$\mathcal{N} = 4$ super Yang-Mills and Integrability

— An Appetizer —

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0. Brief Introduction

◇ 1+1d integrable systems : sources for many insights on field theory

◇ Are there integrable models in 4d? If solved, it may become another paradigm...

◇ Most tractable models are those with i) many supersymmetries and ii) in the large N limit.

◇ Indeed, we have seen recently several signs of integrability in $\mathcal{N} = 4$ super $SU(N)$ gauge theory in the large N limit.

◇ I hope some in the audience will solve this interesting problem...

1. Detailed Introduction

$\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory

◇ **Classical Action**

$$\mathcal{L} = \frac{\mathrm{tr}}{2g^2}\Big(F_{\mu\nu}F_{\mu\nu} + (D_\mu\phi_i)^2 + [\phi_i,\phi_j]^2 + \bar{\psi}^a i\gamma^\mu D_\mu\psi^a + \bar{\psi}^a\Gamma^i{}_{ab}[\phi_i,\psi^b]\Big)$$

where all fields are $N \times N$ hermitean matrices, and

$$D_\mu = \partial_\mu - i[A_\mu,\ \cdot\ ], \quad F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

$\phi_i,\ i = 1,\ldots,6$: **real vector under global symmetry** $SO(6)$

$\psi_a,\ a = 1,\ldots,4$: **Weyl spinor of global** $SO(6)$.

◇ **Four supercharges Q^a, $a = 1,2,3,4$ with**

$$[Q^a, A_\mu] \sim \gamma_\mu\psi^a$$

helicity	# of states	fields
1	1	A_μ
1/2	4	$\psi_1, \psi_2, \psi_3, \psi_4$
0	6	$\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6$
-1/2	4	$\bar\psi_1, \bar\psi_2, \bar\psi_3, \bar\psi_4$
-1	1	A_μ

- **Maximally supersymmetric in four-dimension without gravity.**

- **Obtained from 10d super Yang-Mills: ($\mu, \nu = 0, \ldots, 9$)**

$$\mathcal{L} = \frac{\mathbf{tr}}{g^2}\left(F_{\mu\nu}F_{\mu\nu} + \bar\psi\Gamma^\mu D_\mu\psi\right)$$

by naïve dimensional reduction, i.e.
rewriting $A_4, A_5, \ldots, A_9 \to \phi_1, \phi_2, \ldots, \phi_6$ **and forgetting about** $\partial_4, \partial_5, \ldots, \partial_9$**.**

◇ **The coupling constant g is NOT renormalized to ANY ORDERS in perturbation theory.**

◇ **Believed to define a superconformal theory with symmetry group**

$$\underbrace{SO(5,1)}_{\text{conformal group in 4d}} \times \underbrace{SO(6)}_{\text{global sym.}} \subset PSU(2,2|4)$$

Fermionic generators are Q^a and S^a.

◇ **Believed to have a symmetry under strong–weak coupling duality**

$$\frac{16\pi^2}{g^2} \leftrightarrow g^2,$$

At least, protected states do match.

Bare Basics of Feynman diagrams

◇ **Free Theory:**

$$\langle x_{i_1} x_{i_2} \cdots x_{i_n} \rangle_0 \equiv \int (\prod dx) x_{i_1} x_{i_2} \cdots x_{i_n} e^{-A_{ij} x_i x_j}.$$

This can be calculated using the Wick theorem. Propagators are $\langle x_i x_j \rangle = (A^{-1})_{ij}$.

◇ **Interacting theory:**

$$\langle x_{i_1} x_{i_2} \cdots x_{i_n} \rangle \equiv \int (\prod dx) x_{i_1} x_{i_2} \cdots x_{i_n} e^{-A_{ij} x_i x_j + gP(x)}$$

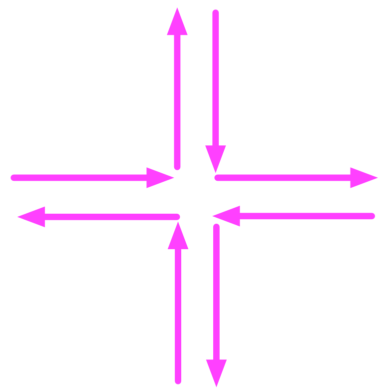
Perturbative calculation is done by expanding $e^{gP(x)}$**:**

$$\langle x_{i_1} x_{i_2} \cdots x_{i_n} \rangle = \langle x_{i_1} x_{i_2} \cdots x_{i_n} \rangle_0 + g \langle x_{i_1} x_{i_2} \cdots x_{i_n} P(x) \rangle_0 + \cdots$$

◇ **For matrix fields, it is useful to draw diagrams in double-line:** e.g.

$\langle \phi_{i\bar{j}}\phi_{k\bar{l}}\rangle \propto \delta_{i\bar{l}}\delta_{k\bar{j}}$ **by**

$\langle \mathbf{tr}\, \phi^4 \rangle$ **by**



◇ **A line connecting two indices signifies that two ends must carry the same index.**

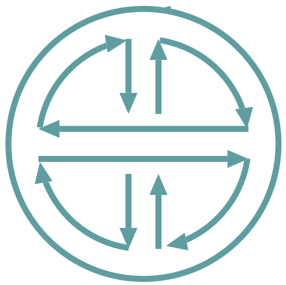
’t Hooft expansion, large N limit

◇ **Thus, each index loop, or surface contributes a factor of $\mathrm{tr}\,1 = N$. Let us denote by E, V, S resp. the number of edges, vertices, surfaces.**

$$\textbf{diagram} \sim (g^2)^{E-V} N^S$$
$$\sim N^{2-2\times\mathbf{genus}} \lambda^{E-V}$$

where $\lambda = g^2 N$: the ’t Hooft coupling .

$\sim N^2\lambda$



$\sim \lambda$

◇ **In the $N \to \infty$ limit with $\lambda = g^2 N$ fixed, only planar diagram contribute.**

◇ **Reminds us of string theory, where genus g contribution $\sim g_s^{-2+2g}$.**

◇ **String theorists have found (in some sense) the dual string description:**

Maldacena Conjecture

$\mathcal{N} = 4$ SU(N) super Yang-Mills theory
is equivalent to
type IIB strings on $AdS_5 \times S^5$.

Supports for the conjecture

◇ **The two systems are given by different ways of taking the low energy limit of one and the same system : N D3 branes in flat $\mathbb{R}^{10}$.**

$$\text{(curvature radius)}^4 \Leftrightarrow \lambda$$
$$\text{string coupling} \Leftrightarrow 1/N^2$$

◇ **Global symmetries match: $SO(5,1) \times SO(6)$ acts geometrically on $AdS_5 \times S^5$.**

$$\text{time translation in } AdS_5 \Leftrightarrow \text{dilatation in 4d}$$

◇ **Many more profound facts...**

- **This conjecture relates two difficult, unsolved objects (strings on AdS and $\mathcal{N} = 4$ SYM) $\rightsquigarrow$ facilitates the analysis of both sides.**

- **Classical strings moving in $AdS_5 \times S^5$ has infinitely many conserved charges of Yangian type.**

Question:

Is there integrability in the $\mathcal{N} = 4$ SYM side?

2. Dilatation operator in $\mathcal{N} = 4$ SYM

- **If $\mathcal{O}$ has dimension $\Delta_0 + \delta\Delta$,**

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{\text{const.}}{|x|^{2(\Delta_0+\delta\Delta)}} = \frac{\text{const.}}{|x|^{2\Delta_0}} - 2\delta\Delta \log(\Lambda|x|) + \cdots$$

In general, there is a mixing

$$\langle \mathcal{O}_a(x)\mathcal{O}_b(0)\rangle = \frac{\text{const.}}{|x|^{2\Delta_0}} - 2\delta\Delta_{ab} \log(\Lambda|x|) + \cdots$$

Multiplicatively renormalized operators are eigenstates of $\delta\Delta_{ab}$.

- **We have to i) calculate $\delta\Delta_{ab}$ perturbatively and ii) diagonalize it.**

- **We take 't Hooft limit and consider planar diagrams only.**

One-loop

- We concentrate on the operators of the form $\mathcal{O}_{i_1 i_2 \cdots i_J} = \mathrm{tr}\, \phi_{i_1} \phi_{i_2} \cdots \phi_{i_J}$.

- One can show that mixing closes in these operators at one-loop.

- $\delta\Delta$ is a $6^J \times 6^J$ matrix $\sim$ some Hamiltonian acting on a chain of J spins with six components.

- One-loop planar diagrams only generate nearest neighbor interactions.

◇ **Combined with $SO(6)$ invariance,**

$$\delta\Delta = \text{const.} + \sum_a (cK_{a,a+1} + c'P_{a,a+1})$$

where $Pv \otimes w = w \otimes v$, $Kv \otimes w = (v,w)\sum_{i=1}^{6} e_i \otimes e_i$.

◇ **Calculating coefficients c, c' is a straightforward exercise in diagrammatics, and we find**

$$\delta\Delta = \text{const.} + \frac{\lambda}{16\pi^2}\sum_{a=1}^{J}(1K_{a,a+1} - 2P_{a,a+1}).$$

◇ **This precise combination of $K - 2P$ implies integrability ! Indeed, define $R_{12}(u)$ acting on $V_1 \otimes V_2$ by ($\dim V_i = n$)**

$$R_{12}(u) = \frac{1}{n-2}\left(u(2u+2-n) - (2u+2-n)P_{12} + 2uK_{12}\right)$$

- **It satisfies the Yang-Baxter relation**

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u).$$

- **Hence the transfer matrix**

$$T_0(u) = R_{01}(u)R_{02}(u)\cdots R_{0J}(u)$$

generates commuting charges H_m by expanding $\log \mathbf{tr}_{V_0}\, T_0(u) = u^m H_m$. One can show

$$H_2 \propto \sum_{a=1}^{J}\left(K_{a,a+1} + \frac{n-2}{2}(1 - P_{a,a+1})\right).$$

- **With $n = 6$, this precisely agrees with $\delta\Delta$. This system can be further analyzed by algebraic Bethe ansatz.**

Aside: Implications for Maldacena Conjecture

- **For large J, the Bethe equation reduces to integral equations for the density of Bethe roots. $\rightsquigarrow$ Anomalous dimensions can be expressed using the complex plane with cuts.**

- **Correspondingly, string state in the AdS side with large J is a string rotating in S^5. Its classical motion can be solved, and is expressed using the same complex plane with cuts.**

- **This gives an intricate test of AdS/CFT correspondence.**

Parity Pair

◇ **We have found commuting charges H_m with $H_2 = \delta\Delta$. Is there any gauge theoretical manifestation for $H_{m\neq 2}$?**

◇ **Consider the outer automorphism of $SU(N)$, parity, given by**

$$\Omega : \phi_1\phi_2\cdots\phi_J \to \phi_J\cdots\phi_2\phi_1.$$

This generates a $\mathbb{Z}_2$ global symmetry, and $[\Delta, \Omega] = 0$. $\rightsquigarrow$ Eigenstates of anomalous dim. can be taken simaltaneously to be parity eigenstates.

◇ **In general, there's no relation between energy eigenvalues for parity even/odd states.**

- **However, in our case we have $\{H_{\mathbf{odd}}, \Omega\} = 0$.**

- **Hence, acting** e.g. **by H_3, we find states of degenerate energy with opposite parity ! "parity pair"**

- **H_1 is a shift operator $\phi_1\phi_2\cdots\phi_J \to \phi_J\phi_1\cdots\phi_{J-1}$. However, due to the cyclicity of tr, this is trivially zero. We need higher charges for this argument.**

- **$\{H_{\mathbf{odd}}, \Omega\} = 0$ is expected generically for integrable chains. Thus, existence of parity pairs is a good touchstone for integrability .**

Extension to Higher-loops

◇ $1/N$ **corrections break degeneracy of parity pairs. Integrability, if any, is a property of the large N limit.**

◇ **On the other hand,** $\lambda = g^2N$ **corrections with** $1/N \to 0$ **preserves parity pairs at least up to** λ^3 **order.**

◇ **At higher order, there appear**

i) **long range spin exchange,**

ii) **three-body and many-body interaction,**

iii) **fluctuation in the length of the spin chain**

◇ **This suggests the existence of a new kind of integrable spin chain.**

◇ **Let us expand the anomalous dimension as**

$$\Delta = \Delta_0 + \lambda \delta\Delta_1 + \lambda^2 \delta\Delta_2 + \cdots .$$

As $[\delta\Delta_1, \delta\Delta_2] \neq 0$, Δ is not the combination of the commuting Hamiltonians of $SO(6)$ Heisenberg spin chain.

◇ **$\delta\Delta$, restricted to $SU(2) \subset SO(6)$ subsector and with terms of ii), iii) dropped by hand, is known to agree with the Inozemtsev spin chain up to λ^3, but disagrees at order λ^4.**

◇ **Inozemtsev spin chain is claimed to be the most general $SU(2)$-invariant integrable chain with two-body interactions only. However, as Δ contains many-body interactions, the disagreement is not so discouraging.**

3. Conclusion

✓ **One-loop anomalous dimensions of the $\mathcal{N} = 4\ SU(N)$ in the 't Hooft limit can be identified with the integrable $SO(6)$ spin chain .**

✓ **This integrability leads to the existence of the parity pairs.**

✓ **Parity pairs persist at least up to three-loops.**

✓ **It is an interesting and important problem to settle whether there is integrability, or at least parity pairs to all orders in λ.**

Appendix.

We list some of the key papers:

- ◇ **before and including BMN...** **please refer to Ideguchi-san's master's thesis !**
- ◇ **Korchemsky et al.** **Integrability in the Regge limit.**
- ◇ **Frolov-Tseytlin** **On spinning strings.**
- ◇ **Minahan-Zarembo** **Finds integrable spin chains in dilatation operator.**
- ◇ **Beisert-Kristjansen-Staudacher** **Higher order analysis of dilatation operator. Parity Pair.**
- ◇ **Beisert-Minahan-Staudacher-Zarembo** **Matching between Bethe ansatz and spinning string.**

- **Beisert, "The dynamical spin chain"** Analysis of dilatation operator from the viewpoint of superconformal group.
- **Beisert-Staudacher** Spin chains for $su(2,2|4)$.
- **Bena-Polchinski-Roiban** Higher charges in the classical strings in $AdS_5 \times S^5$.
- **Kruczenski** Semiclassical limit of spin chain and strings.
- **Arutyunov-Staudacher** Matching of higher charges to one-loop order.
- **Dolan-Nappi-Witten I, II** Shows one-loop dilatation commutes with the Yangian
- **Arutyunov-Russo-Tseytlin** Reduction to Integrable systems of strings in $AdS_5 \times S^5$.
- **Kazakov-Marshakov-Minahan-Zarembo** General matching between strings and spins using complex curves.