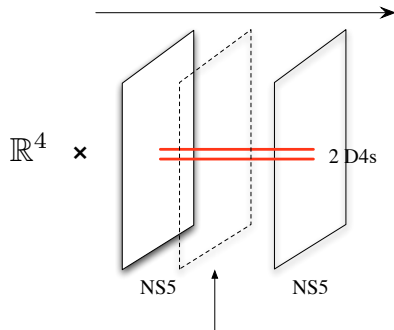
Then we see

$$L_0 |\psi\rangle \sim \frac{u}{\epsilon_1 \epsilon_2} |\psi\rangle, \quad L_1 |\psi\rangle \sim \frac{\Lambda^2}{\epsilon_1 \epsilon_2} |\psi\rangle.$$

Contents

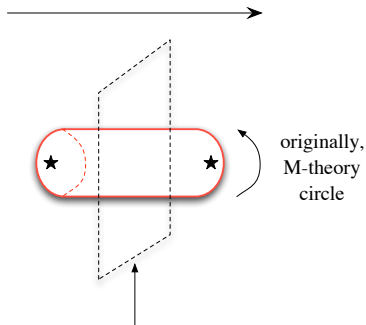
1. What is the phenomenon?
2. Why does it happen?
- 3. How should we generalize?**

Consider this direction as the time!



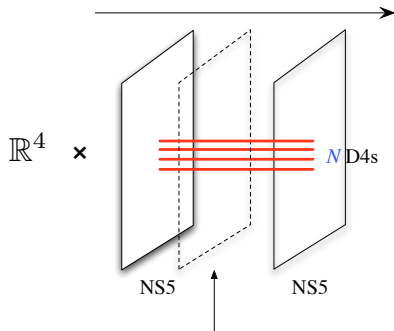
At "constant time", we have instantons

Consider this direction as the time!



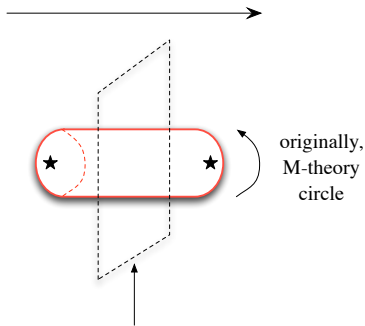
At "constant time", we have the Virasoro Verma module

Consider this direction as the time!



At "constant time", we have instantons

Consider this direction as the time!



At "constant time", we have
the W -algebra Verma module

- For $\mathbf{SU}(N)$, the statistical-mechanical model is

$$Z = \sum_k \Lambda^{2Nk} \int_{M_{N,k}} e^{-\pi(\epsilon_1 H_1 + \epsilon_2 H_2 + \sum a_i J_i)} \mathrm{dvol}$$

where

$M_{N,k}$: the moduli space of k instantons of $\mathbf{SU}(N)$

(a_i) : chemical potentials for $\mathbf{SU}(N)$

(J_i) : Hamiltonians for $\mathbf{SU}(N)$

- Virasoro algebra = $W_2 = W(\hat{\mathbf{S}}\mathbf{U}(2))$ had

$$T(z)$$

corresponding to the Seiberg-Witten curve

$$\lambda^2 + \phi_2(z) = 0$$

- $W_N = W(\hat{\mathbf{S}}\mathbf{U}(N))$ algebra has

$$W_2(z), W_3(z), \dots, W_N(z)$$

corresponding to the Seiberg-Witten curve

$$\lambda^N + \phi_2(z)\lambda^{N-2} + \dots + \phi_N(z) = 0$$

- The instanton partition function

$$\begin{aligned} Z &= \sum_k \Lambda^{2Nk} \int_{M_{N,k}} e^{-\pi(\epsilon_1 H_1 + \epsilon_2 H_2 + \sum a_i J_i)} \mathrm{dvol} \\ &= \langle \psi | \psi \rangle \end{aligned}$$

is the norm of the coherent state $|\psi\rangle$:

$$W_{N,1}|\psi\rangle \sim \frac{\Lambda^N}{(\epsilon_1 \epsilon_2)^{N/2}} |\psi\rangle$$

- For $N = 3$, checked by [Taki].
For $N \geq 4$, not done yet. I don't know why..
- Rumored that [Maulik-Okounkov] have a geometric proof.

More general W -algebras

There are more general W -algebras [de Boer-Tjin '93]

$$W(\hat{\mathbf{S}}\mathbf{U}(N), [n_1, \dots, n_M]) \quad \text{where} \quad n_1 + \dots + n_M = N$$

with the special cases

$$\begin{aligned} W_N &= W(\hat{\mathbf{S}}\mathbf{U}(N), [N]), \\ \hat{\mathbf{S}}\mathbf{U}(N) &= W(\hat{\mathbf{S}}\mathbf{U}(N), [1, \dots, 1]) \end{aligned}$$

- E.g. $W(\hat{\mathbf{SU}}(N), [\overbrace{2, \dots, 2}^m, \overbrace{1, \dots, 1}^n])$ has

spin one	$J_b^a(z),$	$j_j^i(z),$
spin 3/2	$G_i^a(z),$	$\tilde{G}_a^i(z)$
spin 2	$S_b^a(z)$	

where $a, b = 1, \dots, m$ and $i, j = 1, \dots, n$.

- $W([2, 1])$ is called Bershadsky-Polyakov algebra;
 $W([2, 1, \dots, 1])$ is Romans' quasisuperconformal algebra.
 (both 1991)
- Do they appear in 4d gauge theory?

- The answer is Yes! [Braverman-Feigin-Finkelberg-Rybnikov],
[Wyllard], [Kanno-YT]
- Consider N D4-branes on

$$\mathbb{R}_t \times \mathbb{C}_z \times (\mathbb{C}_w / \mathbb{Z}_M)$$

where \mathbb{Z}_M acts on the Chan-Paton indices as

$$(\underbrace{\omega, \dots, \omega}_{n_1}, \underbrace{\omega^2, \dots, \omega^2}_{n_2}, \dots, \underbrace{\omega^M, \dots, \omega^M}_{n_M})$$

- Bulk D0s + fractional D0s bound at the surface operator $w = 0$

- As before, the BPS part of the dynamics reduces to susy QM on the (fractional) instantons, with the Hilbert space

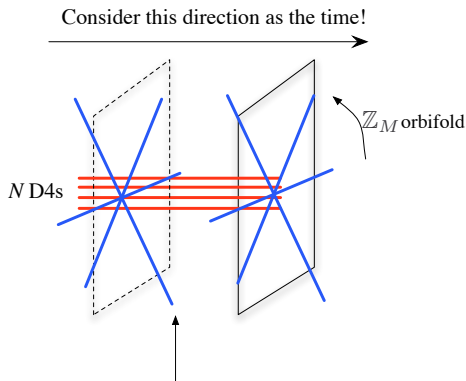
$$\mathcal{H} = H^*(\text{moduli space of instantons on } \mathbb{C}_z \times (\mathbb{C}_w/\mathbb{Z}_M))$$

- This equals the Verma module of the W -algebra

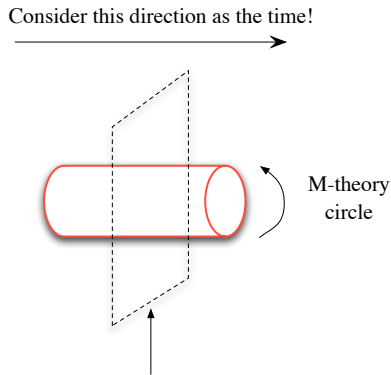
$$W(\hat{\mathbf{SU}}(N), [n_1, \dots, n_M])$$

- Moreover, **the instanton partition function**
= **the norm of the coherent state**
- (only explicitly checked for $W([2, \dots, 2, 1, \dots, 1])$, though)

Summary



At "constant time", we have instantons



At "constant time", we have
the [general W-algebra](#) Verma module

- W_N -algebra reappeared also in the context of higher-spin $\text{AdS}_3/\text{CFT}_2$ last year. [Gaberdiel, Gopakumar, Hartman, ...]
- Might be a good time to study W -algebras again, after 17-year hiatus.
- One good thing is that PC is much faster now. We can do more complicated W -algebra calculation in short time, on this laptop.
- Due to programming complexity, detailed checks were done only for $W([2, \dots, 2, 1, \dots, 1])$. Anyone wants to help me?

- Mathematica code to calculate the **instanton partition function** and the **norm of the W-algebra coherent state** (at least the cases implemented) is available on the preprint webpage. **Enjoy!**

The screenshot shows the arXiv.org interface for the preprint 'Instanton counting with a surface operator and the chain-saw quiver' by Hiroaki Kanno and Yuji Tachikawa. The page is categorized under 'High Energy Physics - Theory'. The title is prominently displayed in bold. Below the title, the authors' names and the submission date (2 May 2011) are listed. The abstract text describes the moduli space of SU(N) instantons and the chain-saw quiver. On the right side, there is a 'Download:' section with links for PDF, PostScript, and Other formats. Below this is an 'Ancillary files (details):' section, which is circled in green, containing links for 'w2m1n.nb' and 'w2m1n.pdf'. Further down, the 'Current browse context:' section shows the hierarchy 'hep-th' and navigation links like '< prev', 'next >', 'new', and 'recent | 1105'. At the bottom of the right sidebar is a link for 'References & Citations'.

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All papers Go!

High Energy Physics - Theory

Instanton counting with a surface operator and the chain-saw quiver

Hiroaki Kanno, Yuji Tachikawa
(Submitted on 2 May 2011)

We describe the moduli space of SU(N) instantons in the presence of a general surface operator of type $N=n_1 + \dots + n_M$ in terms of the representations of the so-called chain-saw quiver, which allows us to write down the instanton partition function as a summation over the fixed point contributions labeled by Young diagrams. We find that the instanton partition function depends on the ordering of n_i which fixes a choice of the parabolic structure. This is in accord with the

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References & Citations

- You should also post your Mathematica / Maple code to the arXiv!