More on $\mathcal{N} = 2$ S-dualities and M5-branes

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based on works in collaboration with

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F. Benini,
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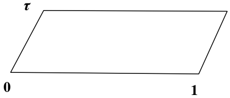
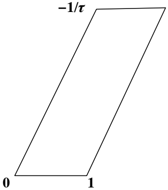
Contents

Montonen-Olive duality

- $\mathcal{N} = 4$ $\mathrm{SU}(N)$ SYM at coupling $\tau = \theta/(2\pi) + (4\pi i)/g^2$ equivalent to the same theory coupling $\tau' = -1/\tau$
- One way to 'understand' it: start from 6d $\mathcal{N} = (2,0)$ theory, i.e. the theory on N M5-branes, put on a torus





- Low energy physics depends only on the complex structure
  → S-duality!

S-dualities in $\mathcal{N} = 2$ theories

- You can wrap N M5-branes on a more general Riemann surface, possibly with punctures, to get $\mathcal{N} = \mathbf{2}$ superconformal field theories
- Different limits of the shape of the Riemann surface gives different weakly-coupled descriptions, giving S-dualities among them
- Anticipated by [Witten,9703166], but not well-appreciated until [Gaiotto,0904.2715]

Contents

Contents

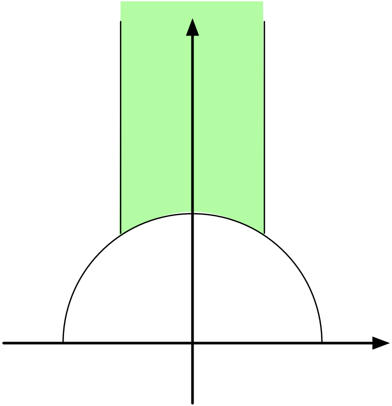
S-duality in $\mathcal{N} = 2$

SU(2) with $N_f = 4$

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$$

$$\tau \to \tau + 1, \qquad \tau \to -\frac{1}{\tau}$$

- Exchanges monopoles and quarks
- Comes from S-duality of Type IIB



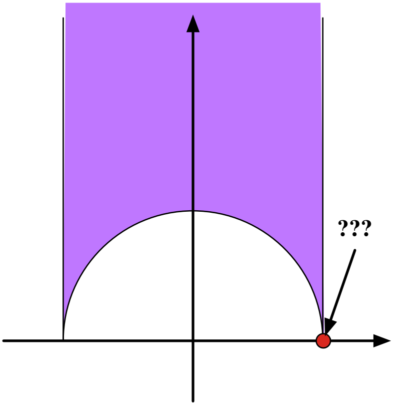
S-duality in $\mathcal{N} = 2$

SU(3) with $N_f = 6$

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$$

$$\tau \to \tau + 2, \qquad \tau \to -\frac{1}{\tau}$$

- Exchanges monopoles and quarks
- Infinitely Strongly coupled at $\tau = 1$



Argyres-Seiberg duality

$\mathrm{SU}(3) + 6$ flavors

at coupling τ

$$\Updownarrow$$

$\mathrm{SU}(2) + 1$ flavor $+ \mathrm{SCFT}[E_6]$

at coupling $\tau' = \dfrac{1}{1-\tau}$, $\mathrm{SU}(2) \subset E_6$ is gauged

[Argyres-Seiberg,0711.0054]

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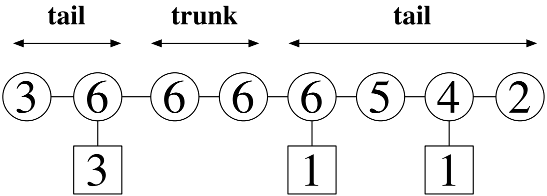
[Argyres-Seiberg,0711.0054]

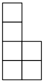
What??? Huh???

Linear quiver and Young diagrams

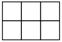
Consider a superconformal linear quiver



Associate Young diagrams to tails:

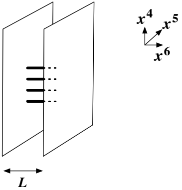


$$\mathbf{6-5 \leq 5-4 \leq 4-2 \leq 2-0}$$



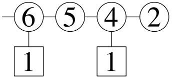
$$\mathbf{6-3 \leq 3-0}$$

Recall



$$\int d^5x F_{\mu\nu}F_{\mu\nu} = L\int d^4x F_{\mu\nu}F_{\mu\nu} \longrightarrow g^2 \propto 1/L$$

Combine them to realize



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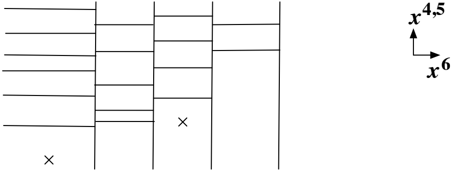
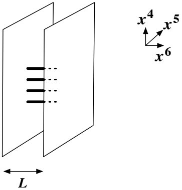


Type IIA

Recall



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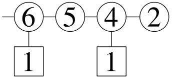
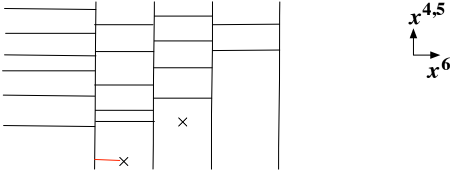
Combine them to realize  :



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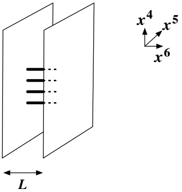

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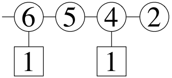
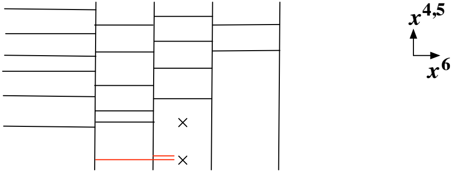
Combine them to realize  :



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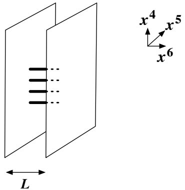

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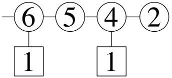
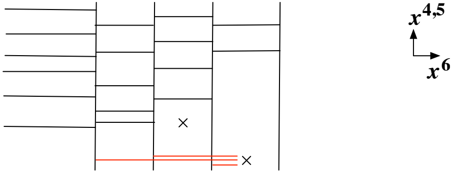
Combine them to realize  :



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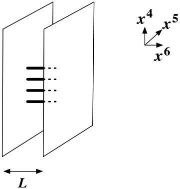

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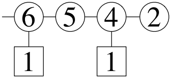
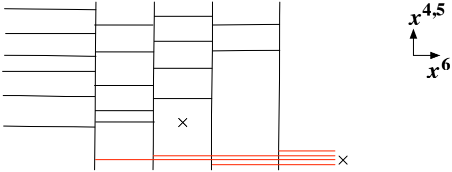
Combine them to realize  :



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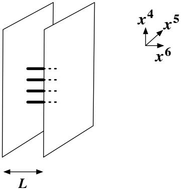

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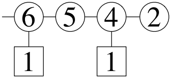
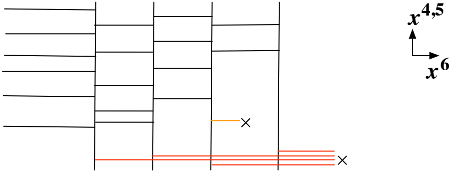
Combine them to realize  :



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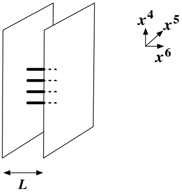

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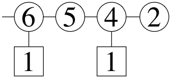
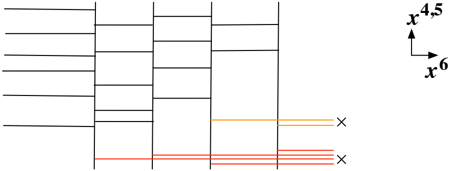
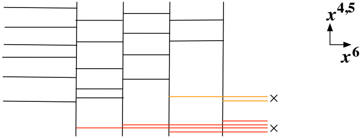
Combine them to realize  :



M-theory

From:



To:

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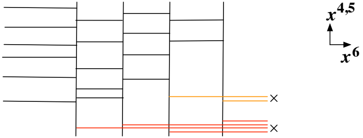

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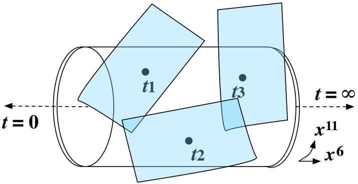


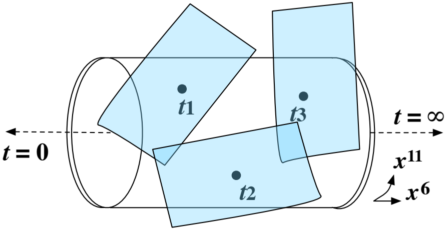
where $t = \exp(x_6 + ix_{11})$, $v = x_4 + ix_5$. SW curve is

$$v^N \prod_i (t - t_i) = 0$$

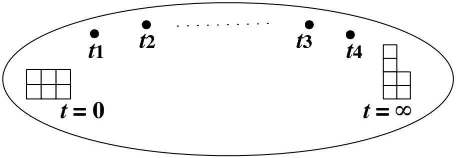
Funny boundary conditions at $t = 0, \infty$.

Gaiotto's curve

From:

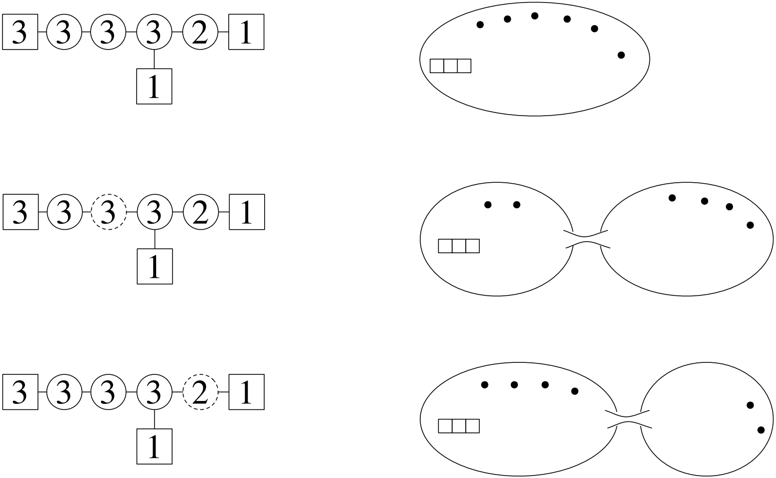


To [Gaiotto,0904.2715]:



Gaiotto's construction

Recall the separation of two points $\sim$ inverse gauge coupling. Then:



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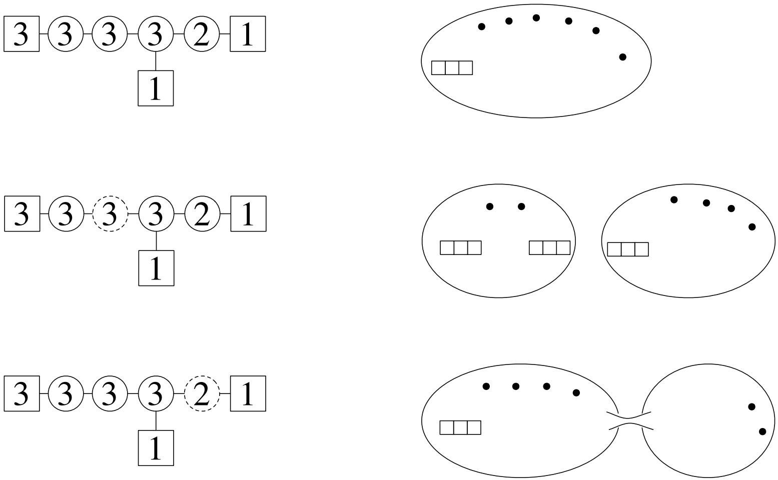

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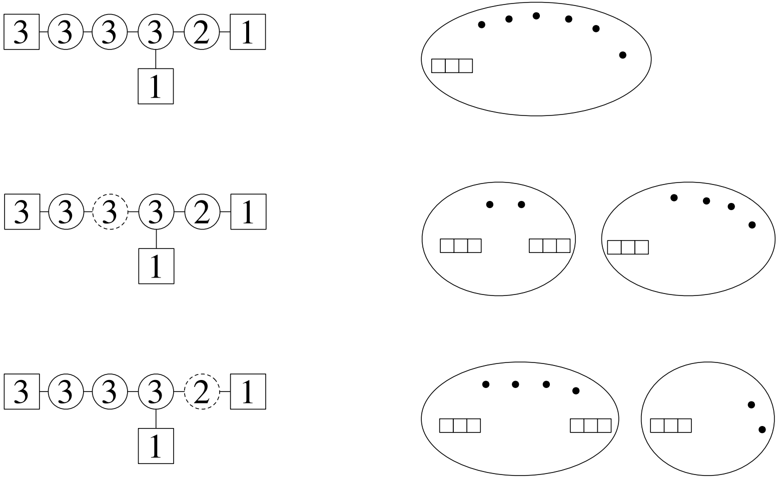

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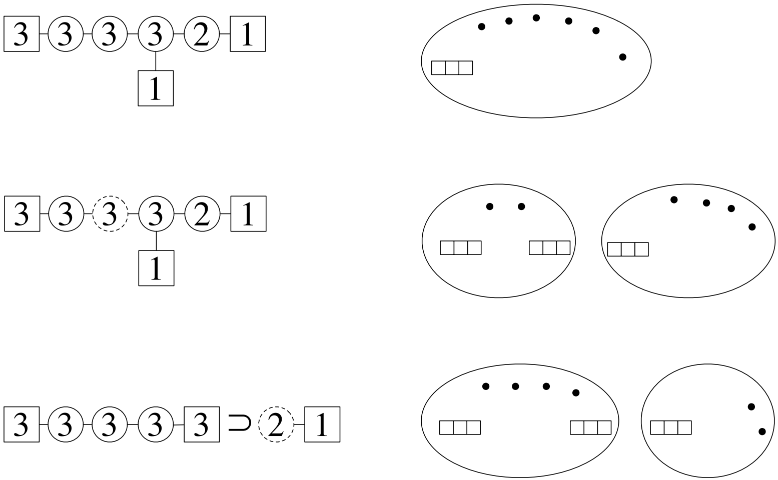
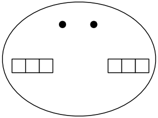


Argyres-Seiberg from Gaiotto



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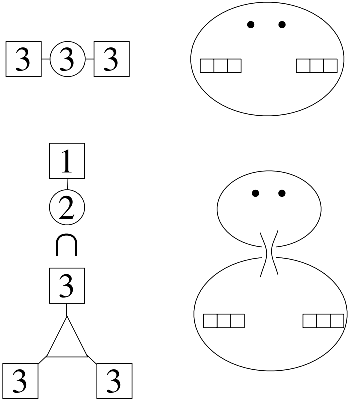

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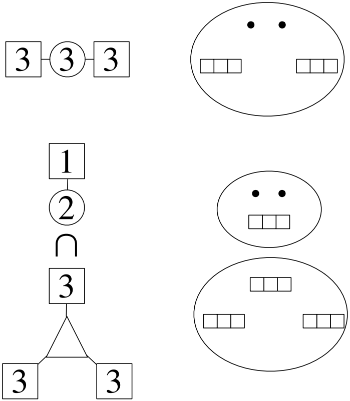
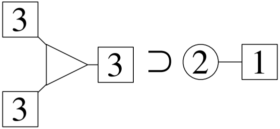


Argyres-Seiberg from Gaiotto

$\mathbf{SU(3)}$ with $\boldsymbol{N_f = 6}$

is S-dual to $\mathbf{SU(2)}$ with $\boldsymbol{N_f = 1}$,
coupled to a strange theory with $\mathbf{SU(3)^3}$ flavor symmetry



$\mathbf{SU(3)} \times \mathbf{SU(3)}$ enhances to $\mathbf{SU(6)}$;

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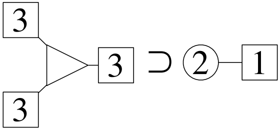

$\mathbf{SU(3)} \times \mathbf{SU(3)}$ enhances to $\mathbf{SU(6)}$; three $\mathbf{SU(3)}$s on the same footing

Argyres-Seiberg from Gaiotto

$\mathbf{SU(3)}$ with $N_f = 6$

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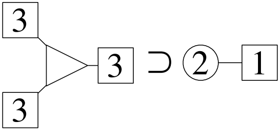
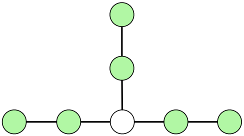


$\mathbf{SU(3)} \times \mathbf{SU(3)}$ enhances to $\mathbf{SU(6)}$; three $\mathbf{SU(3)}$s on the same footing
→ E_6 flavor symmetry!



Effective number of multiplets

- Basic quantities for CFT: central charges
- a and c in 4d $\sim n_v$ and n_h if $\mathcal{N}=2$
- $\mathrm{SU}(3)$ with $N_f=6$:

$$n_v = 8, \qquad n_h = 18$$

- $\mathrm{SU}(2)$ with $N_f=1$ and SCFT[E_6]

$$n_v = 3 + ??, \qquad n_h = 2 + ??$$

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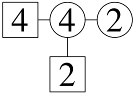
$$n_v = 3 + ??, \qquad n_h = 2 + ??$$

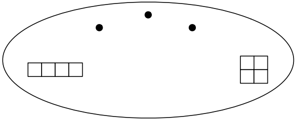
- SCFT[E_6] has

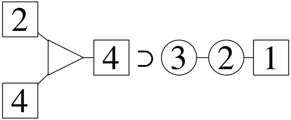
$$n_v = 5, \qquad n_h = 16$$

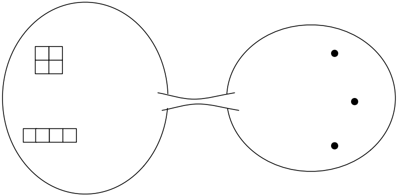
- agrees with other independent calculations [Aharony-YT] [Cheung-Ganor-Krogh]

E_7









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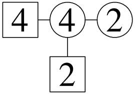
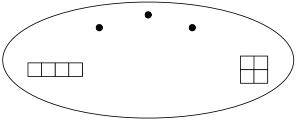
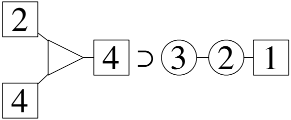
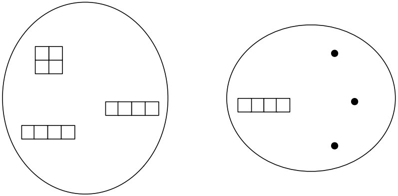


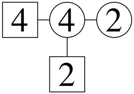




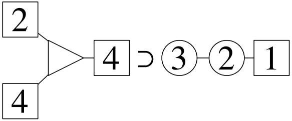


E_7

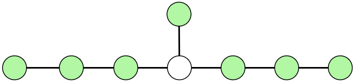
$\mathbf{SU(4)} \times \mathbf{SU(2)}$ with a bifundamental and six flavors for $\mathbf{SU(4)}$



is S-dual to $\mathbf{SU(3)} \times \mathbf{SU(2)}$ with a bifundamental and one flavor for $\mathbf{SU(2)}$, couple to a strange theory with $\mathbf{SU(4)^2} \times \mathbf{SU(2)}$ flavor symmetry

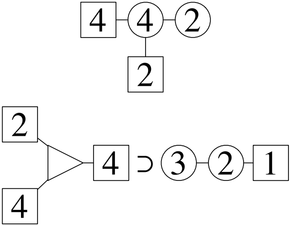


$\mathbf{SU(4)} \times \mathbf{SU(2)}$ enhances to $\mathbf{SU(6)}$; two $\mathbf{SU(4)}$s on the same footing
$\longrightarrow$ $\boldsymbol{E_7}$ flavor symmetry! [Benvenuti-Benini-YT]



E_7

Realization here:



$$n_v = 15 + 3 = 18$$
$$n_h = 16 + 8 + 8 = 32$$

$$n_v = ? + 8 + 3 = 18$$
$$n_h = ?? + 6 + 2 = 32$$

E_7

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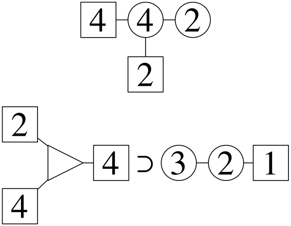
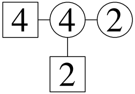


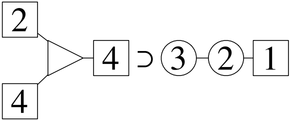
$$n_v = 15 + 3 = 18$$
$$n_h = 16 + 8 + 8 = 32$$

$$n_v = 7 + 8 + 3 = 18$$
$$n_h = 24 + 6 + 2 = 32$$

Realization here:



$$n_v = 15 + 3 = 18$$
$$n_h = 16 + 8 + 8 = 32$$



$$n_v = 7 + 8 + 3 = 18$$
$$n_h = 24 + 6 + 2 = 32$$

Another realization [Argyres-Seiberg]:

USp(4) with six fundamental hypers

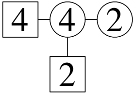
$$n_v = 4 \times 5/2 = 10, \qquad n_h = 4 \times 6 = 24.$$

E_7 theory couple to SU(2)

$$n_v = ? + 3 = 10, \qquad n_h = ?? = 24.$$

E_7

Realization here:



$$n_v = 15 + 3 = 18$$
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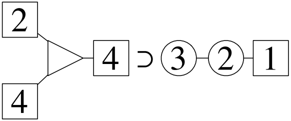

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Another realization [Argyres-Seiberg]:

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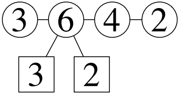
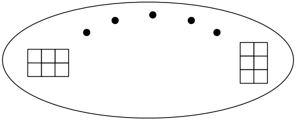
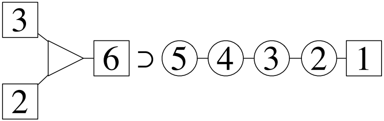
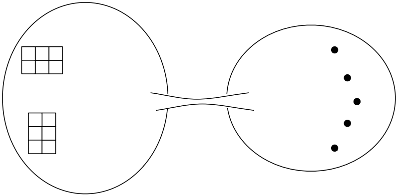
$$n_v = 7 + 3 = 10, \qquad n_h = 24 = 24.$$

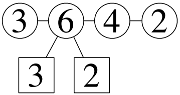
E_8

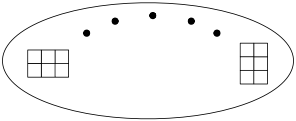


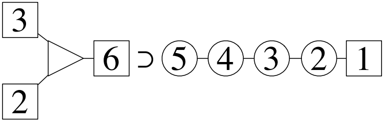


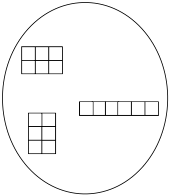


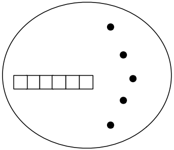



3 6 4 2
3 2



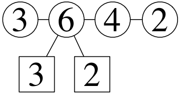

3
2
6 ⊃ 5 4 3 2 1



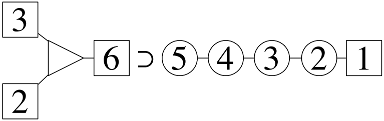


E_8

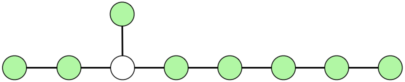
This quiver



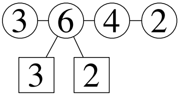
is S-dual to $\mathbf{SU(5)} \times \mathbf{SU(4)} \times \mathbf{SU(3)} \times \mathbf{SU(2)}$ with lots of hypers, couple to a strange theory with $\mathbf{SU(6)} \times \mathbf{SU(3)} \times \mathbf{SU(2)}$ flavor symmetry



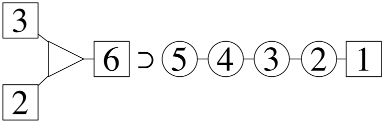
$\mathbf{SU(3)} \times \mathbf{SU(2)}$ enhances to $\mathbf{SU(5)}$. This $\mathbf{SU(5)}$ does not commute with $\mathbf{SU(6)} \longrightarrow \boldsymbol{E_8}$!



One realization:



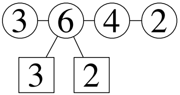
$$n_v = 8 + 35 + 15 + 3 = 61$$
$$n_h = 18 + 30 + 24 + 8 = 80$$



$$n_v = ?? + 24 + 16 + 8 + 3 = 61$$
$$n_h = ?? + 20 + 12 + 6 + 2 = 80$$

E_8

One realization:



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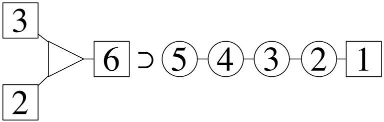

$$n_v = 11 + 24 + 16 + 8 + 3 = 61$$
$$n_h = 40 + 20 + 12 + 6 + 2 = 80$$

[Argyres-Wittig 0712.2028] had this example:

$\mathbf{USp(6)}$ with $\mathbf{11}$ half-hypers in $\mathbf{6}$ +

the three-index antisymmetric traceless tensor $\mathbf{14}$

$\Updownarrow$

E_8 theory coupled to $\mathbf{SO(5)}$ gauge multiplet via

$$\mathbf{SO(5)} \times \mathbf{SO(11)} \subset \mathbf{SO(16)} \subset E_8.$$

Here, the original had

$$n_v = 21, \qquad n_h = 33 + 7 = 40.$$

The dual had

$$n_v = ?? + 10 = 21, \qquad n_h = ?? = 40.$$

E_8

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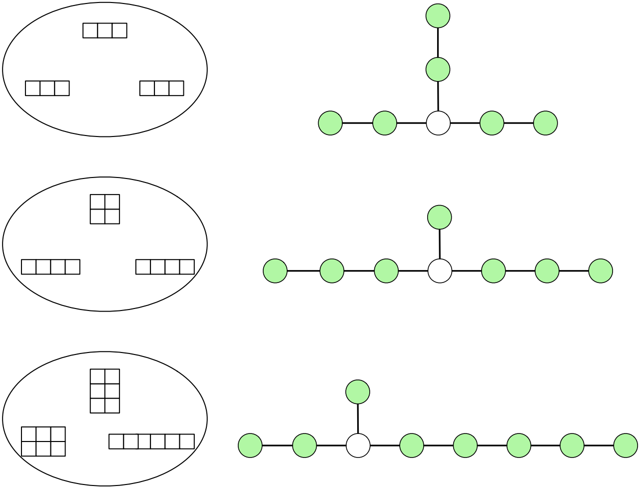
Summary of E's

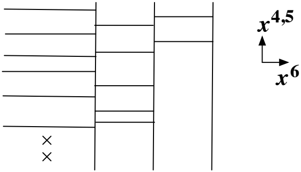


Why does the RHS and LHS look similar? Is there a physical reason?

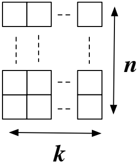
Boundary condition

Consider the Young diagram . As a quiver tail, this is $\begin{array}{c}-\textcircled{6}-\textcircled{4}-\textcircled{2}\\ \boxed{2}\end{array}$.

As a IIA configuration, we have

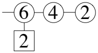


In general,



means that $N = nk$ M5-branes end on $\mathbb{C}^2/\mathbb{Z}_k$.

Boundary condition

Consider the Young diagram . As a quiver tail, this is 
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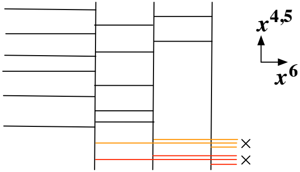

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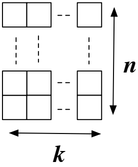

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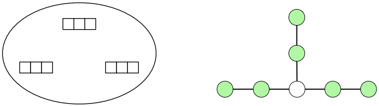
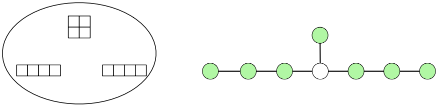
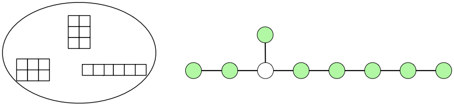
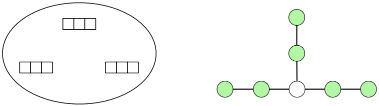


M5-branes wrap a sphere, which touches three $\mathbb{C}^2/\mathbb{Z}_3$.
This is one way to partially resolve an E_6 singularity.
Similarly for E_7, E_8:





Summary of E's



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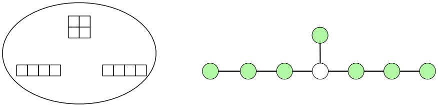
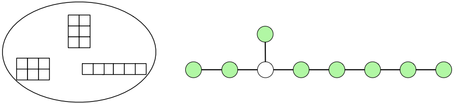
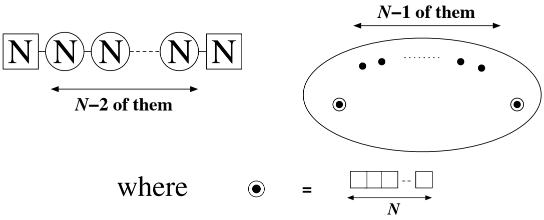




In fact, they are basically dual to the F-theory consideration done in MIT 10 years ago [DeWolfe,Iqbal,Hanany,Zwiebach...]

Contents

The T_N theory



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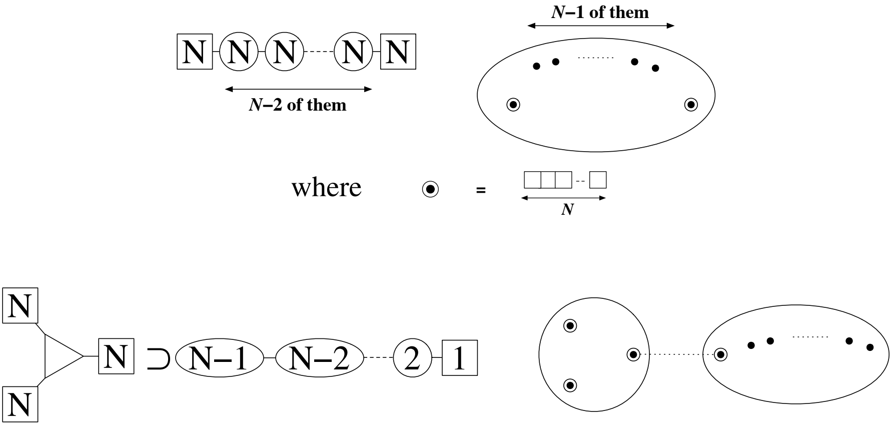

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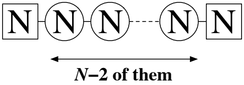
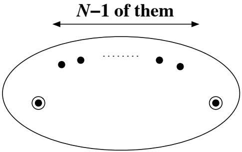
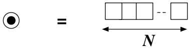
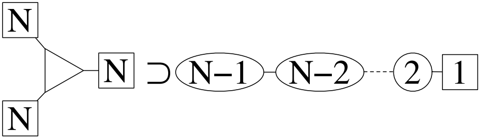
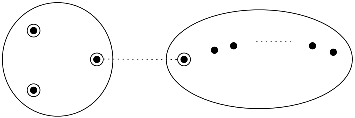




where 

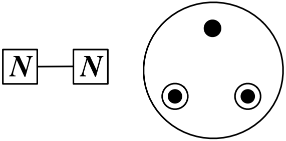




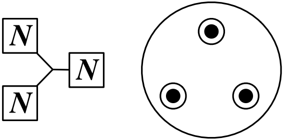
$$n_v = \frac{2}{3}N^{3} - \frac{3}{2}N^2 - \frac{N}{6} + 1, \qquad n_h = \frac{2}{3}N^{3} - \frac{2}{3}N.$$

The T_N theory

- the bifundamental,
  $\mathbf{SU}(N) \times \mathbf{SU}(N) \times \mathbf{U}(1)$



- the T_N theory,
  $\mathbf{SU}(N) \times \mathbf{SU}(N) \times \mathbf{SU}(N)$



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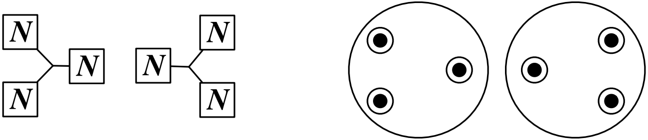

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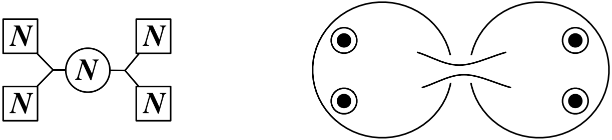

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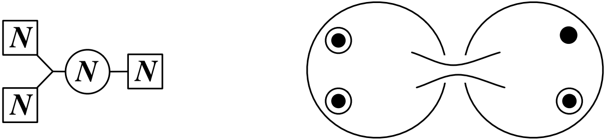
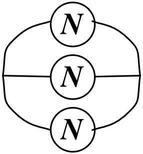
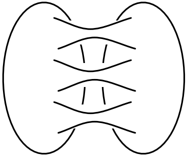
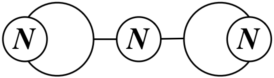
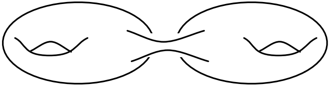


Fun with T_N









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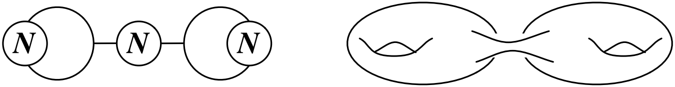
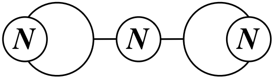
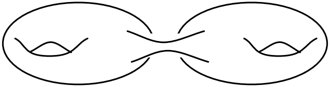


- $\mathbf{2(g-1)}$ copies of $\boldsymbol{T_N}$, $\mathbf{3(g-1)}$ copies of $\mathbf{SU}(\boldsymbol{N})$
  → $\mathbf{3(g-1)}$ marginal couplings!

Fun with T_N

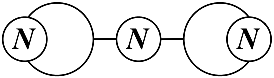


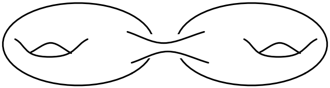


- $\mathbf{2}(g-\mathbf{1})$ copies of T_N, $\mathbf{3}(g-\mathbf{1})$ copies of $\mathbf{SU}(N)$
  → $\mathbf{3}(g-\mathbf{1})$ marginal couplings!

$$n_v = (g-1)\left[\frac{4}{3}N^3 - \frac{N}{3} - 1\right], \qquad n_h = (g-1)\left[\frac{4}{3}N^3 - \frac{4N}{3}\right].$$

Fun with T_N





- $2(g-1)$ copies of T_N, $3(g-1)$ copies of $\mathrm{SU}(N)$
  $\longrightarrow$ $3(g-1)$ marginal couplings!

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- agree with the central charges determined from the gravity solutions [Maldacena-Nuñez,hep-th/0007018] or the anomaly [Harvey-Minasian-Moore,hep-th/9808060]
- Now we know the field theory realization.

Triskelion and Sicily

Gaiotto called these theories “generalized quiver theories,”
but we [Benini-YT-Wecht] didn’t like it.

Triskelion and Sicily

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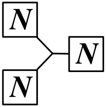
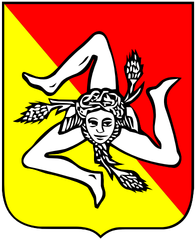
Siciliy’s flag





has in it a triskelion: tri+ skelios (Gk. leg).
We adopted the terminology “Sicilian gauge theories.” Please do.

Contents

4d CFT vs 2d CFT

- We now have a map

$$G_N : \text{ Riemann surface with punctures} \longrightarrow \text{ 4d field theory}$$

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- This morally means that $Z \circ G_N$ gives a 2d CFT.

- More physically, consider N M5-branes wrapped on $X_4 \times \Sigma_2$
- Take a quantity Z for the 6d theory,
- furthermore suppose Z depends only on the complex structure.

$$
\begin{tikzcd}
 & \textbf{6d theory on } X_4 \times \Sigma_2 \arrow[dl, "\text{KK-reduce along } \Sigma_2"'] \arrow[dr, "\text{KK-reduce along } X_4"] & \\
\textbf{4d theory on } X_4 & & \textbf{2d theory on } \Sigma_2
\end{tikzcd}
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6d CFT

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4d CFT vs 2d CFT

- Nekrasov's instanton partition function
  = the Virasoro/W_N conformal block.

- Full partition function
  = the Liouville/Toda correlation function.
  [AGT,Wyllard,Marshakov-Mironov-Morozov,...]

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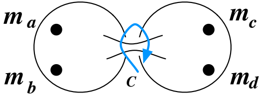
- Superconformal Index = a 2d TQFT
  [Gadde-Pomoni-Rastelli-Razamat]

Seiberg-Witten curve

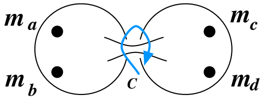
- $\mathbf{SU(2)}$ with four quark pairs with mass parameters $m_{a,b,c,d}$



- The SW curve is $y^2 = \phi_2(z)$ where
  - z is the coordinate of the base sphere
  - SW differential is $y dz$
  - $\phi_2(z)(dz)^2$ is a quadratic differential with (for $i = a, b, c, d$)

$$\phi_2(z)(dz)^2 \sim m_i^2 dz^2/(z - z_i)^2$$

  - $\exp(-2\pi i \tau_{UV})$ is the cross-ratio of $z_{a,b,c,d}$
  - Mass of the W-boson is $\int_C y dz$



- Consider Liouville theory

$$S = \frac{1}{\pi} \int d^2x \sqrt{g} \left(|\partial_\mu \varphi|^2 + \mu e^{2b\varphi} + QR\varphi \right)$$

where $Q = b + 1/b$. $\quad c = 1 + 6Q^2$.

- Its four-point function is

$$\langle V_{m_1} V_{m_2} V_{m_3} V_{m_4} \rangle = \int da\, C_{m_1,m_2,a} C_{a,m_3,m_4} |\mathcal{F}|^2$$

where $V_a(z) = e^{-(Q+2a)\varphi(z)}$ and $C_{\alpha_1,\alpha_2,\alpha_3}$: DOZZ 3pt function

Liouville vs. gauge theory

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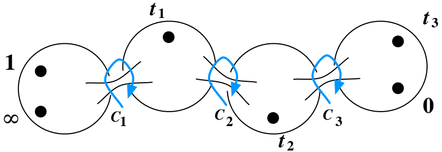
- Conformal block $\mathcal{F}$ is Nekrasov's instanton partition function Z_{inst}
- Product of C's happens to be $|Z_{\text{1-loop}}|^2$

$$\langle V_{m_1} V_{m_2} \cdots \rangle = \int \prod (a_i^2 da_i) |Z_{\text{1-loop}} Z_{\text{inst}}|^2$$

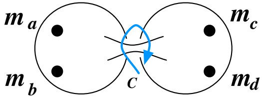
- When $b = 1$ (i.e. $\epsilon_1 = \epsilon_2$, $c = 25$)
  the RHS is the partition function on S^4. [Pestun]

Nekrasov vs. Conformal block



- Both Z_{inst} and $\mathcal{F}$ depend on the decomposition of the Riemann surface into pairs of pants
- Nekrasov's side:
  decomposition determines the S-duality frame
- Conformal block's side:
  decomposition determines the channel
- S-duality is the s-t channel duality!

'Semiclassical' limit



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- Liouville theory has $\quad T(z).$
- Both have spin-2.

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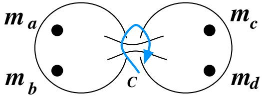

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$$\lim_{m_a \gg \epsilon_1, \epsilon_2} \langle T(z) \rangle = \phi_2(z)$$

Toda

- A_{N-1} Toda theory

$$S = \frac{1}{\pi}\int d^2x\sqrt{g}\left(|\partial_\mu\vec{\varphi}|^2 + \mu\sum_i e^{2b\vec{e}_i\cdot\vec{\varphi}} + Q\vec{\rho}\cdot\vec{\varphi}R\right)$$

  where $Q = b + 1/b$. $\qquad c = (N-1) + Q^2N(N^2-1)$.
- A_1 Toda = Liouville.
- Has W_N symmetry, with generators

$$W_2(z) = T(z), \quad W_3(z), \quad \ldots, \quad W_N(z)$$

‘Semiclassical’ limit

- SW curve was

$$y^N + y^{N-2}\phi_2(z) + y^{N-3}\phi_3(z) + \cdots + \phi_N(z) = 0.$$

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presumably. (nobody has checked...)

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- $\mathbf{SU}(\boldsymbol{N})$ quivers give the $\boldsymbol{A_{N-1}}$ Toda theory [Wyllard,Mironov-Morozov]

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- Anomaly polynomial of N M5-branes minus center of mass
  [Harvey-Minasian-Moore]

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- The former can be derived from the latter. [Alday-Benini-YT] (relation first observed by [Bonelli-Tanzini])

Gauge theory vs. Liouville/Toda

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WHY???