E, F, G of Instantons

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Question

Can we do microscopic instanton calculation in d=4 N=2 exceptional gauge theory ?

Seiberg-Witten Theory

- A way to give the Exact Low Energy Lagrangian for various d=4 N=2 theories

$$\mathcal{L} = \int d^4\theta \mathrm{Im}\, a_i^{D\dagger} a^i + \int d^2\theta \tau_{ij} W_\alpha^i W^{\alpha,j} + c.c.$$

where

$$a_i^D = \partial\mathcal{F}/\partial a^i \quad \text{and} \quad \tau_{ij} = \partial^2\mathcal{F}/\partial a^i \partial a^j$$

- Utilize $\mathcal{F}$ is holomorphic in a^i
- and duality exchanging $a^i \leftrightarrow a_i^D$

Instanton Expansion

- The prepotential can be expanded as

Classical + One-loop + k-instanton

$$\mathcal{F} = \frac{1}{2}\tau_{ij}^{\text{bare}} a^i a^j + \sum_q \pm (q_i a^i)^2 \log \frac{(q_i a^i)^2}{\Lambda^2} + \sum_{k \geq 1} \mathcal{F}_k (\Lambda^b)^k$$

where q_i are charges, and

b is the coeff. of one-loop beta function

Λ^b behaves like $e^{2\pi i \theta}$

Microscopic Calculation

- $\mathcal{F}_k$ should be given by the integral over the k-instanton moduli
- [Nekrasov] identified

  the Integral = Witten Index of the SUSY QM on the moduli

- Thus, one can use localization
- $\mathcal{F}_k$ for SU(n) can be expressed as the summation over ‘fixed’ instantons, labeled by n-tuples of Young tableaux.

A, B, C, D of Instantons

- Enumeration of fixed instantons is difficult except SU(n)
- the ADHM matrix model method is extensible to other classical gauge groups

$$A_n = SU(n+1),\ B_n = SO(2n+1),$$

$$C_n = Sp(2n),\ D_n = SO(2n)$$

[Nekrasov-Okounkov] [Nekrasov-Shadchin]

What about E, F, G ?

- Generic properties is known, but
- No explicit construction of moduli
- One can generalize the approach by [Nakajima-Yoshioka], which only uses generic properties !

Blowup !

fixed instanton

$\mathbb{R}^4$ $\qquad$ $\mathbb{R}^4 \sharp \overline{\mathbb{CP}^2}$

fixed instantons has the same structure as $\mathbb{R}^4$

additional flux in the Cartan

[Nakajima-Yoshioka]

Hence,

$\mathbb{R}^4$ $\qquad\qquad$ $\mathbb{R}^4 \sharp \overline{\mathbb{CP}^2}$

Integration on the moduli

$Z_{\mathbb{R}^4}$

$$Z_{\mathbb{R}^4 \# \overline{\mathbb{CP}^2}} \propto Z_{\mathbb{R}^4} Z_{\mathbb{R}^4} \sum_{\text{flux}} (\bullet)$$

[Nakajima-Yoshioka]

Furthermore,

Standard Fact in Donaldson Theory:

Instanton moduli on $\mathbb{R}^4$ and Instanton moduli on $\mathbb{R}^4 \sharp \overline{\mathbb{CP}^2}$

are almost the same.

Therefore,

$$Z_{\mathbb{R}^4} = Z_{\mathbb{R}^4 \# \overline{\mathbb{CP}^2}} \propto Z_{\mathbb{R}^4} Z_{\mathbb{R}^4} \sum_{\text{flux}} (\bullet)$$

[Nakajima-Yoshioka]

Outcome

$$\frac{\partial^2 \mathcal{F}}{\partial(\log\Lambda)^2} \propto \frac{\partial^2 \mathcal{F}}{\partial a^i \partial \log\Lambda} \frac{\partial^2 \mathcal{F}}{\partial a^j \partial \log\Lambda} \frac{\partial}{\partial \tau_{ij}} \log \Theta_E(0|\tau_{ij})$$

where

$$\Theta_E(0|\tau_{ij}) = \sum_{k \in \text{coroot lattice}} \exp(\pi i \tau_{ij} k^i k^j + \pi i \rho_i k^i)$$

- This is the Contact Term Equation derived using low-energy TQFT argument by [Losev-Nekrasov-Shatashvili]
- Recursively determines the prepotential

[Nakajima-Yoshioka]

What we did

- Extension to E, F, G
- Extension to theories with massless hypers
- Checks against one-instanton result by [Ito-Sasakura]
- Checks against one-instanton result for SU(n), Sp(n) with (anti)-symmetric

n.b. SW curves are not hyperelliptic.

Application

n D3-branes probing O7

Sp(n) with an antisymmetric

↓ Instanton Calculation

$$\mathcal{F}(a_1, \ldots, a_n) = \mathcal{F}_{SU(2)}(a_1) + \cdots + \mathcal{F}_{SU(2)}(a_n)$$

They move independently !

Conclusion

- The Contact Term Equation is derived using microscopic instanton calculus for exceptional gauge groups
- Generalizable to theories with hypermultiplets
- It recursively determines the instanton expansion of the prepotential
- It can be and has been checked against Seiberg-Witten-type analysis

Outlook

- Derive the contact term equation for E, F and G from the Seiberg-Witten curve
- Whitham-type analysis on non-hyperelliptic is necessary
- If you are interested, please tell me !