

# E, F, G of Instantons

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# Question

Can we do microscopic  
instanton calculation in  $d=4$   
 $N=2$  exceptional gauge theory ?



# Seiberg-Witten Theory

- A way to give the Exact Low Energy Lagrangian for various **d=4 N=2 theories**

$$\mathcal{L} = \int d^4\theta \operatorname{Im} a_i^D{}^\dagger a^i + \int d^2\theta \tau_{ij} W_\alpha^i W^{\alpha,j} + c.c.$$

where

$$a_i^D = \partial \mathcal{F} / \partial a^i \quad \text{and} \quad \tau_{ij} = \partial^2 \mathcal{F} / \partial a^i \partial a^j$$

- Utilize  $\mathcal{F}$  is **holomorphic** in  $a^i$
- and **duality** exchanging  $a^i \leftrightarrow a_i^D$



# Instanton Expansion

- The prepotential can be expanded as

Classical + One-loop + k-instanton

$$\mathcal{F} = \frac{1}{2} \tau_{ij}^{\text{bare}} a^i a^j + \sum_q \pm (q_i a^i)^2 \log \frac{(q_i a^i)^2}{\Lambda^2} + \sum_{k \geq 1} \mathcal{F}_k (\Lambda^b)^k$$

where  $q_i$  are charges, and

$b$  is the coeff. of one-loop beta function

$\Lambda^b$  behaves like  $e^{2\pi i \theta}$



# Microscopic Calculation

- $\mathcal{F}_k$  should be given by the integral over the k-instanton moduli

- [Nekrasov] identified

the Integral =

Witten Index of the  
SUSY QM  
on the moduli

- Thus, one can use localization
- $\mathcal{F}_k$  for  $SU(n)$  can be expressed as the summation over 'fixed' instantons, labeled by n-tuples of Young tableaux.



# A, B, C, D of Instantons

- Enumeration of fixed instantons is difficult except  $SU(n)$
- the ADHM matrix model method is extensible to other classical gauge groups

$$A_n = SU(n+1), \quad B_n = SO(2n+1),$$

$$C_n = Sp(2n), \quad D_n = SO(2n)$$

[Nekrasov-Okounkov] [Nekrasov-Shadchin]



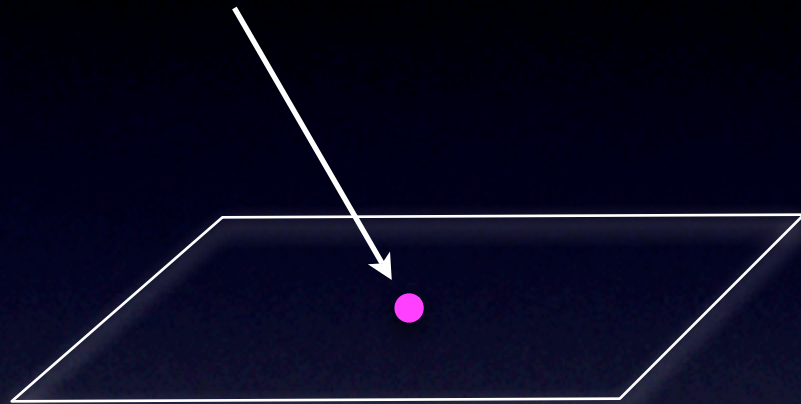
# What about E, F, G ?

- Generic properties is known, but
- No explicit construction of moduli
- One can generalize the approach by [Nakajima-Yoshioka], which only uses generic properties !

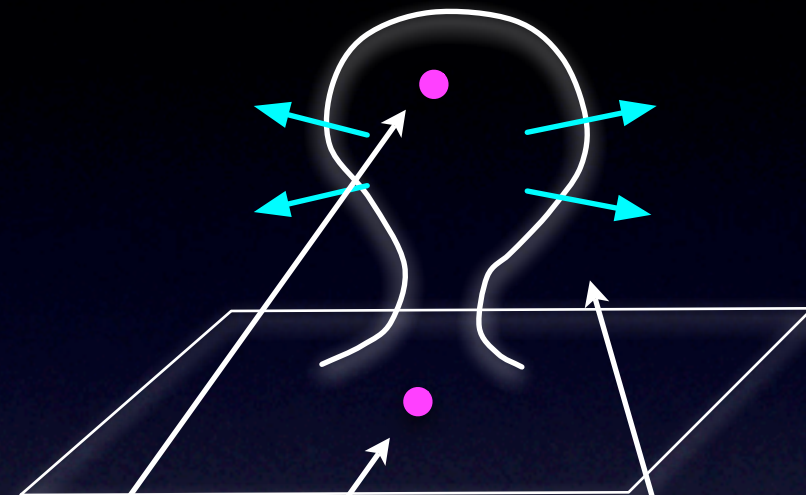


# Blowup !

fixed instanton



$\mathbb{R}^4$



$\mathbb{R}^4 \# \overline{\mathbb{CP}^2}$

fixed instantons has the same  
structure as  $\mathbb{R}^4$

additional flux in the Cartan

[Nakajima-Yoshioka]

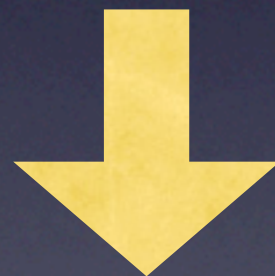


# Hence,


 $\mathbb{R}^4$ 

 $Z_{\mathbb{R}^4}$ 

Integration  
on the moduli

 $\mathbb{R}^4 \# \overline{\mathbb{CP}^2}$ 


$$Z_{\mathbb{R}^4 \# \overline{\mathbb{CP}^2}} \propto Z_{\mathbb{R}^4} Z_{\mathbb{R}^4} \sum_{\text{flux}} (\bullet)$$

[Nakajima-Yoshioka]



# Furthermore,

Standard Fact in Donaldson Theory:

Instanton moduli on

$$\mathbb{R}^4$$

and

Instanton moduli on

$$\mathbb{R}^4 \# \overline{\mathbb{CP}^2}$$

are almost the same.

Therefore,

$$Z_{\mathbb{R}^4} = Z_{\mathbb{R}^4 \# \overline{\mathbb{CP}^2}} \propto Z_{\mathbb{R}^4} Z_{\mathbb{R}^4} \sum_{\text{flux}} (\bullet)$$

[Nakajima-Yoshioka]



# Outcome

$$\frac{\partial^2 \mathcal{F}}{\partial (\log \Lambda)^2} \propto \frac{\partial^2 \mathcal{F}}{\partial a^i \partial \log \Lambda} \frac{\partial^2 \mathcal{F}}{\partial a^j \partial \log \Lambda} \frac{\partial}{\partial \tau_{ij}} \log \Theta_E(0 | \tau_{ij})$$

where

$$\Theta_E(0 | \tau_{ij}) = \sum_{k \in \text{coroot lattice}} \exp(\pi i \tau_{ij} k^i k^j + \pi i \rho_i k^i)$$

- This is the **Contact Term Equation** derived using low-energy TQFT argument by [Losev-Nekrasov-Shatashvili]
- **Recursively** determines the prepotential

[Nakajima-Yoshioka]



# What we did

- Extension to E, F, G
  - Extension to theories with massless hypers
  - Checks against one-instanton result by [Ito-Sasakura]
  - Checks against one-instanton result for  $SU(n)$ ,  $Sp(n)$  with (anti)-symmetric
- n.b. SW curves are not hyperelliptic.



# Application

$n$  D3-branes probing  $O7$

$Sp(n)$  with an antisymmetric



Instanton  
Calculation

$$\mathcal{F}(a_1, \dots, a_n) = \mathcal{F}_{SU(2)}(a_1) + \dots + \mathcal{F}_{SU(2)}(a_n)$$

They move independently !



# Conclusion

- The Contact Term Equation is derived using **microscopic** instanton calculus for **exceptional** gauge groups
- Generalizable to theories **with hypermultiplets**
- It recursively determines the instanton expansion of the prepotential
- It can be and has been **checked** against Seiberg-Witten-type analysis



# Outlook

- Derive the contact term equation for  $E$ ,  $F$  and  $G$  from the Seiberg-Witten curve
- Whitham-type analysis on non-hyperelliptic is necessary
- If you are interested, please tell me !