Moduli Fixing and the Statistics of Vacua

— a review + some original material —

Yuji Tachikawa

collaborated with T. Eguchi
(Univ. of Tokyo, Hongo)

Januray, 2005

1. Introduction

Quantizing Gravity...

- **Gravitational loops measurable in ten years**

- **Two candidates:**

  - **Loop Quantum Gravity: cannot yet incorporate the SM.**

  - **String Theory : extremely rich & models which look like the SM.**

String Theory

- **Theory in 10 dimensions : only a few !**
  - **Type II : supergravity in 10d + gauge fields on branes**
  - **Type I / heterotic: supergravity + gauge fields in 10d**
- **Compactify on some 6d space : huge # of theories in 4d**
  - **Many topological types**
  - **Form continuous family**
  - **Config. of branes**

- **How many are there ?**
- **Is the SM one of them?**
- **Is the SM the only one under some criteria?**
  - **with $\mathcal{N} = 1$ susy?**
  - **with inflationary period?**
  - **with tiny cosmological constant**
- **But before discussing these, we need to study...**

the Moduli Problem

- ◇ The **extra** dimensions : continuously **deformable**
- ◇ The **positions** of branes : continuously **changeable**

⤳ Massless Neutral Scalar Particles **"Moduli"**

- mediating the **FIFTH force**
- they'll obtain their mass through SUSY breaking...
- but even with it, they **destroy BBN**

Moduli are BAD.

Moduli Fixing

- To give them mass ~ potential ~ superpotential...
- Hard because of the perturbative **non-renormalization theorem**

- Difficult but Possible

  [Kachru-Kallosh-Linde-Trivedi], [Denef-Douglas]

  - Uses **Flux Superpotential** [Gukov-Vafa-Witten]
  - and Superpotential from **D-brane Instantons**

    [Witten], [Gorlich-Kachru-Tripathy-Trivedi]

$\mathcal{N} = 1$ **supergravity vacua with no moduli !**

Discretuum

- Again: **How many are there ?**
  - (Topological types of the extra dimension) $\sim 10^4$
  - $\times$ (Ways of introducing Fluxes) $\sim 10^{200}$
  - $\times$ (Several Vacua for each choice of fluxes) ~ 1000

$\rightsquigarrow$ **Milliards** of vacua !

- Discrete but forms almost Continuum : **DISCRETUUM**

- **You can’t study each vacua one by one.**
- **Then: How are vacua distributed ?**
  - **Singular extra dimensions favored,**
  - **Cosmological Constant is uniformly distributed, etc.**
- **What is the correct a-priori probability?**

CONTENTS

2. Moduli Fixing

Calabi-Yau compactification

- 6-dimensional CY = the holonomy $SU(3) \subset SO(6)$

⇝ CY : complex mfd $x^1, x^2, x^3, x^4, x^5, x^6 \to z^1, z^2, z^3, \bar{z}^{\bar{1}}, \bar{z}^{\bar{2}}, \bar{z}^{\bar{3}}$,
with Kähler form ω, everywhere nonzero $(3,0)$ form Ω

- 6d spinor $4 = 3 \oplus 1$ under $SU(3)$

⇝ **1/4 of SUSY** remain

⇝ Type IIB/CY : $\mathcal{N} = 2$ and Heterotic/CY : $\mathcal{N} = 1$ in 4d

- Concentrate on Type IIB/CY.
- No YM, need to put **D-branes**

⇝ breaks SUSY to $\mathcal{N} = 1$

Moduli in CY compactification

- **CYs come in various topological types:**
  - $h_{1,1}$ **two-cycles**, $h_{1,1}$ **four-cycles**
  - $2h_{1,2} + 2$ **three-cycles: call them** $A_0, A_1, \dots, A_{h_{12}}$

and $B_0, B_1, \dots, B_{h_{12}}$ **so that** $A_i \cdot B_j = \delta_{ij}$ **and** $A_i \cdot A_j = B_i \cdot B_j = 0$

- **CYs can be continuously deformed, parametrized by**
  - $\rho_i = \int_{C_i} \omega \wedge \omega$ **: sizes of four-cycles for** $i = 1, \dots, h_{11}$
  - $z_i = \int_{A_i} \Omega$ **: periods of three-cycles for** $i = 1, \dots, h_{12}$

- **The metric of CY varies as ρ_i and z_i:** $g_{mn}(\rho_i, z_i)$
- **Take an ansatz for the 10d metric as**

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn}(\rho_i(x^\mu), z_i(x^\mu)) dx^m dx^n$$

- **Plug this into** $S = \int dx^{10} \sqrt{g_{(10)}} R_{(10)} \rightsquigarrow$

$$S = \int dx^4 \sqrt{g_{(4)}} R_{(4)} + \int dx^4 \sqrt{g_{(4)}} g^{\mu\nu}_{(4)} G_{i\bar{j}} \partial_\mu \rho^i \partial_\nu \bar{\rho}^{\bar{j}} + \int dx^4 \sqrt{g_{(4)}} g^{\mu\nu}_{(4)} G'_{i\bar{j}} \partial_\mu z^i \partial_\nu \bar{z}^{\bar{j}}$$

- ρ^i **combines with** $\int_{C_i} C^{(4)}$ **to become a complex scalar**

$$\rho^i_{\text{complex}} = i \int_{C_i} \omega \wedge \omega + \int_{C_i} C^{(4)}$$

For brevity, we always mean ρ^i_{complex} **by** ρ^i **from now on**

- $h_{11} + h_{12}$ **massless complex scalars in total**
  - ρ_i: **called size moduli or Kähler moduli**
  - z_i: **called shape moduli or complex structure moduli**
- **Complexified string coupling** $\tau = ie^{-\phi} + C^{(0)}$ **is also a massless scalar.**
- **Massless scalars corresp. to the motion of D-branes inside the CY.**
- **Need orbifolding. So the presentation above is a bit less precise.**

Superpotentials for Moduli

- **The system is $\mathcal{N} = 1$ supergravity $\rightsquigarrow$ the σ model metric is Kähler:**

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$$

and the potential is:

$$V = e^K (g^{i\bar{j}} D_i W \bar{D}_{\bar{j}} \bar{W} - 3 W \bar{W})$$

where $D_i = \partial_i + (\partial_i K)$ with W holomorphic

- **$\partial_i K$ term signifies that W is better understood as a holomorphic section of the Hodge bundle:**

$$\text{Under } K \to K + f + \bar{f}, \quad W \to e^{-f} W$$

- **Just compactifying on CY leads to $W = 0$.**
- **Masses to all moduli $\rightsquigarrow$ we need W depending all variables τ, ρ_i, z_i.**
  - **Fluxes give W for τ and z_i's**
  - **Instanton corrections give W for ρ's**
- **Let's see each in detail.**

Flux superpotential

- Type IIB has **2-form potentials** $C_{\mathbf{NSNS}}$ and $C_{\mathbf{RR}}$ with **3-form** field **strengths** $H_{\mathbf{NSNS}}$ and $H_{\mathbf{RR}}$
- Quantized fluxes through **three-cycles**
- They give rise to

$$
\begin{aligned}
W &= \int_{CY} \Omega \wedge (H_{\mathbf{RR}} + \tau H_{\mathbf{NSNS}}) \\
&= \sum_{i=0}^{h_{12}} \left[\int_{A_i} \Omega \int_{B_i} (H_{\mathbf{RR}} + \tau H_{\mathbf{NSNS}}) - \int_{B_i} \Omega \int_{A_i} (H_{\mathbf{RR}} + \tau H_{\mathbf{NSNS}}) \right] \\
&= \sum_{i=0}^{h_{12}} \left[z_i (N_i^{\mathbf{RR}} + \tau N_i^{\mathbf{NSNS}}) - \frac{\partial F}{\partial z_i} (M_i^{\mathbf{RR}} + \tau M_i^{\mathbf{NSNS}}) \right]
\end{aligned}
$$

Comments

- **This depends on string coupling and shape, not on the size**

.

- N_i **and** M_i **are the number of fluxes, hence integers**

- **Linear in Fluxes.**

- **The relation** $\partial F/\partial z_i = \int_{B_i} \Omega$ **with** $z_i = \int_{A_i} \Omega$ **is the defining relation for the so-called special geometry:**

$$e^{-K} = \int \Omega \wedge \bar{\Omega} = \sum_{i=0}^{h_{12}} \left[z_i \frac{\partial \bar{F}}{\partial \bar{z}_{\bar{\imath}}} - \bar{z}_{\bar{\imath}} \frac{\partial F}{\partial z^i} \right]$$

- **This form for W : obtainable by a standard KK reduction;**
- **or, by considering the domain-wall tension [Gukov]:**

  - **A (p, q) 5-brane wrapped on A_i is a BPS domain wall in 4d point of view. $\rightsquigarrow$ The tension is $\left| W|_{x^3=\infty} - W|_{x^3=-\infty} \right|$ from supergravity analysis.**

  - **This sources p units of $H_{\mathbf{RR}}$ and q units of $H_{\mathbf{NSNS}}$ through B_i.**

  - **The tension is given by $\left| (p + \tau q) \int_{A_i} \Omega \right|$, which can be seen from the (p, q)-brane action.**

  - **One can read off W !**

Constraint on N_i and M_i

- A term $\int C^{(4)} \wedge H_{\text{NSNS}} \wedge H_{\text{RR}}$ in type IIB sugra.
- Of course there is a coupling $\int_{\text{D3}} C^{(4)}$.
- Another coupling $-\int_{\text{O3}} C^{(4)}$ to Orientifold planes.

⇝ EOM for $C^{(4)}$ leads

$$\begin{aligned} \#_{O3} &= \#_{D3} + \int H_{\text{RR}} \wedge H_{\text{NSNS}} \\ &= \#_{D3} + \sum_{i=0}^{h_{12}} \left[N_i^{\text{RR}} M_i^{\text{NSNS}} - M_i^{\text{RR}} N_i^{\text{NSNS}} \right] \end{aligned}$$

- $\#_{\text{O3}}$ is fixed by the geometry of CY.

Instanton corrections

- **Superpotentials for the size moduli ρ^i : How?**

- **Consider wrapping N D7-branes on a 4-cycle C_i**

$\rightsquigarrow$ $\mathcal{N} = 1$ $U(N)$ **gauge theories with coupling constant ρ^i**

$\rightsquigarrow$ **Superpotential $\sim e^{-i\rho^i/N}$ associated with gaugino condensation.**

- **Consider D3-brane instantons wrapping C_i.**

$\rightsquigarrow$ **Contributes $\propto e^{-i\rho^i}$ to the superpotential if the # of the fermionic zero-modes is appropriate.**

- **One can find CYs where sufficiently generic instanton corrections arise [Denef-Douglas].**

- **Discussions based on [Witten]: which neglects $H_{\mathbf{RR}}$ and $H_{\mathbf{NSNS}}$**

$\rightsquigarrow$
e.g.[Gorlich-Kachru-Tripathy-Trivedi]. No definite treatment yet.

Closed string moduli are FIXED !

CONTENTS

3. Statistics of Vacua: Theory

- We used **fluxes** $H_{\mathbf{RR}}$ and $H_{\mathbf{NSNS}}$.
- In a typical CY, there're **100∼200 3-cycles** to put fluxes;
- LHS of the tadpole constraint

$$\#_{O3} = \#_{D3} + \int H_{\mathbf{RR}} \wedge H_{\mathbf{NSNS}}$$

is of order 1000∼5000.

- To have SUSY vacua, $\#_{D3} \geq 0$ and the quadratic form becomes effectively positive definite ⇝ $\sqrt{4000} \sim 100$ **choices** for **each** three-cycle

$$10^{100} \sim 10^{200} \textit{ choices of fluxes!}$$

◇ **Gauge group & matter contents** :
determined by the topology of the CY and the config. of branes.
⇝ **Determine the form of the low energy lagrangian.**

◇ **Coupling constants depend on the moduli**
◇ **which are fixed at different position for different flux**
⇝ **Determine the coefficient of the low energy lagrangian**

◇ **Once you construct the SM (+ susy + inflatons etc.), there'll be plethora of vacua with slightly differing Yukawas!**

Philosophical Considerations Aside,

- Need the distribution of Yukawas / Cosmological constants
- which are determined by the moduli

⇝ We need **the distribution of the moduli** !

However,

- The position of the moduli surely depends on the flux ...
- **How** on earth **are the fluxes** H_{RR} and H_{NSNS} **distributed ?**

We don’t know yet.

- As we saw, fluxes **change** when we cross **domain walls**.

⇝ **Flux distribution is tied to the dynamics of domain walls in the early universe. Extremely early universe before inflation matters !**

- **So we can’t study realistic distribution of flux. Period.**

As a zeroth approximation,

- **We try a gaussian ensemble of the fluxes $H_{\mathbf{RR}}$ and $H_{\mathbf{NSNS}}$:**

$$N_i = \int_{A_i} (H_{\mathbf{RR}} + \tau H_{\mathbf{NSNS}}), \qquad M_i = \int_{B_i} (H_{\mathbf{RR}} + \tau H_{\mathbf{NSNS}}).$$

- **Under a large fluctuation, we have monodromies acting on (N_i, M_i):**

$$\begin{pmatrix} N_i \\ M_i \end{pmatrix} \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} N_i \\ M_i \end{pmatrix}$$

which respects the pairing $(N_i, M_i) \cdot (N_i{}', M_i{}') = \sum_i (N_i M_i{}' - N_i{}' M_i)$

- **Assume the ensemble to be monodromy invariant.**

- **How, then, does**

$$W(z) = N_i z_i - M_i \frac{\partial F}{\partial z_i}$$

distribute? Result:

$$\langle W(z) W(w)^* \rangle \propto \sum_i \left[z_i \left(\frac{\partial F}{\partial w_i} \right)^* - {w_i}^* \left(\frac{\partial F}{\partial z_i} \right) \right]$$

$$= e^{-K(z,w^*)},$$

$$\langle W(z) W(w) \rangle = 0$$

$$\langle W(z)^* W(w)^* \rangle = 0$$

- **The distribution $\langle W(z)W(w)^* \rangle \propto e^{-K(z,w^*)}$ is a very natural guess: transforms covariantly under the Kähler transform:**

$$K(z, z^*) \to K + f(z) + f^*(z^*), \qquad W(z) \to e^{-f(z)} W(z)$$

- **We can study the behavior of $\mathcal{N} = 1$ supergravity system with random superpotential $\langle W(z)W(w)^* \rangle \propto e^{-K(z,w^*)}$.**

- **Huge literature on systems with random potential (not superpotential) in condensed matter physics. We should utilize them...**

Distribution of Vacua

- **Supersymmetric Vacua are defined by $D_i W = 0$.**

⇝ **Expected number of vacua at z_i is given by**

$$\rho(z,\bar{z}) = \left\langle \delta(D_i W(z))\delta(\bar{D}_{\bar{\imath}} W(\bar{z})^*) \left| \det \begin{pmatrix} \partial_i D_j W & \partial_i D_{\bar{\jmath}} W^* \\ \partial_{\bar{\imath}} D_j W & \partial_{\bar{\imath}} D_{\bar{\jmath}} W^* \end{pmatrix} \right| \right\rangle$$

- **Determinant needed to count each vacua with weight $+1$.**
- **Absolute value makes evaluation harder; instead consider**

$$\tilde{\rho}(z,\bar{z}) = \left\langle \delta(D_i W(z))\delta(\bar{D}_{\bar{\imath}} W(\bar{z})^*) \det \begin{pmatrix} \partial_i D_j W & \partial_i D_{\bar{\jmath}} W^* \\ \partial_{\bar{\imath}} D_j W & \partial_{\bar{\imath}} D_{\bar{\jmath}} W^* \end{pmatrix} \right\rangle$$

- **This counts vacua with signs ± 1.**

- $\tilde{\rho}$ **can be calculated by expressing the Dirac delta using exponential & using the Wick theorem.**

- **The result is,**

$$\tilde{\rho}(z)\prod_i dz^i \wedge d\bar{z}^{\bar{i}} \propto \det \frac{1}{2\pi}(R^i{}_j + \delta^i{}_j \omega)$$

where

$$R^i{}_j = R^i{}_{ik\bar{l}}\, dz^k \wedge d\bar{z}^{\bar{l}}, \qquad \omega = \frac{i}{2} g_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}}$$

is the curvature and the Kähler form of the moduli space.

A mathematical comment

◇ **Let** M **compact** n **dim'l Kähler and nonsingular,** E **a** n **dim'l vector bundle on** M**.** $\rightsquigarrow$ **A generic section of** L **will have** $\int_M c_n(E)$ **zeros when counted with signs. This is almost the definition of the Chern classes.**

◇ $c_n(E) = \det R_E$ **via the Chern-Weil homomorphism.**

◇ **As** $D_i W$ **is a section of** $TM \otimes H$**, it has precisely**

$$\int_M \det R_{TM\otimes H} = \int_M \det(R_{TM} + R_H) = \int_M \det(R_{TM} + \omega)$$

zeroes.

◇ **Of course in supergravity the target space of** σ **model is noncompact and has singularity !**

Physical Comments

- **Suppose there're no curvature : $R = 0$.**

$\rightsquigarrow$ $\tilde{\rho} \propto \det \omega$, **that is, the vacua distribute uniformly following the volume. Reminiscent of modular cosmology of [Horne-Moore]...**

- **Vacua tends to cluster around where the curvature R is large.**

- **Recall we're discussing the curvature of the moduli space.**
- **But it is known that the curvature of the moduli is large when the curvature of the CY is large.**

$\rightsquigarrow$ **Strongly curved extra dimension is favored .**

CONTENTS

4. Statistics of Vacua: Examples

- **To visualize $\tilde{\rho}$,**
- **We need to calculate $g_{i\bar{j}}$and ${R^i}_j$:**

$$g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K, \qquad {R^i}_{jk\bar{l}} = \partial_{\bar{l}} g_{j\bar{m}} \partial_k g^{\bar{m}i}$$

- **Utilize that we're dealing CY: the Kähler manifold is special, i.e.**

$$e^{-K} = \bar{X}_i F_i - \bar{F}_i X_i \quad \textbf{where} \quad X_i = \int_{A_i} \Omega (= z_i), \quad F_i = \int_{B_i} \Omega \left(= \frac{\partial F}{\partial z_i} \right).$$

- **Need to do three-dimensional integral.**

⋄ **Quite formidable; but various techniques devised and now standard tools for those who practice mirror symmetry...**

- **Use Picard-Fuchs equations.**
- **Direct 3d integral in the case of K3 fibration**
  [Billó-Denef-Frè-Presando-Troost-van Proeyen-Zanon]

$$\int_{A_i} \Omega = \int_{\text{path in the base}} \int_{\text{2-cycle in K3}} \Omega$$

⋄ **Strominger's formula**

$$R_{i\bar{j}k\bar{l}} = F_{ikm}\bar{F}_{\bar{j}\bar{l}\bar{n}}g^{\bar{n}m} + g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}} \quad \text{where} \quad F_{ijk} = \int \Omega \wedge \partial_i \partial_j \partial_k \Omega$$

is extremely useful to reduce the amount of calculation.

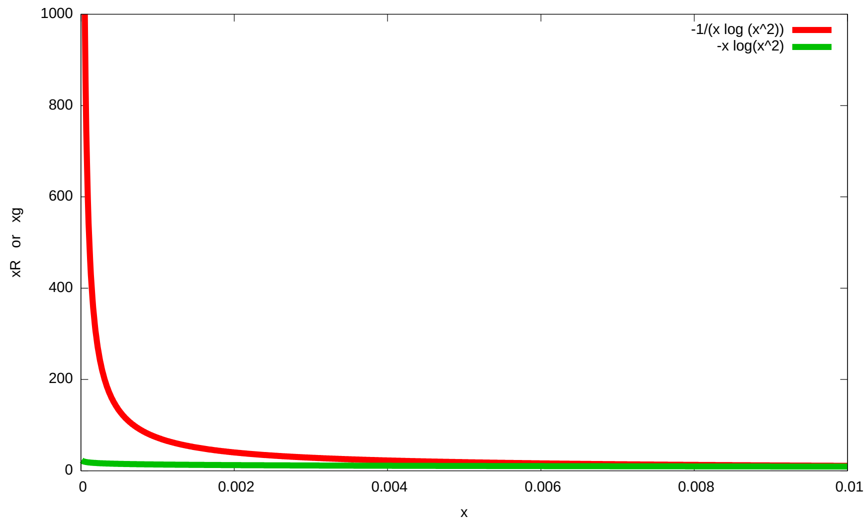
- **We're mostly interested in the curvature singularities.**
- **Behavior of X_i and F_i around them are determined by the types of the singularity, and well studied.**
- **Full info on the CY is not necessary .**

⇝ **Consult the mirror symmetry literature,**
⇝ **Plug into the formula for $\tilde{\rho}$,**
⇝ **Now you have another paper !**

Near Conifold Singularity [Giryavets-Kachru-Tripathy]

- **where a 3-cycle collapses. Call it A_1.**
- **Let $\phi \equiv X_1 \rightsquigarrow F_1 \sim \phi \log \phi$:**

$$g_{\phi\phi^*} \sim \log|\phi|^2, \qquad R_{\phi\phi^*} \sim \frac{1}{|\phi|^2(\log|\phi|)^2} \gg g_{\phi\phi^*}$$



What we did

- **Took two-modulus CY: degree 8 hypersurfaces in $\mathbb{WCP}^4_{1,1,2,2,2}$ with**

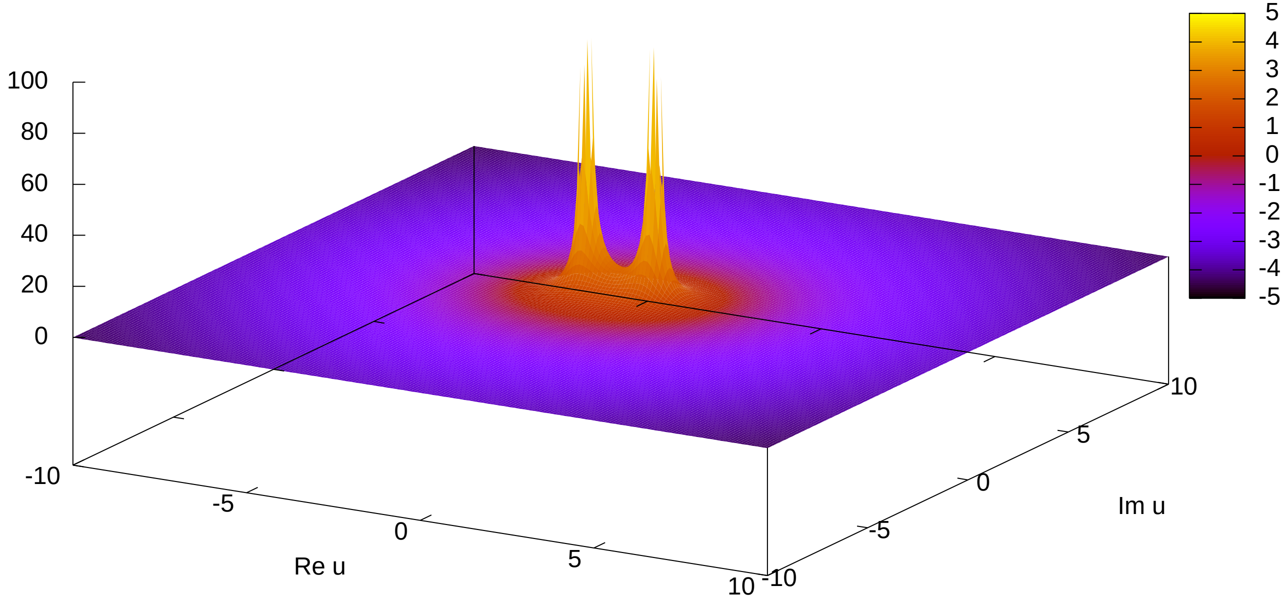
$$\frac{1}{8}x_1^8 + \frac{1}{8}x_2^8 + \frac{1}{8}x_3^4 + \frac{1}{8}x_4^4 + \frac{1}{8}x_5^4 - \psi_0 x_1 x_2 x_3 x_4 x_5 - \frac{1}{4}\psi_s (x_1 x_2)^4 = 0$$

- **Studied the behavior near the geometric engineering limit where pure $SU(2)$ SYM decouples from supergravity.**
- **Denote $\epsilon = 1/(2\psi_s)$ and $u = \psi + \psi_0^4$. When $\epsilon \to 0$,**

  $\epsilon^{1/2}$: **Dynamical Scale of SYM measured in Planck units;**

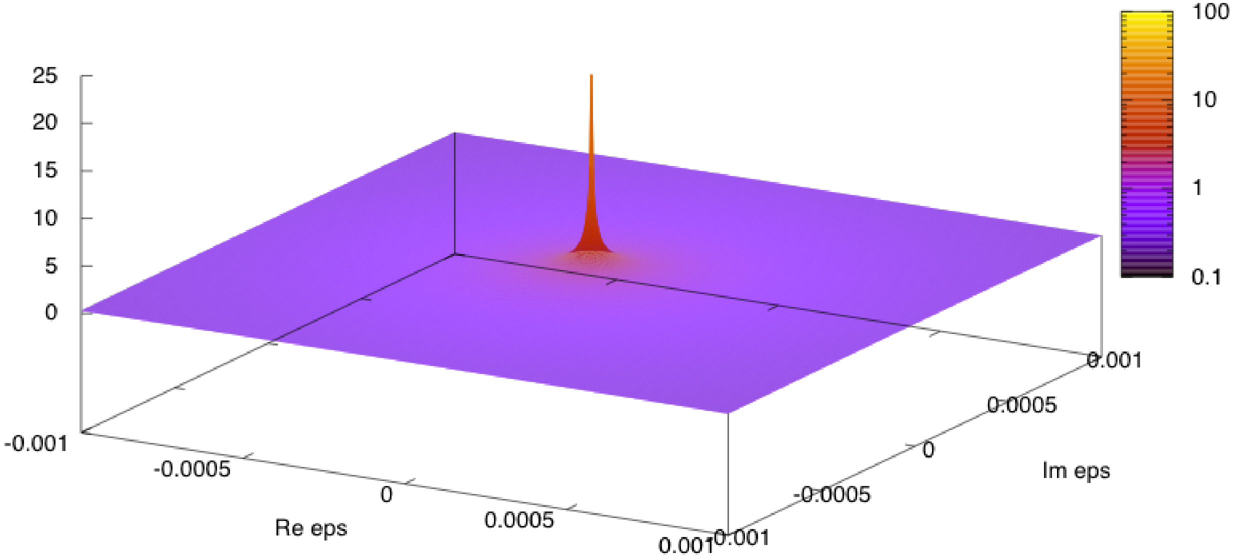
  u : **Seiberg-Witten's u.**

$\epsilon = 0.001$, u:finite



- **Just two conifold singularities at $u = \pm 1$.**

$u = 5$, vary ϵ



- $\det(R+\omega) \sim \dfrac{1}{|\epsilon|^1(\log|\epsilon|)^3}$ **if** $1/\epsilon \gg u \gg 1 \rightsquigarrow \sim \dfrac{1}{|\epsilon|^3(\log|\epsilon|)^3}$

5. Conclusion & Outlook

- ✓ Moduli can now be **fixed**.
- ✓ Fixing needs **fluxes**.
- ✓ Flux introduces **huge number of vacua** .
- ✓ Vacuum distribution can be studied.
- ✓ We saw some examples.

Outloook

- Moduli fixing in string theory other than type IIB

⇝ Already large literature exists.

- Behavior around various singularities in the moduli.

⇝ We had done some. Nothing spectacular so far...

- Pre-inflationary cosmology.

⇝ We must stay inside physics, must not do theology...