Five-dimensional Chern-Simons terms and Nekrasov's Instanton Counting...

with a review!

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1. Introduction

◇ **Consider type IIA on a CY**

$$X : \underbrace{\text{ALE singularity}}_{\text{Produces YM in 6d}} \text{ fibered over sphere}$$

$\rightsquigarrow \mathcal{N} = 2$ **SYM in 4d.**

◇ **Prepotential $\mathcal{F}_{g=0}$ for this theory can be studied by two methods:**

i) **from the world-sheet point of view, or**
ii) **from the space-time point of view.**

World-Sheet

◇ **Consider topological A-model on the same X:**

$$\mathcal{F}_{g=0} = \text{ **genus zero free energy of this theory**}$$

◇ **For higher-genus amplitudes, there's a correspondence**

$$\log Z^{(\textbf{all-genus})}_{\textbf{topological theory}} = g_s^{-2}\mathcal{F}_{g=0} + \mathcal{F}_{g=1} + g_s^2\mathcal{F}_{g=2} + \cdots$$

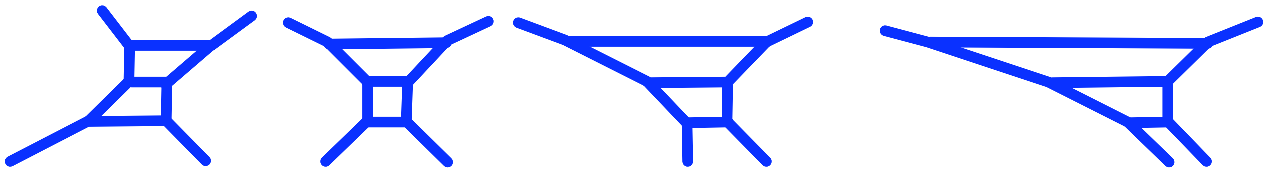
$$\mathcal{F}^{\textbf{graviphoton corrected}}_{\textbf{physical theory}} = \hbar^{-2}\mathcal{F}_{g=0} + \mathcal{F}_{g=1} + \hbar^2\mathcal{F}_{g=2} + \cdots .$$

($\hbar$: magnitude of the bkg. graviphoton field strength)

(Bershadsky-Cecotti-Ooguri-Vafa, 9309140)

◇ $\mathcal{F}_g$ **is exactly calculable by a world-sheet technique called topological vertex for toric Calabi-Yau** X.

◇ **(Iqbal-KashaniPoor 0306032) (Eguchi-Kanno 0310235) calculated** $\mathcal{F}$ **for the following local toric CYs** X_N^κ**: ex) for** $N = 3$



$k =$ **0** **1** **2** **3**

◇ **They are ALE fibrations over** S^2**, hence give 4d** $U(N)$ **super Yang-Mills.**

Space-Time point of view

- Can we calculate $\mathcal{F}_g$ by spacetime instanton calculation ?

  Nekrasov's " Instanton Counting"

- It was found that

$$\mathcal{F} \text{ for } X_N^{\kappa=0} = \mathcal{F}^{\text{graviphoton}}_{\text{corrected}} \text{ of } \overbrace{\text{5d } U(N) \text{ SYM compactified on } S^1}^{\text{We call them 5d SYM with some abuse...}},$$

proposed in (Nekrasov 0206161).

(cf. $\mathcal{F}$ for 4d theory does not agree. S^1 is the M-theory circle.)

- However, $\mathcal{F}$ for $X_N^{\kappa\neq 0}$ does not agree.

◇ **This is as expected in hindsight, as M theory on CY comes with 5d Chern-Simons interaction:**

$$\int_{\mathbb{R}^5 \times CY} C \wedge dC \wedge dC = \int_{\mathbb{R}^5} c^{ijk} A_i \wedge dA_j \wedge dA_k$$

where

$$c^{ijk} = \int_{CY} \omega^i \wedge \omega^j \wedge \omega^k, \qquad C = A_i \omega^i$$

◇ **For $X_N^{\kappa \neq 0}$ in the limit where the gauge group enhances, they give**

$$\kappa \int CS[A] \text{ where } \quad dCS[A] \propto \mathrm{tr} F \wedge F \wedge F$$

(Intriligator-Morrison-Seiberg 9702198)

◇ **$\mathcal{F}$ for $X_N^{\kappa \neq 0}$ can be reproduced by extending Nekrasov's instanton counting to include this non-abelian CS.** **(Y.T. 0401184)**

A comment on Geometric Engineering

It may be obvious to the audience, but as it was surprising at least to the speaker, so please allow me to stress:

$$\text{type IIA} \xrightarrow{\text{on non-compact CY}} \underbrace{\text{theory without gravity}}_{\text{In 5d compactified on } S^1 \text{ !}}$$

Because type IIA IS M-theory at finite g_s.

Indeed, we might have discovered M-theory by comparing

topological A model on local Calabi-Yaus
and
Seiberg-Witten theory for 5d SU(2) SYM...

Another Comment: on Type II – Heterotic Duality

◇ **The fact that the topological vertex and the instanton counting gave exactly the same sequence of integers (Gopakumar-Vafa invariants) is another stringy miracle. This indicates the Type II – Heterotic duality holds for higher genera !**

◇ **It is natural to conjecture the extension to**

$$\mathbb{C}^2/\Gamma \text{ fibered over } S^2 \quad \Leftrightarrow \quad \text{4d } \Gamma \text{ instantons}$$

for any $\Gamma = A_n, D_n, E_n$

◇ **Another interesting avenue would be to study compact CYs. Indeed, Heterotic was extremely powerful for counting curves of higher genera which do not wrap base $\mathbb{CP}^1$. (Mariño-Moore)**

YAC: this time on History (highly subjective)

Physical Camp: Dorey, Hollowood, Italian group,. . .
Determined the measure of the instanton moduli by honest but tedius calculation...
Knew what they were computing,
but didn't know how to compute.

Topological Camp: Moore, Nekrasov, . . .
Studied the geometry of the instanton moduli from mathematical viewpoint ...
Knew how to compute,
but didn't know what they were computing.

$\Downarrow$

Realized they were computing
the SAME THING ($\sim$ Jan. 2002)

CONTENTS

2. Instanton Counting and Fixed-Point theorem

Graviphoton-background

Firstly let us recall what's the graviphoton:

$$g_{\mu\nu} \text{ in 5d} \xrightarrow{S^1\text{-cptfy}} g_{\mu\nu} \text{ in 4d and } A_\mu = g_{5\mu}$$

Hence, constant background field strength for the graviphoton corresponds to the 5d geometry

$$ds^2 = (dx_\mu)^2 + (dx_5 + A_\mu dx_\mu)^2$$

with $\Omega = dA$ constant. $A_\mu = \frac{1}{2}\Omega_{\mu\nu}x^\nu$

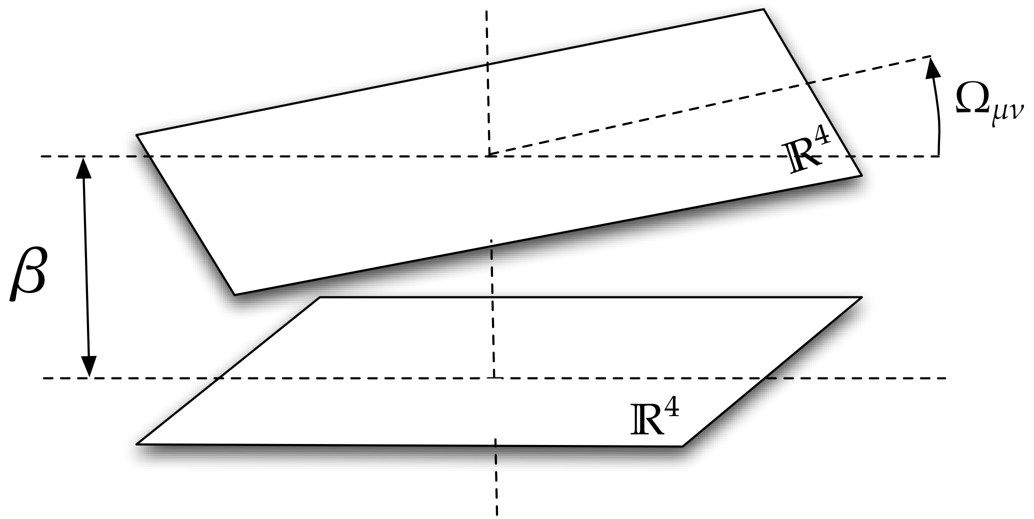
- S^1 **is fibered over** $\mathbb{R}^4$.
- **Denote the eigenvalues of** Ω **by** $\pm\hbar$.

Ω-background

Consider another 5d background with metric

$$ds^2 = (dx^\mu + A_\mu dx^5)^2 + (dx^5)^2$$

with the same A_μ as before. $\mathbb{R}^4$ is fibered over S^1.



$$(x^5 \in [0, \beta])$$

◇ **We can show that**

$$e^{-\hbar^{-2}\mathcal{F}_{\text{graviphoton corrected}}} = Z_{\text{on }\Omega\text{ bkg}} \qquad \cdots\bigstar$$

for any 5d 8-susy theory,

◇ **while $Z_{\text{on }\Omega}$ can be expressed as**

$$Z_{\text{on }\Omega} = \mathrm{tr}(-)^F e^{-\beta H} e^{i\beta\hbar J}$$

by considering the 5th direction as 'time'.

None other than the Witten Index!

(J is one of the Lorentz generators.)

Derivation of the relation ★

When one puts 8-susy 5d theory on this background, the graviphoton-corrected prepotential is given by

$$\hbar^{-2}\mathcal{F} = \sum_{i,r} N_{i,r}\mathcal{F}_r(a_i) \quad \textbf{where} \quad \mathcal{F}_r(a) = \sum_{k>0} \frac{1}{k}(2\sinh\frac{k\hbar}{2})^{2r-2}e^{-ka}$$

and

$N_{i,r}$ **:# of 'hypermultiplets' with central charge** a_i **and left spin content** I_r**.**

- $I_r = \Big((\tfrac{1}{2}) \oplus 2(0)\Big)^r$
- **Derived by free field calculation (Gopakumar-Vafa).**

◇ **Put the same 5d theory on this background.**
The partition function should have the form

$$Z = \prod_{i,r} Z_r(a_i)^{N_{i,r}}.$$

◇ **Free field calculation shows that, surprisingly,**

$$Z_r(a) = e^{-\mathcal{F}_r(a)}.$$

Hence,

$$Z_{\textbf{on } \Omega \textbf{ bkg.}} = \exp(-\hbar^{-2}\mathcal{F}_{\textbf{on graviphoton bkg.}})$$

i.e., in order to calculate $\mathcal{F}$, one can calculate $Z_{\textbf{on } \Omega}$ instead.

End of Derivation

Application to the SYM

◇ We have to **encode the VEV** of the adjoints.

◇ We can accomplish this by **introducing Wilson lines** around the 5th direction, because

4d adjoint scalar (complex) $=$

5d adjoint scalar (real) $+$ Wilson lines.

◇ Hence, the object to calculate is

$$e^{-\hbar^{-2}\mathcal{F}(a)} = Z_{\textbf{on }\Omega} = \textbf{tr}(-)^F e^{-\beta H} e^{i\beta\hbar J} e^{i\beta a_i J_i}$$

where J_i: generators of global gauge rotation.

Reduction onto the instanton moduli

Supersymmetries commuting $\hbar J + a_i J_i$ remains
$\Longrightarrow$ only BPS configurations, i.e. 4d instantons contribute.
For k-instanton configuration,

$$\text{energy} = k\tau.$$

Thus,

$$Z_{\text{on }\Omega} = \sum_{k\geq 0} e^{-\beta\tau k}\,\mathrm{tr}(-)^F e^{-\beta H_k} e^{i\beta\hbar J} e^{i\beta a_i J_i}$$

Now H_k is the hamiltonian for

susy QM on the k-instanton moduli.

SUSY QM on the instanton moduli

What kind of susy QM is it?

$4+1$ **d**	$\Rightarrow$	$0+1$ **d**
kin. term	$\longrightarrow$	**kin. term**
CS term	$\longrightarrow$	**coupling to external** $U(1)$

withDimension of k instanton moduli = $4Nk$ (bosonic) + $4Nk$ (fermionic)

$\rightsquigarrow 2Nk$ **fermionic oscillators generate the spin bundle.**

A comment

If $\mathcal{N} = 4$, this will calculate the Euler number of the moduli. This is essentially (Vafa-Witten 9408074). Hence,

Instanton counting = $\mathcal{N} = 2$ version of the "Strong Coupling Test".

Fixed point theorem for the Witten index

In order to calculate $Z_{\text{on } \Omega}$, we utilize the

Atiyah-Bott-Lefshetz fixed point theorem.

[Setup]

- M: Spin manifold, $J \to M$: a line bundle on M
- an action of a group element $g = e^a$ on M
- consider a susy QM with

  its Hilbert space $=$ sections of (spin bundle of M) $\otimes J$
- We want to calculate

$$Z = \mathrm{tr}(-)^F e^{-\beta H} e^a.$$

- This is β independent.

◇ **In $\beta \longrightarrow \infty$ limit, this becomes the character of the vacuum under g.**

◇ **In $\beta \longrightarrow 0$ limit, centrifugal force restricts the contribution to that from the neighborhood of the fixed points of g.**

◇ **Evaluation of Gaussian fluctuation reveals**

$$Z = \sum_{p:\text{f.p.}} e^{w} \prod_{\alpha=1}^{d/2} \frac{1}{e^{i\theta_\alpha/2} - e^{-i\theta_\alpha/2}}.$$

where

w: **eigenvalue of a on $J|_p$**

$i\theta_\alpha$: **eigenvalues of a on $TM|_p$.**

For physical proof, see (Goodman-Witten NPB271(1986)21)

An Example

◇ **Take $M = S^2$ and J: charge n monopole configuration**

◇ **Consider $g = e^{i\theta}$: rotating around the z direction**

◇ **The vacua forms spin n representation of $SU(2)$**

$$\rightsquigarrow \qquad Z = e^{in\theta} + e^{i(n-1)\theta} + \cdots + e^{-in\theta}$$

◇ **The fixed point theorem tells, as the fixed points are north and south poles,**

$$Z = \frac{e^{i(n+1/2)\theta}}{e^{i\theta/2} - e^{-i\theta/2}} + \frac{e^{-i(n+1/2)\theta}}{e^{-i\theta/2} - e^{i\theta/2}}$$

They Agree!

Summary of this (rather lengthy) chapter

✓ $e^{-\hbar^2 \mathcal{F}(a)} = Z_{\textbf{on }\Omega}$.

✓ $Z_{\textbf{on }\Omega} = \sum_{k>0}$ **(Witten index of the k-instanton moduli).**

✓ **(Witten index)** $= \sum_{\textbf{f.p.}}$ **(contrib. from each f.p.).**

◇ **Hence, we need only to study**

i) **the structure of instanton moduli $M_{N,k}$,**
ii) **the structure of $U(1)$ bundle J on them,**
iii) **and the group action on TM and J.**

3. Fixed points in the Instanton Moduli

ADHM construction of the Instanton Moduli

Consider the following system of branes:

	0	1	2	3	4	5	6	7	8	9
k **D3's**	—	—	—	—	•	•	•	•	•	•
N **D7's**	—	—	—	—	—	—	—	—	•	•

$\mathcal{N} = 2$ **field theory on D3 is described by**

$$\mathcal{L} = \int d^4\theta(\Phi^\dagger\Phi + B_1^\dagger e^V B_1 + B_2^\dagger e^{-V} B_2)$$
$$+ \int d^2\theta\, \mathrm{tr}(I\Phi J) + \int d^2\theta\, \mathrm{tr}(B_1[\Phi, B_2]) + \int d^2\theta\, \mathrm{tr}\, W_\alpha W^\alpha + c.c.$$

where

$$\begin{array}{rll} \Phi\text{:} & k \times k, & \textbf{position of D3s along 8,9} \\ B_1, B_2\text{:} & k \times k, & \textbf{position of D3s along 4,5,6,7} \\ I, J^\dagger\text{:} & k \times N, & \textbf{from D3-D7 strings} \end{array}$$

The susy vacua describes

Coulomb phase: **D3's and D7's are separated along 8,9**

Higgs phase: **D3's are absorbed on the D7s as instantons in 4,5,6,7 directions**

$$\Downarrow$$

$$U(N)\ k\text{-}\textbf{Instanton moduli} \sim \frac{\left\{ \begin{array}{r} [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0 \\ [B_1, B_2] + IJ = 0. \end{array} \right\}}{U(k)\ \textbf{gauge invariance}}$$

Group Action on the instanton moduli

More invariantly, the ADHM data B_1, B_2, I, J can be considered as:

$$\begin{pmatrix} B_1 & -B_2^\dagger \\ B_2 & B_1^\dagger \end{pmatrix} \in \boldsymbol{V}^* \otimes \boldsymbol{V} \otimes \mathbb{R}^4, \qquad (I, J^\dagger) \in \boldsymbol{V}^* \otimes \boldsymbol{W} \otimes \boldsymbol{S}^+$$

where

- $\boldsymbol{S}^\pm$: **the Weyl spinor of $SO(4)$**
- $\boldsymbol{V}$: **Dirac zero modes in the fund. of $U(N)$; there are $\boldsymbol{k}$ of them**
- $\boldsymbol{W}$: **$U(N)$ fiber at spatial infinity with dimension $\boldsymbol{N}$**

V transforms as fundamental under $U(k)$.

5d Chern-Simons term and the determinant line bundle

◇ **Next we need to identify the $U(1)$ bundle J coming from the 5d CS term, but please recall...**

This is known from the classic work on anomaly.

e.g. (Sumitani) (Alvarez-Gaumé and Witten).

◇ **5d non-abelian Chern-Simons term with unit coefficient determines the determinant line bundle on the moduli space:**

$$J\big|_{A_\mu(x)\in M_{N,k}} = \bigwedge^{\text{highest}} \mathrm{Ker}(\partial_\mu + A^a_\mu(x)T^a_{\textbf{fund.}}) = \det V.$$

Naïve analysis of the fixed points

- **Eigenvalues of B_i are the centers of instantons**
- **They are rotated by J $\rightsquigarrow$ All centers should be at the origin**
- **Extremely singular: needs some regularization**

$$\Downarrow$$

$$\begin{array}{r} U(N) \text{ non-commutative} \\ k\text{-Instanton moduli} \end{array} \sim \frac{\left\{ \begin{array}{r} [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta_r, \\ [B_1, B_2] + IJ = \zeta_c. \end{array} \right\}}{U(k) \text{ gauge invariance}}$$

- **Without loss of generality, we can set $\zeta_c = 0$, $\zeta_r \neq 0$.**

Noncommutative $U(1)$, k-instanton case

◇ **Let us concentrate on the eq.** $[B_1, B_2] + IJ = 0$

◇ **One can show** $J = 0$**.**

◇ $\mathcal{I} = \{P(z,w) \in \mathbb{C}[z,w] \big| P(B_1, B_2) = 0\}$ **establishes 1-1 correspondence**

$$\text{codimension } k \text{ ideal of } \mathbb{C}[z,w] \;\Leftrightarrow\; U(1) \text{ NC } k\text{-instanton}$$

◇ **rotationally inv. ideal** $\mathcal{I}$ **are generated by monomials** $z^a w^b$**.**

$\rightsquigarrow \{(a,b) \big| z^a w^b \neq \mathcal{I}\}$ **makes a Young tableau with** k **boxes**

Example.

z^0w^0	z^0w^1	z^0w^2
z^1w^0	z^1w^1	z^1w^2
z^2w^0		
z^3w^0		

$$\mathbb{C}[z,w]/\mathcal{I} = \mathrm{span}\{1, w, w^2, z, zw, zw^2, z^2, z^3\}$$

$U(N)$ k-instanton case.

- **nonzero adjoint VEV $a_i \neq 0$ $\Leftrightarrow$ D7's are seperated along 8,9**

$$\Downarrow$$

- **each of the D3s is absorbed on one of D7s**

$$\Downarrow$$

- **$U(1)$ k_i-instanton for i-th D7 with $\sum k_i = k$**

$$\Downarrow$$

- **N-tuples of Young tableaux with total k boxes**

Another view

◇ **Take a f.p. p in $M_{N,k}$ and the ADHM data x which projects to p:**

$$\begin{array}{ccccc} \begin{array}{c} \boldsymbol{V^* \otimes V \otimes \mathbb{R}^4} \\ \boldsymbol{\oplus\, V^* \otimes W \otimes S^+} \end{array} & \ni & \boldsymbol{x} & \overset{\text{only upto U(k)}}{=} & \boldsymbol{e^g x} \\ \Big\downarrow /\!/\!/\boldsymbol{U(k)} & & \Big\downarrow & & \Big\downarrow \\ \boldsymbol{M_{N,k}} & \ni & \boldsymbol{p} & = & \boldsymbol{e^g p} \end{array}$$

$$(\boldsymbol{a} = (\hbar, \{\boldsymbol{a_i}\}) \in \boldsymbol{so}(4) \times \boldsymbol{u}(N))$$

i.e. we need $u(k)$ rotation $\phi(g)$ to achieve $x = e^{\phi(g)} e^g x$.

◇ **k eigenvalues of $\phi(g)$ is readily depicted using the Young tableaux:**

Example.

a_1	$a_1 + \hbar$	$a_1 + 2\hbar$
$a_1 - \hbar$	a_1	$a_1 + \hbar$
$a_1 - 2\hbar$		
$a_1 - 3\hbar$		

a_2	$a_2 + \hbar$	$a_2 + 2\hbar$	$a_2 + 3\hbar$
$a_2 - \hbar$	a_2	$a_2 + \hbar$	
$a_2 - 2\hbar$	$a_2 - \hbar$		

◇ **Note $\phi(g)$ can be identified with the adjoint vev ϕ.**

Nekrasov's formula without 5d Chern-Simons

◇ **We have first to determine the eigenvalues of $\hbar J + a_i J_i$ on the tangent space.**

◇ **They can be calculated by some hard work... One gets, for models without 5d Chern-Simons,**

$$e^{-\hbar^{-2}\mathcal{F}(a_i)} = \sum_k e^{-\beta\tau k} \sum_{\substack{(Y_1,\dots,Y_N): \\ N \text{ Young tableaux} \\ \text{with total } k \text{ boxes}}} \prod_{(i,m)\neq(j,n)} \frac{\sinh\frac{\beta}{2}(a_i - a_j + \hbar(y_{i,n} - y_{j,m} + m - n))}{\sinh\frac{\beta}{2}(a_i - a_j + \hbar(m-n))}$$

where $y_{i,n}$ is the length of the i-th row of the N-th Young tableau.

Nekrasov's formula with 5d Chern-Simons

- **Recall the fixed point formula**

$$Z = \sum_{p:\text{f.p.}} e^{w} \prod_{\alpha=1}^{d/2} \frac{1}{e^{i\theta_\alpha/2} - e^{-i\theta_\alpha/2}}.$$

- **Thus, in the presence of 5d Chern-Simons:**

$$e^{-\hbar^{-2}\mathcal{F}(a_i)} = \sum_{k} e^{-\beta\tau k} \sum_{\substack{(Y_1,\ldots,Y_N): \\ N \text{ Young tableauxwith total } k \text{ boxes}}} e^{\clubsuit}$$

$$\prod_{(i,m)\neq(j,n)} \frac{\sinh\frac{\beta}{2}(a_i - a_j + \hbar(y_{i,n} - y_{j,m} + m - n))}{\sinh\frac{\beta}{2}(a_i - a_j + \hbar(m-n))}$$

where $\clubsuit$ = **eigenvalue of** $U(N) \times SO(4)$ **action on** $J|_{\text{f.p.}}$.

◇ **Recall** $J|_{\text{f.p.}} = (\bigwedge^{\text{highest}} \mathrm{Ker} D)^{\otimes \kappa} = \det V^{\otimes \kappa}$**. At a f.p.**

$$p = e^a p \in M \text{ with } a \in u(N) \oplus so(4),$$

we saw

$$x = e^{\phi(a)} e^a x \text{ with } \phi(a) \in u(k)$$

for the corresponding ADHM data. Hence,

$$\clubsuit = \kappa \mathrm{tr} \phi(a) = \kappa \sum_{l=1}^{N} \left(a_i \sum_{(i,j)\in Y_l} 1 + \hbar \sum_{(i,j)\in Y_l} (i-j) \right)$$

◇ **With this factor,** $\mathcal{F}$ **from Instanton Counting precisely agrees with** $\mathcal{F}$ **for** X_N^κ **calculated using topological vertex.**

4. Conclusion

Summary

- ✓ **Prepotential $=$ the (generalized) Witten index of the theory.**
- ✓ **The Witten index can be evaluated by the localization.**
- ✓ **The fixed points are labeled by Young tableaux.**
- ✓ **5d Chern-Simons corresponds to the determinant line bundle in this framework.**
- ✓ **With this, the results agree with the results from the top. vertex.**

Outlook

- ◇ **Can one devise Instanton Counting method for any local toric CY?**
- ◇ **Can one devise Instanton Counting method for compact CYs?**

5. Seiberg-Witten curve from the Young Tableaux

Does the Nekrasov's formula agree with good old result by Seiberg and Witten?

We will see that they indeed agree, by transforming the Nekrasov's formula to classical many body systems.

Let us firstly recall the celebrated

Seiberg-Witten theory

Exact low energy Lagrangian for $d=4, \mathcal{N}=2$ $SU(N)$ SYM...

- ◇ **VEVs of the adjoint scalar breaks $SU(N) \to U(1)^{N-1}$**
- ◇ **$\mathcal{N}=2\,U(1)^{N-1}$ theory is described by a single holomorphic function $\mathcal{F}(a_1,\ldots,a_{N-1})$:**

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{Im}\left(\int d^4\theta \frac{\partial\mathcal{F}}{\partial a_i} a_i^\dagger + \int d^2\theta \frac{1}{2}\frac{\partial\mathcal{F}}{\partial a_i \partial a_j} W_i W_j\right)$$

- ◇ **$\mathcal{F}$ can be determined by exploiting holomorphy and the duality symmetry.**

Take a hyperelliptic

$$y + \frac{1}{y} = \frac{P(x)}{\Lambda^N}$$

with $\deg P(x) = N$ and take

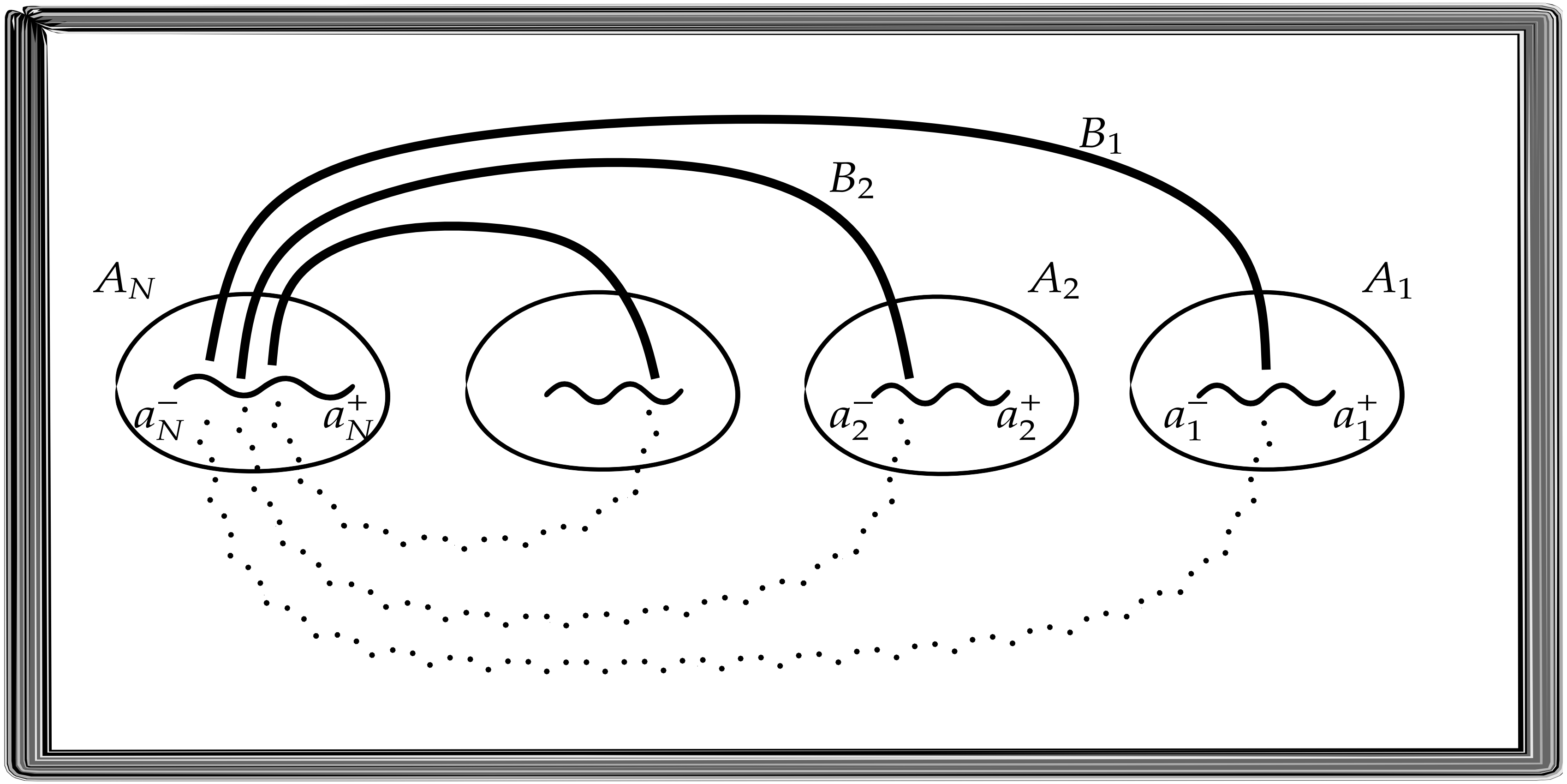
$$a_i = \oint_{A_i} dS, \qquad a_D^i = \oint_{B^i} dS$$

where A_i and B^i form canonical bases of one cycles,

$dS = x dy/y$: the SW differential .

Then, $\mathcal{F}$ can be determined from

$$a_D^i = \frac{\partial \mathcal{F}}{\partial a_i}.$$


B_1
B_2
A_N
A_2
A_1
a_N^-
a_N^+
a_2^-
a_2^+
a_1^-
a_1^+

$\mathcal{F}$ **is defined highly implicitly. After some work, we get**

$$\mathcal{F} = \frac{N}{\pi i}\sum a_k^2 \qquad \text{: Bare term}$$

$$+ \frac{i}{4\pi}\sum_{i<j}(a_i - a_j)^2 \log\frac{(a_i - a_j)^2}{\Lambda^2} \qquad \text{: One-loop term}$$

$$+ \sum_{m=1}^{\infty}\frac{\Lambda^{2Nm}}{2m\pi i}\mathcal{F}^{(m)}(a) \qquad \text{: Instanton corrections}$$

◇ **From the RG eq. ,**

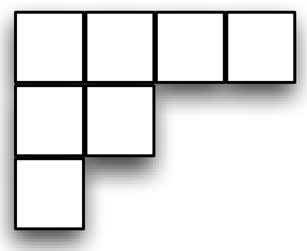
$$\Lambda = \Lambda_0 e^{\frac{\pi}{N} i\left(\frac{\theta}{2\pi} + i\frac{4\pi}{g_0^2}\right)}, \quad \textbf{thus} \quad \Lambda^{2Nk} \sim e^{ik\theta}$$

which is appropriate for k**-instanton contribution.**

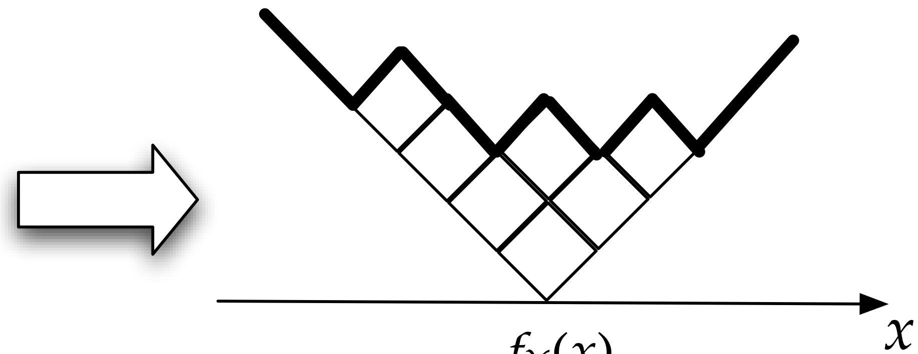
◇ **Can we calculate** $\mathcal{F}^{(m)}$ **by instanton calculus?**

- **Define $f_Y(x)$ for a Young tableau by**

$$f_Y(x) = |x| + \sum_i \Big(|x - \hbar y_i + \hbar(i-1)| - |x - \hbar y_i + \hbar i| + |x + \hbar i| - |x + \hbar(i-1)| \Big).$$



$Y = (4 \geq 2 \geq 1)$



$f_Y(x)$

- **Furthermore, define for $\vec{Y} = (Y_1, \ldots, Y_N)$ the function $f_{\vec{Y}}$ by**

$$f_{\vec{Y}} = \sum_i f_{Y_i}(x - a_i).$$

- **Then, Nekrasov's formula can be cast in the form**

$$Z_{\text{pert}} Z_{\text{on }\Omega} = \sum_{f_{\vec{Y}}} \exp\left(-\frac{1}{4}\int dx dy f''(x) f''(y) \gamma_\hbar(x-y)\right)$$

where

$$\log\frac{x}{\Lambda} = \gamma_\hbar(x+\hbar) + \gamma_\hbar(x-\hbar) - 2\gamma_\hbar(x)$$

$$Z_{\text{pert}} = \exp\left(\sum_{k,l} \gamma_\hbar(a_l - a_n)\right).$$

◇ **In the limit $\hbar \to 0$, it becomes**

$$Z_{\text{pert}} Z_{\text{on } \Omega} = \sum_{f_{\vec{Y}}} \exp\left(-\frac{1}{\hbar^2} E[f_{\vec{Y}}]\right)$$

where

$$E[f_{\vec{Y}}] = \frac{1}{8} \int dx dy f''(x) f''(y) (x-y)^2 \left(\log\left(\frac{x-y}{\Lambda} - \frac{3}{2} \right) \right)$$

◇ **Hence, the configuration f_0 which maximizes $E[f]$ dominates the sum and**

$$\mathcal{F}(a_i) = \lim_{\hbar \to 0} \hbar^2 \log Z = E[f_0].$$

◇ **Note that a_i's and N are encoded in the asymptotic behavior of $f_{\vec{Y}}$**

◇ **It is convenient to Legendre-transform in a_i.**
Using the function

$$\xi(x) = \begin{cases} \xi_l x & x \text{ is near } a_l \\ \text{smoothly interpolating} & \text{otherwise} \end{cases}$$

We get $\sum_i \xi_i a_i = \int \xi(x) f''(x) dx$. **Hence we have to minimize**

$$S[f] = -\frac{1}{8}\int dx dy f''(x) f''(y)(x-y)^2 \left(\log\left(\frac{x-y}{\Lambda}\right) - \frac{3}{2}\right) + \frac{1}{2}\int \xi(x) f''(x) dx.$$

i.e.

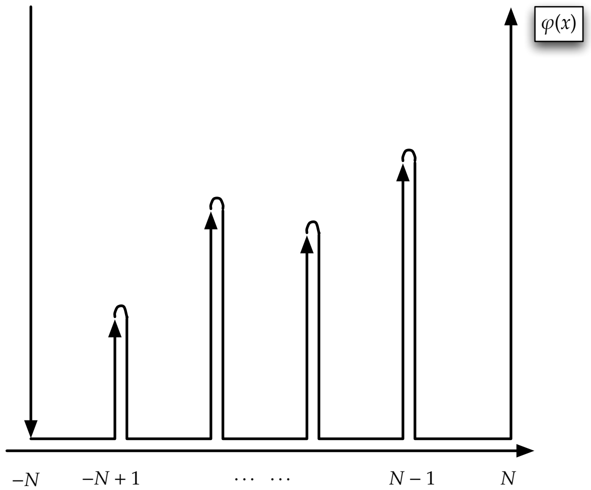
$$\underbrace{\int dy (y-x)\left(\log\left(\frac{|x-y|}{\Lambda}\right) - 1\right) f''(y)}_{\equiv g(x)} = \xi'(x) \qquad \cdots \clubsuit.$$

- **Consider $\phi(x) = f'(x) + g'(x)/(\pi i)$. Differentiating ♣, we get**

$$\phi(x)' = \underbrace{\mathrm{Im}\phi(x)'}_{=f''(x)} + i \int \frac{dy}{x-y} \mathrm{Im}\phi(y)'.$$

$\rightsquigarrow$ $\phi(x)$ **can be holomorphically extended to upper half plane.**

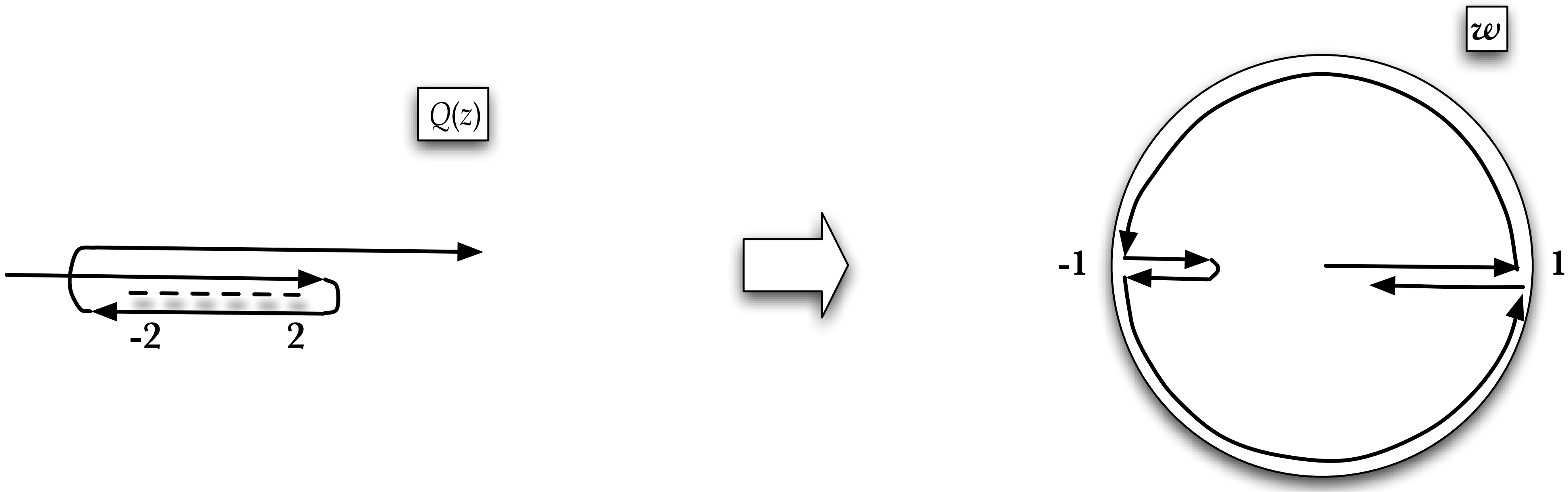
- **Boundary behavior of $\phi(x)$ is fixed by ♣:**



Hence,

$$\phi(z) = \frac{2}{\pi i}\log w \qquad \textbf{where } w + \frac{1}{w} = \frac{Q(z)}{\Lambda^N}$$

for some N.



- a_i **can be recovered from** f**:**

$$a_i = \frac{1}{2}\int_{\textbf{near } a_i} x f(x)'' dx = \frac{1}{2\pi i}\oint_{A_i} \underbrace{z\frac{dw}{w}}_{\text{SW differential !}}$$

- **The SW curve appeared as one solves the maximization problem.**