On S-duality of 5d SYM on S^1

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based on [arXiv:1009.0339]
and on an unpublished, quite unfinished work

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Introduction

$\mathcal{N} = 4$ super Yang-Mills with group G,

or, in general, $\mathcal{N} = 2$ gauge theory with zero one-loop beta function:

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \text{ doesn't run, and tunable.}$$

(n.b. with $\mathcal{N} = 1$ or less, higher order effects drive the coupling to zero.)

What happens when it becomes very strong? ⟶ Dual descriptions.

$\mathcal{N} = 4$: Montonen-Olive duality

$$G \text{ at } \tau \text{ is equivalent to } {}^LG \text{ at } \tau' = -\frac{1}{n_G \tau}$$

- $n_G = 1$: $A_{N-1} = \mathbf{SU}(N)$, $D_N = \mathbf{SO}(2N)$, E_N
- $n_G = 2$: $B_N = \mathbf{SO}(2N+1)$, $C_N = \mathbf{Sp}(N)$, F_4
- $n_G = 3$: G_2

$$B_N \leftrightarrow C_N, \text{ otherwise } G = {}^LG$$

(as far as the Lie algebra is concerned, that is.)
[Montonen-Olive,1977] [Olive-Witten,1978] [Osborn,1979]

What's ${}^{L}G$?

- Called the Goddard-Nuyts-Olive dual [1977],
  or the Langlands dual of G [circa early '70s].

- $\mathcal{N} = 4$ SYM has six adjoint scalars.

- Give one a vev Φ in the Cartan of g and break G to $U(1)^r$.

- Off-diagonal components give the roots:

$$[\Phi, E_\alpha] = (\alpha \cdot \Phi) E_\alpha.$$

e.g. for $\mathbf{SU}(N)$, let $\Phi = \mathbf{diag}(a_1, \ldots, a_N)$.
Then E_{ij}: a matrix with 1 at (i, j) and 0 otherwise, satisfies

$$[\Phi, E_{ij}] = (a_i - a_j) E_{ij}.$$

Let $e_i = \mathbf{diag}(0, 0, .., 1, ..., 0)$ where 1 is at the i-th position. i.e. $\alpha = e_i - e_j$ is a root for $E_\alpha = E_{ij}$.

- $\sigma^+ = E_\alpha, \sigma^- = E_{-\alpha}$, and $\sigma^3 = \alpha^\vee = \dfrac{2\alpha}{\alpha \cdot \alpha}$ satisfy the standard commutation rules. Compare

$$[\alpha^\vee, E_\alpha] = 2E_\alpha, \qquad \text{and} \qquad [\sigma^3, \sigma^+] = 2\sigma^+.$$

- $\alpha^\vee$ is called a co-root.

- Each E_α gives a W-boson. The mass comes from $|D_\mu \langle \Phi \rangle|^2$:

$$m = \alpha \cdot \langle \Phi \rangle.$$

- Each E_α gives a monopole:
  start from a standard BPS $\mathbf{SU(2)}$ monopole.
  Recall E_α, $E_{-\alpha}$ and $\alpha^\vee$ form an $\mathbf{SU(2)}$ subalgebra.
  You can think of the $\mathbf{SU(2)}$ monopole as a monopole in the gauge group G, with mass

$$m = \frac{4\pi}{g^2} \alpha^\vee \cdot \langle \Phi \rangle = \frac{4\pi}{g^2} \frac{2\alpha}{\alpha \cdot \alpha} \cdot \langle \Phi \rangle.$$

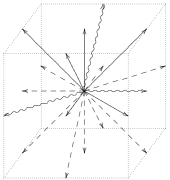
Co-roots of G form roots of $^L G$.

$G \leftrightarrow {}^L G$ $\qquad$ $\alpha \leftrightarrow \alpha^\vee = 2\alpha/|\alpha|^2$

$\tau \leftrightarrow -1/(n_G \tau)$ $\qquad$ W-boson $\leftrightarrow$ monopoles

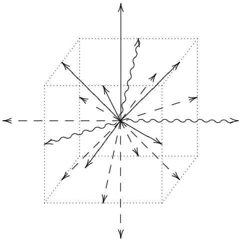
SO(7)

$\pm e_i \pm e_j, \pm e_i$



Sp(3)

$\pm m_i \pm m_j, \pm 2 m_i$



$n_G = 1$: simply-laced ; $n_G = 2, 3$: non-simply-laced

Simply-laced case

- G at $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$ is equivalent to G at $-\frac{1}{\tau}$.
- $G = A, D$ or E.
- Geometric explanation: there's 6d theory with "non-Abelian self-dual tensor fields": instead of $F^a_{\mu\nu}$, one "has" $F^a_{\mu\nu\rho}$ with

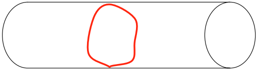
$$F_{\mu\nu\rho} = \frac{1}{6}\epsilon_{\mu\nu\rho}{}^{\alpha\beta\gamma}F_{\alpha\beta\gamma}$$

- This theory is **conformal**, non-Lagrangian, decoupled from gravity, with A, D or E.
- Has "self-dual strings" coupled to $F^a_{\mu\nu\rho}$.

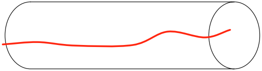
- Compactified on S^1 of radius L, we have 5d maximally-supersymmetric YM with gauge group G, with coupling

$$\frac{1}{g_{5d}^2} \propto \frac{1}{L}$$

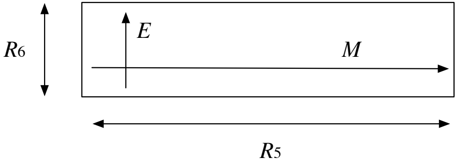
- Wrapped strings are W-bosons:



- Unwrapped strings are monopole-strings



Compactification on the torus



- 6d theory of type G on a torus.

$$\int d^5x \frac{1}{g_{5d}^2} F_{\mu\nu}F_{\mu\nu} = \int d^4x \frac{R_5}{g_{5d}^2} F_{\mu\nu}F_{\mu\nu} = \int d^4x \frac{R_5}{R_6} F_{\mu\nu}F_{\mu\nu}$$

→ 4d coupling is $\tau = iR_5/R_6$.

- Strings wrapped around E : W-bosons, mass $\propto R_6$
- Strings wrapped around M: Monopoles, mass $\propto R_5$
- Invariance under $\tau \to -1/\tau$ manifest: $R_5 \leftrightarrow R_6$.

Non-simply-laced case?

- String/M theory only gives 6d theory with A_{N-1}, D_N or E_N, i.e. **simply-laced**.
- [Henningson,2004] showed that from purely 6d perspective.
- We shouldn't have 6d B_N non-abelian-tensor theory anyway, because that would predict $B_N \leftrightarrow B_N$.

- To get 4d **non**-simply-laced theory, one needs a twist. [Vafa,1997]

- Today's objective: explore this system in detail, from the point of view of **5d theory on S^1**.

- I think it's worth while, given recent importance of 6d theory on Riemann surfaces, giving rise to $\mathcal{N}=2$ dualities of Gaiotto.
- Some very recent works in Dec. 2010 by [Douglas], [Lambert–Papageorgakis–Schmidt-Sommerfeld], [Terashima-Yagi]

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5d maximally-supersymmetric YM

- gauge group G
- $S = \int d^5x \frac{1}{g_{5d}^2} F_{\mu\nu}F_{\mu\nu} + \cdots$
- has 16 supercharges.
- has five adjoint scalars.

- When $G = C_N = \mathbf{Sp}(N)$, you can choose $\theta_{5d} = 0$ **or** π associated to $\pi_4(\mathbf{Sp}(N)) = \mathbb{Z}_2$.

- 4d theta angle $\leftrightarrow \pi_3(G) = \mathbb{Z}$.
  5d theta angle $\leftrightarrow \pi_4(G)$, which is nontrivial only for $\mathbf{Sp}(n)$.
- The only new material in my work is for $\mathbf{Sp}(N)$ with $\theta_{5d} = \pi$, but I won't be able to talk about it today!

5d maximally-supersymmetric YM on S^1

- Compactify it on S^1
  → 4d $\mathcal{N} = 4$ theory with gauge group G **with KK towers.**
- 4d $\mathcal{N} = 4$ has **six** adjoint scalars:
  five from 5d scalars, **another** from $\int_{S^1} dx^4 A_4$.
- Set $A_5 = 0$ at the asymptotic infinity for simplicity.
  Higgs via one of 5d scalar which we call Φ.
- The tower of W-bosons are labeled by

$$\mathring{\alpha} = (k, \alpha)$$

where $k \in \mathbb{Z}$ is the KK momentum, α is a root. The mass formula is

$$m = \sqrt{(\frac{2\pi k}{R_5})^2 + |\alpha \cdot \Phi|^2} = |\mathring{\alpha} \cdot \mathring{\Phi}|$$

where $\mathring{\Phi} = (2\pi/R_5, \Phi)$.

How about monopoles?

Let's review how to get 4d monopoles first.

The $\mathbf{SU(2)}$ BPS monopole is the solution to

$$B_i^a = D_i \phi^a$$

where $a = 1, 2, 3$; $i, j = 1, 2, 3$. Explicitly,

$$A_i^a = \epsilon_{aij} \hat{r}^j \left(\frac{1}{r} - \frac{u}{\sinh ur} \right), \qquad \phi^a = \hat{r}^a \left(\frac{1}{r} - u \coth ur \right)$$

where $u = \langle \phi \rangle$ is the vev at the asymptotic infinity.

The parameter u is the only scale; as such, the mass is proportional to u.

To embed into larger G with vev $\langle \Phi \rangle$, choose α, and set

$$\Phi = \langle \Phi \rangle_{\perp} + E_{\alpha}\phi^{+} + \alpha^{\vee}\phi^{3} + E_{-\alpha}\phi^{-}$$

for $u = \alpha \cdot \Phi$.

$\langle \Phi \rangle_{\perp}$ is the projection of $\langle \Phi \rangle$ orthogonal to α.

The mass is then

$$m = \frac{4\pi}{g^2}\frac{\alpha^{\vee} \cdot \alpha^{\vee}}{2}u = \frac{4\pi}{g^2}\alpha^{\vee} \cdot \Phi.$$

What happens in 5d? Of course, we can lift the solution we just got to 5d:

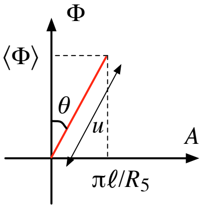
$$\boldsymbol{\Phi}(\vec{x}, x^4) \equiv \boldsymbol{\Phi}(\vec{x})$$

We want to make it dependent on x^4:

$$\boldsymbol{\Phi}(\vec{x}, x^4) = e^{i\pi\ell\frac{x^4}{R_5}\alpha^\vee}\boldsymbol{\Phi}(\vec{x})e^{-i\pi\ell\frac{x^4}{R_5}\alpha^\vee}$$

This in itself is not a solution. We use the following trick:

Turning on a linear combination of the 4d scalars $\boldsymbol{\Phi}$ and A_4.



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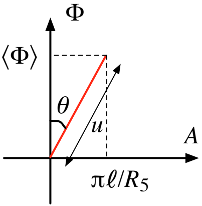
(recall in 5d we have $\boldsymbol{\Phi_1} = \boldsymbol{\Phi}$, $\boldsymbol{\Phi_{2,\dots,5}}$ and $A_{0,1,2,3,4}$.)



We let

$$\begin{aligned} \Phi &= \langle\Phi\rangle_{\perp} + \quad \cos\theta(E_{\alpha}\phi^{+} + \alpha^{\vee}\phi^{3} + E_{-\alpha}\phi^{-}), \\ A_4 &= \qquad\quad + \quad \sin\theta(E_{\alpha}\phi^{+} + \alpha^{\vee}\phi^{3} + E_{-\alpha}\phi^{-}). \end{aligned}$$

We need to cancel $\langle A_4 \rangle$ via a large gauge transformation

$$g(x^4) = e^{-\pi i \ell \frac{x^4}{R_5}\alpha^{\vee}}$$

To be consistent, we need ℓ to be an **integer**.

Now it's a solution, with mass

$$m = \frac{4\pi}{g_{5d}^2} R_5 \frac{\alpha^\vee \cdot \alpha^\vee}{2} \sqrt{(\frac{2\pi\ell}{R_5})^2 + |\alpha \cdot \langle\Phi\rangle|^2}$$
$$= \frac{4\pi}{g_{5d}^2} R_5 \frac{2}{\alpha \cdot \alpha} |\mathring{\alpha} \cdot \mathring{\Phi}| \qquad = \frac{4\pi}{g_{5d}^2} R_5 |\mathring{\alpha}^\vee \cdot \mathring{\Phi}|$$

where

$$\mathring{\alpha} = (\ell, \alpha), \quad \mathring{\Phi} = (\frac{2\pi}{R_5}, \Phi)$$

and

$$\mathring{\alpha}^\vee = \frac{2}{\alpha \cdot \alpha}(\ell, \alpha).$$

These solutions were studied extensively in the context of calorons. [Kraan-van Baal,'98] [K.M. Lee,'98] [Hanany-Troost,'01] [Kim-Lee-Yee-Yi,'04]

KK W-bosons & Monopoles

- So, towers of W-bosons are labeled by $\mathring{\alpha} = (k, \alpha)$ with mass

$$m = |\mathring{\alpha} \cdot \mathring{\Phi}|.$$

k is the KK momentum.

- Towers of monopoles are labeled by $\mathring{\alpha}^\vee \equiv \frac{2}{|\alpha|^2}(\ell, \alpha)$ with mass

$$m = \frac{4\pi}{g_{5d}^2} R_5 |\mathring{\alpha}^\vee \cdot \mathring{\Phi}|$$

ℓ is **not** the KK momentum; remember $D_4 \Phi = 0$.

- Instead, ℓ is the **instanton charge**: In 5d theory on $\mathbb{R}_t \times \mathbb{R}^3 \times S^1$,

$$\#\text{inst} = \int_{\mathbb{R}^3 \times S^1} \mathbf{tr}\, F \wedge F$$

$$\propto \int d\theta d\varphi B_{\theta\varphi} \int dr dx^4 \partial_r A_4 \propto (\alpha^\vee \cdot \alpha^\vee)\ell.$$

For simply-laced G, i.e. $G = A, D$ or E, $\alpha^\vee = \alpha$.
Then towers of **KK W-bosons** labeled by (k, α) with mass

$$m = \sqrt{\left(\frac{2\pi k}{R_5}\right)^2 + |\alpha \cdot \Phi|^2}$$

and **towers of instantonic monopoles** with mass

$$m = \frac{4\pi}{g_{5d}^2} R_5 \sqrt{\left(\frac{2\pi \ell}{R_5}\right)^2 + |\alpha \cdot \Phi|^2}$$

are nicely exchanged when we exchange

$$g_{5d}^2 \leftrightarrow R_5.$$

What happens when G is **not** simply laced?

Consider $C_n = \mathbf{Sp}(n)$ theory.

- The roots are $\pm e_i \pm e_j$ $(i \neq j)$ and $\pm 2e_i$.
- KK W-bosons are

$$\mathring{\alpha} = (k, \pm e_i \pm e_j), \qquad (k, \pm 2e_i)$$

- then instantonic monopoles are

$$\mathring{\alpha}^\vee = (k, \pm e_i \pm e_j), \qquad (k, \pm e_i), \qquad (k + \frac{1}{2}, \pm e_i).$$

- $k = 0$ gives the roots $\pm e_i \pm e_j$ and $\pm e_i$, i.e. roots of $\mathbf{SO}(2n+1)$.
- But what are those with $k + 1/2$?
- Is it the KK spectrum of any 5d theory? YES!

“Twisted” gauge theory

Consider 5d $\mathbf{SO(2n+2)}$ theory on S^1 with a twist

$$\Phi(x^4 = 0) = P\Phi(x^4 = R_5)P^{-1}$$

where P is the parity transformation of the gauge group $\mathbf{SO(2n+2)}$.

- P is a kind of Wilson line, but not quite.
- It's slightly outside of $\mathbf{SO(2n+2)}$. It's only in $\mathbf{O(2n+2)}$.
- Called an **outer automorphism** of $\mathbf{SO(2n+2)}$.

Parity $P = \mathbf{diag}(1, 1, \ldots, 1, -1)$ breaks

$$\mathbf{SO}(2n+2) \to \mathbf{SO}(2n+1) + \underline{2n+1},$$

with weights

$$\mathbf{SO}(2n+1) : \pm e_i \pm e_j, \qquad \pm e_i$$
$$\underline{2n+1} : \pm e_i$$

P multiplies $\underline{2n+1}$ by -1.
So the spectrum of KK W-bosons under the b.c.

$$\Phi(x^4 = 0) = P\Phi(x^4 = R_5)P^{-1}$$

is

$$(k, \pm e_i \pm e_j), \qquad (k, \pm e_i), \qquad (k + \frac{1}{2}, \pm e_i).$$

This agrees with the spectrum of instantonic monopoles of $\mathbf{Sp}(n)$.

It can also be checked that

the instantonic monopoles of $\mathbf{SO}(2n+2)$ theory with P twist

form the same tower as

the KK W-bosons of $\mathbf{Sp}(n)$.

Therefore, if we only care about the bottom of the tower, we see

$$C_n = \mathbf{Sp}(n) \leftrightarrow B_n = \mathbf{SO}(2n+1)$$

but if we also include the tower, we see

$$C_n = \mathbf{Sp}(n) \leftrightarrow (D_{n+1} = \mathbf{SO}(2n+2), P)$$

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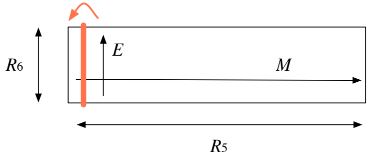
5d $\mathbf{SO}(2n+2)$ with P twist in 6d description

- We know how to realize 5d D_{n+1} with P twist.
- 6d D_{n+1} on S^1 → 5d $\mathbf{SO}(2n+2)$ theory.
- Impose

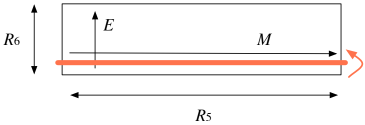
$$\Phi(x^4 = R_5) = P\Phi(x^4 = 0)P^{-1}$$

where $P = \mathbf{diag}(1, 1, \ldots, 1, -1)$: parity
→ 4d $\mathbf{SO}(2n+1)$ theory.



5d Sp(n) in 6d description

Interchange the role of two directions:



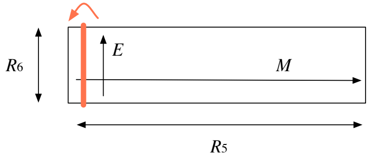
→ 6d D_{n+1} theory on S^1 with this twist = 5d $\mathbf{Sp}(n)$.

KK momentum ↔ instanton number

We saw that the KK momentum and the instanton number were interchanged

$$(k, \alpha) \leftrightarrow (l, \alpha)^\vee$$

In the 6d description, the instanton number **is** the 6d KK momentum!



Recall 6d theory on S^1 is a gauge theory and has the action

$$\sim \int d^5x \frac{1}{R_6} \,\mathrm{tr}\, F_{\mu\nu} F_{\mu\nu},$$

and therefore an instanton has the mass $\sim 1/R_6$.

2d interpretation

- Given the set of roots α of G,
  the set of co-roots $\alpha^\vee = \dfrac{2}{\alpha \cdot \alpha}\alpha$ form the roots of ${}^L G$.
- This is the inversion of the arrows in the Dynkin diagram:

$$
\begin{aligned}
B_n = \mathbf{SO}(2n+1): &\qquad \circ - \circ - \circ - \cdots - \circ \Rightarrow \circ \\
C_n = \mathbf{Sp}(n): &\qquad \circ - \circ - \circ - \cdots - \circ \Leftarrow \circ
\end{aligned}
$$

2d interpretation

- Given the set of roots α of G,
  the set of (k, α) is the roots of the affine Lie algebra $\mathcal{L}(G)$:

$$j^a(z).$$

- Its Dynkin diagram is the extended corresponding Dynkin diagram:

$$\begin{array}{ll} B_n^{(1)}: & \circ - \overset{\overset{\circ}{|}}{\circ} - \cdots - \circ \Rightarrow \circ \\ C_n^{(1)}: & \circ \Rightarrow \circ - \circ - \cdots - \circ \Leftarrow \circ \end{array}$$

- The set of co-roots, $(2/|\alpha|^2)(k, \alpha)$ is also the roots of a Kac-Moody algebra, whose Dynkin diagram is obtained as before:

$$\begin{array}{r} \circ - \overset{\overset{\circ}{|}}{\circ} - \cdots - \circ \Leftarrow \circ \\ \circ \Leftarrow \circ - \circ - \cdots - \circ \Rightarrow \circ \end{array}$$

2d interpretation

- The set of co-roots, $(2/|\alpha|^2)(k, \alpha)$ is also the roots of a Kac-Moody algebra, whose Dynkin diagram is obtained as before:

$$
\begin{array}{rc}
A^{(2)}_{2n+1}: & \begin{array}{c} \qquad\;\, \circ \\ \qquad\;\, | \\ \circ - \circ - \cdots - \circ \Leftarrow \circ \end{array} \\
D^{(2)}_{n+1}: & \circ \Leftarrow \circ - \circ - \cdots - \circ \Rightarrow \circ
\end{array}
$$

- They are the Dynkin diagram of a **twisted affine Lie algebra**, defined by

$$j^a(z) = P j^a(e^{2\pi i} z) P^{-1}$$

- Summarizing, 4d S-duality exchanges

$$B_n : \circ - \circ - \circ \Rightarrow \circ \qquad \longleftrightarrow \qquad C_n : \circ - \circ - \circ \Leftarrow \circ$$

- For 5d B_n theory on S^1, one first extends it and then take the dual

$$B_n^{(1)} : \circ - \overset{\circ}{\underset{\circ}{|}} - \circ \Rightarrow \circ \qquad \longleftrightarrow \qquad A_{2n+1}^{(2)} : \circ - \overset{\circ}{\underset{\circ}{|}} - \circ \Leftarrow \circ$$

and get 5d A_{2n+1} theory on S^1 with a twist by P.

- The appearance of the **Langlands dual** of **the affine version** of G has been known for quite some time, e.g. [Martinec-Warner,'95], [Braverman,'04]

- In the 4d $\mathcal{N} = \mathbf{2}$ gauge theory / 2d CFT correspondence, what appears on 2d for a 4d theory with gauge group G is

  the **W-algebra** of
  the **Langlands dual** of
  the **affine version** of G.

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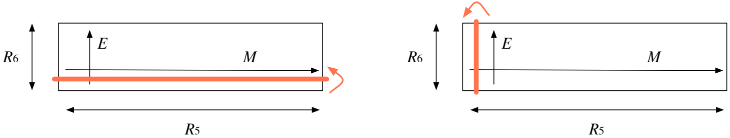
Summary

- 5d maximally supersymmetric theory on S^1 has an S-duality, which is finer than 4d S-duality, e.g.

$$C_n = \mathbf{Sp}(n) \leftrightarrow (D_{n+1} = \mathbf{SO}(2n+2), P)$$

- In the 6d description, this comes from the exchange of the direction with the twist.



- In the 2d description, this is the Langlands duality of the affine algebras.

$$C_n^{(1)} : \circ \Rightarrow \circ - \circ \Leftarrow \circ \quad \longleftrightarrow \quad D_{n+1}^{(2)} : \circ \Leftarrow \circ - \circ \Rightarrow \circ$$