Distribution of Vacua

in

Calabi-Yau Compactification

Yuji Tachikawa
(Particle Theory Group, Univ. of Tokyo, Hongo)

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by Tohru Eguchi and YT

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1. String Theory Landscape & Swampland

- Quantization of gravity
  - because it's challenging
  - because it will be needed soon

    $\Leftarrow$ spectral index of primordial fluctuation

- Candidates (generally covariant + quantum mechanical):
  - String (or M) theory
  - Loop Quantum Gravity ... Pure metric theory.

String / M theory

- Not originally meant to quantize gravity
- Worldlines $\Rightarrow$ Worldsheets
- Consistency $\Rightarrow$ 10 D + graviton
- Many higher-dim'l solitons, branes, which support gauge fields

Compactification

- 10D $\Rightarrow$ 4D Minkowski + very small 6D space
- Many consistency conditions.
- Semi-realistic models:
  - Supersymmetrized Standard Models +
  - Hidden sector for dynamically breaking SUSY
  - Axion, etc..

which is a triumph for string theory.

- Presence of Moduli.

Status

- ◇ No experimental tests.
- ◇ Rich as a theoretical model
  - natural setting for various QFT phenomena
    (ADHM, Seiberg-Witten, Montonen-Olive duality etc.)
  - natural setting for various higher-dim'l SUGRA
  - microscopic account of entropy of BPS black holes
  - predicted many nontrivial mathematical results
- ◇ Unified most of the research on QFT & SUGRA practitioners

Moduli Fields

- ◇ Neutral, light field with only Planck-suppressed interaction
- ◇ How light ? $\Rightarrow$ massless or SUSY br. or Hubble
- ◇ Corresponding to the 'moduli' of the compactification manifold
- ◇ moduli (pl.) modulus (singl.) :
  parameter(s) in the pure math jargon.
- ◇ VEV of moduli field determines
  the shape & size of the internal manifold.
- ◇ Shape & size determines the Yukawa/gauge couplings.

Moduli Problem

◇ Massless scalar $\Rightarrow$ 5th force

◇ Susy breaking will make them massive $\sim M_{sb}$,

- Overproduced in preheating
- decay after BBN

etc.

◇ Need to make it much heavier !

Moduli Fixing in String theory

◇ Vexing problems for a long time

$\Leftarrow$ Consistency forbids introduction of potentials by hand

◇ Flux compactification + D-brane Instanton Correction saved the day.

◇ Roughly speaking,

- Flux inside internal mfd. $\Rightarrow$ Tend to spread
- D-brane wrapping inside internal mfd. $\Rightarrow$ Tend to shrink

$\Rightarrow$ Shape & Size fixed.

◇ # of choices of flux are **HUGE** !!

- Holes in Calabi-Yau: $100 \sim 200$
- Flux per hole is integral,
- with upper bound ~ 100

$\Rightarrow$ 100^{100} of choices

◇ Flux given $\Rightarrow$ Moduli fixed

$\Rightarrow$ Shape & size fixed $\Rightarrow$ Yukawa & gauge coupling

◇ Huge # of densely-distributed realizable couplings.

◇ Huge landscape of 4d vacua.

Really?

◇ **Opinion varies:**

- **Yet-to-unknown consistency condition $\Rightarrow$ unique solution ?**

$\Uparrow$

- **Let's analyze models at hand statistically !**

$\Downarrow$

- **Any 4d Lagrangian can be UV-completed with gravity !**

Swampland (Vafa)

Q. Which 4d Lagrangian is OK ?

- we'd like to argue without the long detour into
  10d string, Calabi-Yau, fluxes and all that messy stuffs.
- Anomaly cancellation.
  $\Rightarrow$ Certain gauge groups & matter contents are not allowed.
- Upperbound on the rank of gauge groups
- Gravity should be weaker than gauge coupling
  (Arkani-Hamed-Motl-Nicolis-Vafa, hep-th/0601001)
- Positivity of certain dimension > 4 operators $\Leftarrow$ Causality.
  (Adams-Arkani-Hamed-Dubovsky-Nicolis-Rattazzi , hep-th/0602178)

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2. Flux Compactification

$d = 4, \mathcal{N} = 1$ Supergravity

- $\{Q_\alpha, Q_\beta\} = \gamma^\mu_{\alpha\beta} P_\mu$
  - $(g_{\mu\nu}, \psi_\mu)$
  - $(A^a_\mu, \lambda^a_\alpha)$
  - (ψ^i_α, ϕ^i)
- P_μ gauged $\Rightarrow$ Q_α gauged
- ϕ^i are complex scalars, $G_{i\bar{\jmath}}$ and V restricted in

$$\int d^4x \sqrt{g} \left(G_{i\bar{\jmath}}(\phi, \bar{\phi}) \partial_\mu \phi^i \partial_\mu \bar{\phi}^{\bar{\jmath}} + V(\phi, \bar{\phi}) \right)$$

- $K(\phi,\bar\phi)$: Kähler potential , $W(\phi)$: superpotential $\Rightarrow$

$$G_{i\bar\jmath}(\phi,\bar\phi) = \partial_i\bar\partial_{\bar\jmath}K(\phi,\bar\phi),$$
$$V(\phi,\bar\phi) = e^K\left(G^{i\bar\jmath}D_iW(\phi)\bar D_{\bar\jmath}\bar W(\bar\phi) - 3|W(\phi)|^2\right)$$
$$D_iW(\phi) = (\partial_i + (\partial_i K))W$$

- Kähler transformation:

$$K(\phi,\bar\phi) \to K(\phi,\bar\phi) + f(\phi) + \bar f(\bar\phi)$$
$$W(\phi) \to e^{-f(\phi)}W(\phi)$$
$$D_iW(\phi) \to e^{-f(\phi)}D_iW(\phi)$$

leaves $G_{i\bar\jmath}$ and $V(\phi,\bar\phi)$ invariant.

10d IIB supergravity

$$e^{-\phi}, \quad C, \quad g_{\mu\nu},$$

$$H^{\text{NSNS}}_{[\mu\nu\rho]} = \partial_{[\mu} B^{\text{NSNS}}_{\nu\rho]}, \qquad H^{\text{RR}}_{[\mu\nu\rho]} = \partial_{[\mu} B^{\text{RR}}_{\nu\rho]},$$

$$F_{[\mu\nu\rho\sigma\tau]} = \partial_{[\mu} C_{\nu\rho\sigma\tau]} \text{ with constraint } F_{[\mu\nu\rho\sigma\tau]} = \epsilon_{\mu\nu\rho\sigma\tau\alpha\beta\gamma\delta\epsilon} F^{[\alpha\beta\gamma\delta\epsilon]},$$

+fermions

An important coupling: $\int C_{(4)} \wedge H^{\text{NSNS}}_{(3)} \wedge H^{\text{RR}}_{(3)}$

Branes

◇ point-like objects couple to A_μ via $\int_{\text{worldline}} dx^\mu A_\mu$

◇ objects extended in p-direction couple to $(p+1)$-form fields via

$$\int_{\text{worldvolume}} dx^{\mu_0} \cdots dx^{\mu_p} C_{[\mu_0 \cdots \mu_p]}$$

- C $\Leftarrow$ D(-1) brane = D-instanton
- B^{NSNS} $\Leftarrow$ F1 brane = string
- B^{RR} $\Leftarrow$ D1 brane = D-string
- $C_{(4)}$ $\Leftarrow$ D3 brane

◇ $\int C_{(4)} \wedge H_{(3)}^{\text{NSNS}} \wedge H_{(3)}^{\text{RR}} \Rightarrow H^{\text{NSNS}} \wedge H^{\text{RR}}$ has D3-brane charge

Calabi-Yau compactification

◇ 10=4+6

◇ 6-dimensional CY = the holonomy $SU(3) \subset SO(6)$

$\Rightarrow$ CY : complex mfd $\quad x^1, x^2, x^3, x^4, x^5, x^6 \longrightarrow z^1, z^2, z^3, z^{\bar{1}}, z^{\bar{2}}, z^{\bar{3}},$

with Kähler form ω, everywhere nonzero $(3,0)$ form Ω

◇ 6d spinor $4 = 3 \oplus 1$ under $SU(3)$

$\Rightarrow$ 1/4 of SUSY remain $\quad \Rightarrow$ Type IIB/CY : $\mathcal{N} = 2$

◇ No gauge fields $\Rightarrow$ put D-branes $\quad \Rightarrow$ breaks SUSY to $\mathcal{N} = 1$

Moduli in CY compactification

- CYs come in various topological types:
  - $h_{1,1}$ two-cycles, $h_{1,1}$ four-cycles
  - $2h_{1,2}+2$ three-cycles: $A_0, A_1, \ldots, A_{h_{12}}$ and $B_0, B_1, \ldots, B_{h_{12}}$ so that $A_i \cdot B_j = \delta_{ij}$ and $A_i \cdot A_j = B_i \cdot B_j = 0$
- CYs can be continuously deformed, parametrized by
  - $\rho_i = \int_{C_i} \omega \wedge \omega$: sizes of four-cycles for $i = 1, \ldots, h_{11}$
  - $z_i = \int_{A_i} \Omega$: periods of three-cycles for $i = 1, \ldots, h_{12}$

◇ The metric of CY varies as ρ_i and z_i : $g_{mn}(\rho_i, z_i)$

⇒ 10d metric :

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn}(\rho_i(x^\mu), z_i(x^\mu)) dx^m dx^n$$

◇ Plug this into $S = \int dx^{10} \sqrt{g_{(10)}} R_{(10)}$ ⇒

$$S = \int dx^4 \sqrt{g_{(4)}} R_{(4)} +$$

$$+ \int dx^4 \sqrt{g_{(4)}} g_{(4)}^{\mu\nu} G_{i\bar{j}} \partial_\mu \rho^i \partial_\nu \bar{\rho}^{\bar{j}} + \int dx^4 \sqrt{g_{(4)}} g_{(4)}^{\mu\nu} G'_{i\bar{j}} \partial_\mu z^i \partial_\nu \bar{z}^{\bar{j}}$$

◇ ρ^i combines with $\int_{C_i} C^{(4)}$ to become a complex scalar

$$\rho^i_{\text{complexified}} = i \int_{C_i} \omega \wedge \omega + \int_{C_i} C^{(4)}$$

- $h_{11} + h_{12}$ massless complex scalars in total
  - ρ_i: called size moduli or Kähler moduli
  - z_i: called shape moduli or complex structure moduli
- Axio-dilaton $\tau = ie^{-\phi} + C^{(0)}$ is also a modulus.

Superpotentials for Moduli

◇ Just compactifying on CY leads to $W = 0 \Rightarrow V = 0$.

◇ Masses to all moduli
$\Rightarrow$ We need W depending all variables τ, ρ_i, z_i.

- Fluxes give W for τ and z_i's
- Instanton corrections give W for ρ's

(Kachru-Kallosh-Linde-Trivedi hep-th/0301240)

◇ Let's see each in detail.

Flux superpotential

- Type IIB has 2-form potentials B_{NSNS} and B_{RR} with 3-form field strengths H_{NSNS} and H_{RR}
- Quantized fluxes through three-cycles
- They give rise to

$$
\begin{aligned}
W &= \int_{CY} \Omega \wedge (H_{\text{RR}} + \tau H_{\text{NSNS}}) \\
&= \sum_{i=0}^{h_{12}} \left[\int_{A_i} \Omega \int_{B_i} (H_{\text{RR}} + \tau H_{\text{NSNS}}) - \int_{B_i} \Omega \int_{A_i} (H_{\text{RR}} + \tau H_{\text{NSNS}}) \right] \\
&= \sum_{i=0}^{h_{12}} \left[z_i (N_i^{\text{RR}} + \tau N_i^{\text{NSNS}}) - \frac{\partial F}{\partial z_i} (M_i^{\text{RR}} + \tau M_i^{\text{NSNS}}) \right]
\end{aligned}
$$

Comments

$$W = \sum_{i=0}^{h_{12}} \left[z_i (N_i^{\text{RR}} + \tau N_i^{\text{NSNS}}) - \frac{\partial F}{\partial z_i} (M_i^{\text{RR}} + \tau M_i^{\text{NSNS}}) \right]$$

- This depends on string coupling and shape, not on the size .
- N_i and M_i are the number of fluxes, hence integers
- Linear in Fluxes.

◇ This form for W : obtainable by a standard **KK reduction**;

◇ or, from the domain-wall tension (Gukov):

- Wrap (p,q) 5-brane on A_i :

⇒ a BPS domain wall in 4d point of view.

⇒ The tension should be $\left| W|_{\infty} - W|_{-\infty} \right|$ from 4d SUGRA.

- The tension is $\left| (p+\tau q) \int_{A_i} \Omega \right|$, from the (p,q)-brane action.

- p units of H_{RR} and q units of H_{NSNS} through B_i.

⇒ W !

Constraint on N_i and M_i

- A term $\int C^{(4)} \wedge H_{\mathrm{NSNS}} \wedge H_{\mathrm{RR}}$ in type IIB sugra.
- Of course there is a coupling $\int_{\mathrm{D3}} C^{(4)}$.
- Another coupling $-\int_{\mathrm{O3}} C^{(4)}$ to Orientifold planes.

⇒ EOM for $C^{(4)}$ leads

$$
\begin{aligned}
\#_{O3} &= \#_{D3} + \int H_{\mathrm{RR}} \wedge H_{\mathrm{NSNS}} \\
&= \#_{D3} + \sum_{i=0}^{h_{12}} \left[N_i^{\mathrm{RR}} M_i^{\mathrm{NSNS}} - M_i^{\mathrm{RR}} N_i^{\mathrm{NSNS}} \right]
\end{aligned}
$$

- $\#_{\mathrm{O3}}$ is fixed by the geometry of CY.

Instanton corrections

◇ Superpotentials for the size moduli ρ^i : How?

◇ wrapping N D7-branes on a 4-cycle C_i

⇒ $\mathcal{N} = 1\, U(N)$ gauge theories with coupling constant ρ^i

⇒ Superpotential $\sim e^{-i\rho^i/N}$

associated with gaugino condensation.

◇ D3-brane instantons wrapping C_i.

⇒ Contributes $\propto e^{-i\rho^i}$ to the superpotential

if the # of the fermionic zero-modes is appropriate.

◇ $\exists$ CYs with sufficiently generic instanton corrections (Denef-Douglas).

$\Downarrow$

Closed string moduli are FIXED !

◇ Caveats:

- Their discussion was based on (Witten): in which $H_{\mathrm{RR}} = H_{\mathrm{NSNS}} = 0$.
- No definite treatment yet on D-brane instantons with nonzero H.
- Correction to $K(\rho, \bar{\rho})$ might have bigger effects. (Conlon-Quevedo)

◇ **Further caveats:**

- **Fluxes + Instantons make 4d supersymmetric AdS solutions.**
- **Some other mechanism necessary to make de Sitter vacua.**
- **which is unfortunately less controllable.**

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3. Statistics of Vacua: Theory

- We used **fluxes** H_{RR} and H_{NSNS}.
- In a typical CY, there're **100∼200 3-cycles** to put fluxes;
- LHS of the tadpole constraint

$$\#_{O3} = \#_{D3} + \int H_{\text{RR}} \wedge H_{\text{NSNS}}$$

is of order 1000∼5000.

- SUSY requires $\#_{D3} \geq 0$ and the quadratic form positive definite

$\Rightarrow$ $\sqrt{4000} \sim 100$ **choices** for **each** three-cycle

$$10^{100} \sim 10^{200} \text{ choices of fluxes!}$$

- ◇ Gauge group & matter contents : $\Leftarrow$ topology of the CY
  - Form of the low energy lagrangian.
- ◇ Coupling constants $\Leftarrow$ the moduli $\Leftarrow$ Flux
  - Coefficients of the low energy lagrangian
- ◇ Once you construct the SM (+ susy + inflatons etc.), there'll be plethora of vacua with slightly differing Yukawas!

◇ Need the distribution of Yukawas / Cosmological constants

◇ which are determined by the moduli

⇒ We need the distribution of the moduli !

◇ Fixed moduli depends on the flux ...

⇒ Need the distribution of H_{RR} and H_{NSNS}.

We don't know yet.

◇ Fluxes change when we cross domain walls.

⇒ Flux distribution is tied to the dynamics of domain walls in the extremely early universe before inflation!

◇ So we can't study realistic distribution of flux. Period.

As a zeroth approximation,

- We try a gaussian ensemble of the fluxes H_{RR} and H_{NSNS}:

$$N_i = \int_{A_i} (H_{\text{RR}} + \tau H_{\text{NSNS}}), \qquad M_i = \int_{B_i} (H_{\text{RR}} + \tau H_{\text{NSNS}}).$$

- Under a large fluctuation, we have monodromies acting on (N_i, M_i):

$$\begin{pmatrix} N_i \\ M_i \end{pmatrix} \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} N_i \\ M_i \end{pmatrix}$$

which respects the pairing $(N_i, M_i) \cdot (N_i', M_i') = \sum_i (N_i M_i' - N_i' M_i)$

- Assume the ensemble to be monodromy invariant.

◇ **Distribution of** $W(z) = N_i z_i - M_i \frac{\partial F}{\partial z_i}$ $\Rightarrow$

$$\begin{aligned}
\langle W(z) W(w)^* \rangle &\propto \sum_i \left[z_i \left(\frac{\partial F}{\partial w_i} \right)^* - {w_i}^* \left(\frac{\partial F}{\partial z_i} \right) \right] \\
&= e^{-K(z,w^*)}, \\
\langle W(z) W(w) \rangle &= 0 \\
\langle W(z)^* W(w)^* \rangle &= 0
\end{aligned}$$

◇ $\langle W(z)W(w)^*\rangle \propto e^{-K(z,w^*)}$ is very natural ,

because it transforms covariantly under the Kähler transform:

$$K(z,z^*) \to K + f(z) + f^*(z^*), \qquad W(z) \to e^{-f(z)}W(z)$$

◇ We can study the behavior of $\mathcal{N}=1$

supergravity system with random superpotential

with $\langle W(z)W(w)^*\rangle \propto e^{-K(z,w^*)}$.

◇ Huge literature on systems with random potential (not superpotential) in condensed matter physics. We should utilize them...

Distribution of Vacua

◇ Supersymmetric Vacua are defined by $D_i W = 0$.

⇒ Expected number of vacua at z_i is given by

$$\rho(z,\bar{z}) = \langle \delta(D_i W(z))\delta(\bar{D}_{\bar{\imath}} W(\bar{z})^*) \left| \det \begin{pmatrix} \partial_i D_j W & \partial_i D_{\bar{\jmath}} W^* \\ \partial_{\bar{\imath}} D_j W & \partial_{\bar{\imath}} D_{\bar{\jmath}} W^* \end{pmatrix} \right| \rangle$$

◇ Determinant needed to count each vacua with weight $+1$.

◇ Absolute value makes evaluation harder; instead consider

$$\tilde{\rho}(z,\bar{z}) = \langle \delta(D_i W(z))\delta(\bar{D}_{\bar{\imath}} W(\bar{z})^*) \det \begin{pmatrix} \partial_i D_j W & \partial_i D_{\bar{\jmath}} W^* \\ \partial_{\bar{\imath}} D_j W & \partial_{\bar{\imath}} D_{\bar{\jmath}} W^* \end{pmatrix} \rangle$$

◇ This counts vacua with signs ± 1.

- $\tilde{\rho}$ can be calculated using Wick's theorem.

- The result is,

$$\tilde{\rho}(z)\prod_i dz^i \wedge d\bar{z}^{\bar{\imath}} \propto \det \frac{1}{2\pi}(R^i{}_j + \delta^i{}_j \omega)$$

where

$$R^i{}_j = R^i{}_{ik\bar{l}}\, dz^k \wedge d\bar{z}^{\bar{l}}, \qquad \omega = \frac{i}{2} g_{i\bar{\jmath}}\, dz^i \wedge d\bar{z}^{\bar{\jmath}}$$

is the curvature and the Kähler form of the moduli space .

A mathematical comment

◇ Let M compact n dim'l Kähler and nonsingular,

◇ E a n dim'l vector bundle on M.

⇒ A generic section of E have $\int_M e(E)$ zeros,

when counted with signs, where $e(E)$ is the Euler class.

◇ $e(E) = \det R_E$ via the Chern-Weil homomorphism.

◇ $D_i W$ is a section of $TM \otimes H$ ⇒

$$\int_M \det R_{TM\otimes H} = \int_M \det(R_{TM} + R_H) = \int_M \det(R_{TM} + \omega)$$

◇ In supergravity M is noncompact and singular !

Physical Comments

◇ Suppose there're **no curvature** : $R = 0. \Rightarrow \tilde{\rho} \propto \det \omega$

$\Rightarrow$ the vacua distribute **uniformly following the volume**.

◇ Vacua tends to cluster around where the curvature R is large.

◇ Recall we're discussing the **curvature of the moduli space**.

◇ Curvature of the moduli is large $\Leftrightarrow$ the curvature of the CY is large.

$\Rightarrow$ **Strongly curved** extra dimension is **favored** .

Examples

- To visualize $\tilde{\rho}$,
- We need to calculate $g_{i\bar{\jmath}}$ and ${R^i}_j$:

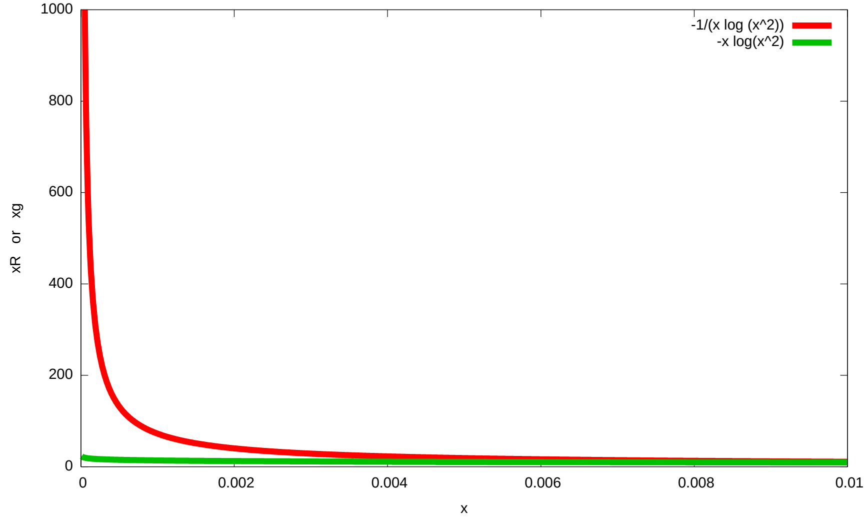
$$g_{i\bar{\jmath}} = \partial_i \bar{\partial}_{\bar{\jmath}} K, \qquad {R^i}_{jk\bar{l}} = \partial_{\bar{l}} g_{j\bar{m}} \partial_k g^{\bar{m}i}$$

⇒ Consult the mirror symmetry literature,
⇒ Plug into the formula for $\tilde{\rho}$,
⇒ Now you have a distribution of vacua !

Near Conifold Singularity (Denef-Douglas, Giryavets-Kachru-Tripathy)

- **where a 3-cycle collapses. Call it A_1.**
- **Let $\phi \equiv X_1 \Rightarrow F_1 \sim \phi \log \phi$:**

$$g_{\phi\phi^*} \sim \log(|\phi|^2), \qquad R_{\phi\phi^*} \sim \frac{1}{|\phi|^2(\log|\phi|)^2} \gg g_{\phi\phi^*}$$



Two param. example (Eguchi-Y.T., unpublished)

◇ Took two-modulus CY: degree 8 hypersurface in $\mathbb{WCP}^4_{1,1,2,2,2}$ with

$$\frac{1}{8}x_1^8 + \frac{1}{8}x_2^8 + \frac{1}{8}x_3^4 + \frac{1}{8}x_4^4 + \frac{1}{8}x_5^4 - \psi_0 x_1 x_2 x_3 x_4 x_5 - \frac{1}{4}\psi_s (x_1 x_2)^4 = 0$$

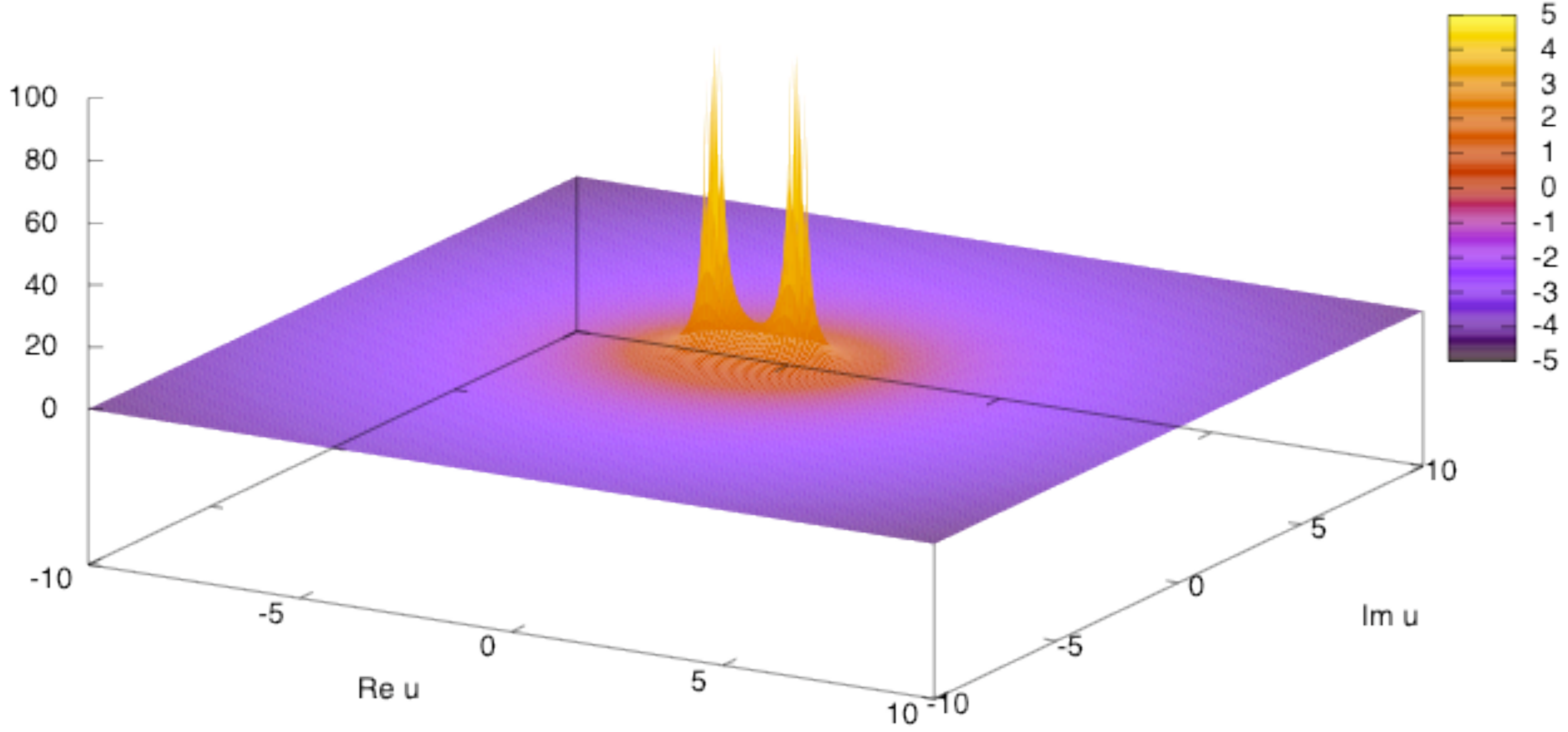
◇ **geometric engineering limit** where the pure $SU(2)$ SYM decouples from supergravity.

◇ Denote $\epsilon = 1/(2\psi_s)$ and $u = \psi + \psi_0^4$. When $\epsilon \to 0$,

$\epsilon^{1/2}$: Dynamical Scale of SYM measured in Planck units;

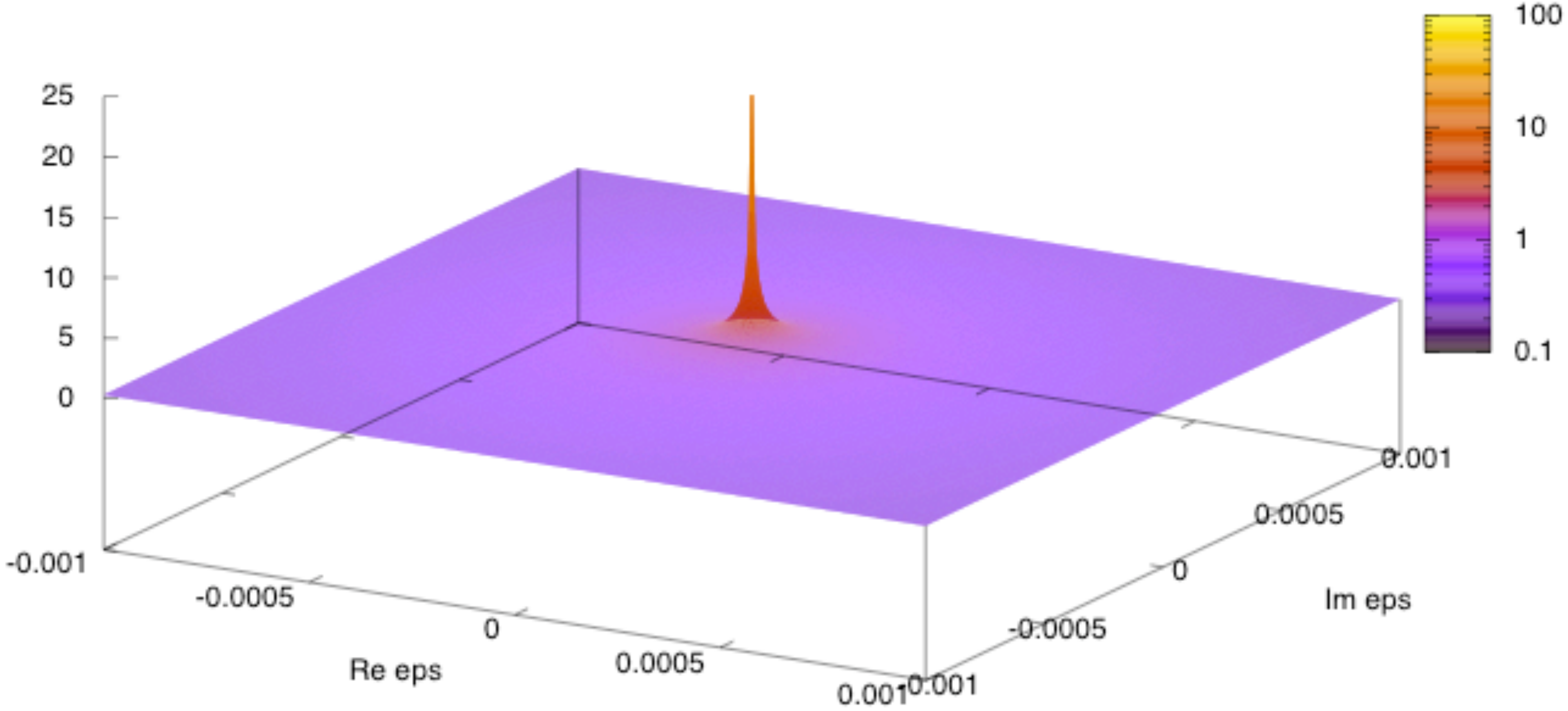
u : Seiberg-Witten's u.

$\epsilon = 0.001$, u : finite



◇ **Just two conifold singularities at $u = \pm 1$.**

$u = 5$, vary ϵ



◇ $\det(R+\omega) \sim \dfrac{1}{|\epsilon|^1(\log|\epsilon|)^3}$ if $1/\epsilon \gg u \gg 1$

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4. Monodromy and Vacuum Density

Singularity in Moduli

- Related to the singularity in CY
- Example: Conifold Singularity

$$x^2 + y^2 + z^2 + w^2 = \epsilon$$

where $x, y, z, w \in \mathbb{C}$

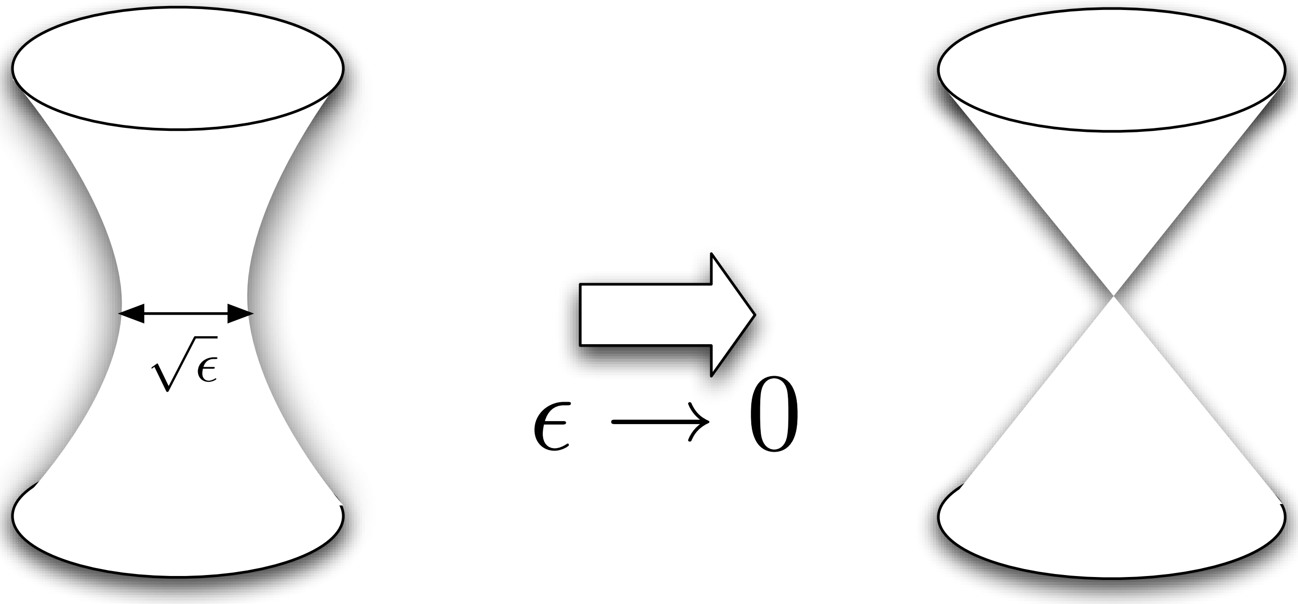
- Easier Example: A_1 Singularity

$$x^2 + y^2 + z^2 = \epsilon$$

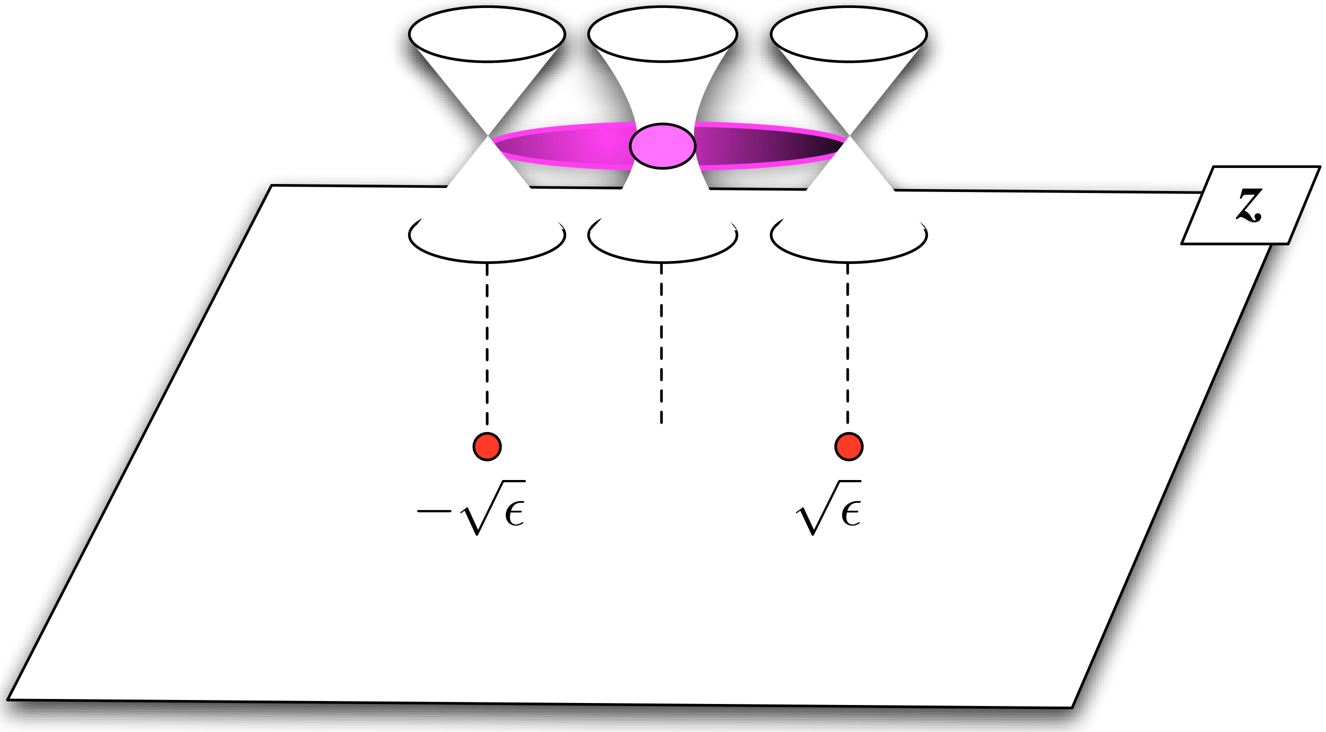
- **Much easier example:**

$$x^2 + y^2 = \epsilon$$

Suppose $\epsilon \in \mathbb{R}_{>0} \Rightarrow \begin{cases} \operatorname{Re} x^2 + \operatorname{Re} y^2 = \epsilon & \Rightarrow \quad \textbf{Circle;} \\ \operatorname{Re} x^2 - \operatorname{Im} y^2 = \epsilon & \Rightarrow \quad \textbf{Hyperbola} \end{cases}$

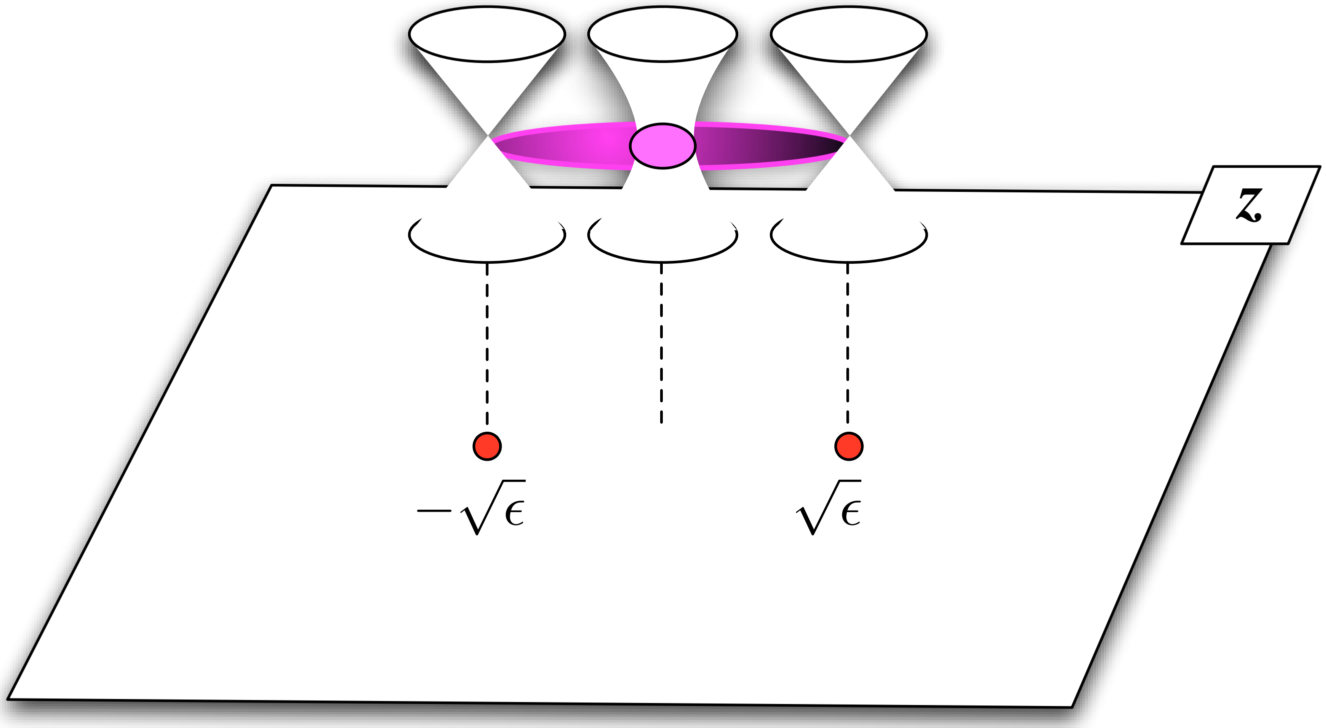


$$x^2 + y^2 + z^2 = \epsilon \longrightarrow x^2 + y^2 = \epsilon - z^2$$



S^2 **of size** $\sqrt{\epsilon}$

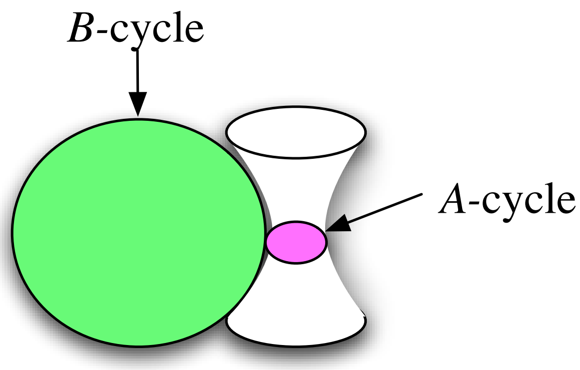
$$x^2 + y^2 + z^2 + w^2 = \epsilon \longrightarrow x^2 + y^2 + w^2 = \epsilon - z^2$$

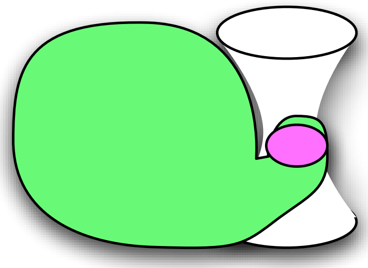


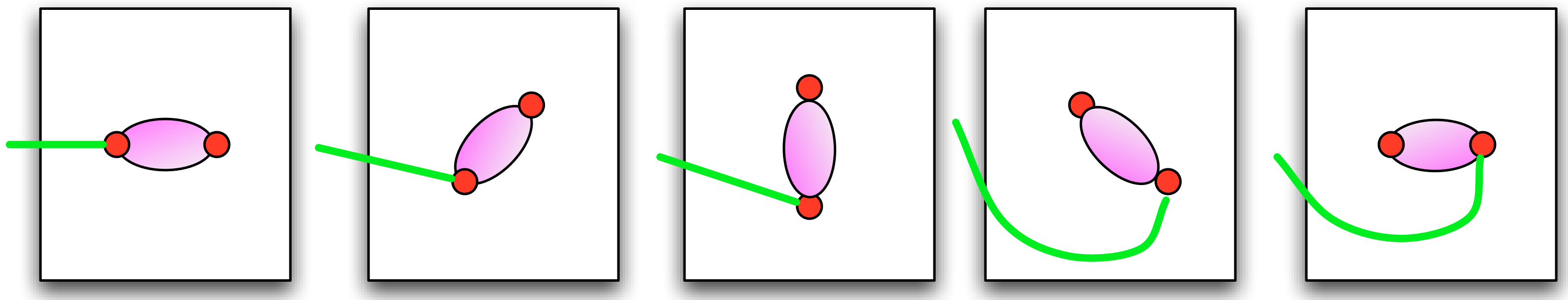
S^3 **of size** $\sqrt{\epsilon}$

Monodromy

$$A \to A$$
$$B \to B + A$$







$\epsilon = 1$ $\epsilon = i$ $\epsilon = -1$ $\epsilon = -i$ $\epsilon = 1$

$$
\begin{aligned}
z &= \int_A \Omega, & A &\to A; \\
F_z &= \int_B \Omega, & B &\to A + B.
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
z &\to z, \\
F_z &\to z + F_z.
\end{aligned}
$$

As $z \sim \epsilon + O(\epsilon^2)$,

$$
\begin{aligned}
z &\sim \epsilon, \\
F_z &\sim \frac{\epsilon}{2\pi i} \log \epsilon.
\end{aligned}
$$

Special Kähler geometry

◇ Existence of special coordinates $X_0, \cdots, X_n$ and the prepotential $F(X)$ so that

$$e^{-K} = \bar{X}_I F_I - \bar{F}_I X_I, \qquad \text{where} \quad F_I = \frac{\partial F}{\partial X_I}.$$

◇ For the complex structure moduli of Calabi-Yau,

$$X_I = \int_{A_I} \Omega, \quad F_i = \int_{B_I} \Omega.$$

where $A_I \cdot A_J = B_I \cdot B_J = 0$, $A_I \cdot B_J = \delta_{IJ}$

◇ Parameters are $z_i = X_i / X_0$, ($i = 1, 2, \ldots, n$).

Vacuum counting in Calabi-Yau moduli

◇ Singularity in CY

⇒ Singularity in the moduli

⇒ **monodromy** in X and F

⇒ the divergent behavior of X and F from holomorphy

⇒ $e^{-K} = \bar{X}_I F_I - \bar{F}_I X_I$

⇒ $g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$

⇒ Curvature.

- For Kähler manifolds with $g_{i\bar{\jmath}} = \partial_i \bar{\partial}_{\bar{\jmath}} K$,

$$R_{i\bar{\jmath}k\bar{l}} = g_{i\bar{m}} \partial_k g^{\bar{m}n} \bar{\partial}_{\bar{l}} g_{n\bar{\jmath}}$$

- For Special Kähler manifolds, Strominger's formula states

$$R_{i\bar{\jmath}k\bar{l}} = -e^{2K} F_{ikm} \bar{F}_{\bar{\jmath}\bar{l}\bar{n}} g^{\bar{n}m} + g_{i\bar{\jmath}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{\jmath}}$$

where

$$F_{ijk} = X_I \partial_i \partial_j \partial_k F_I - F_I \partial_i \partial_j \partial_k X_I$$

Comments

- Special Kähler geometry emerged **independently from** :
  - study of the 2d $\mathcal{N} = (2,2)$ supersymmetric CFT
  - study of 4d $\mathcal{N} = 2$ supergravity
  - study of **singularities** in complex manifolds
- String theory provides the reason of this 'coincidence'.

- Special Kähler gemetry was crucial to
  - Mirror symmetry
  - Seiberg and Witten's solution of $\mathcal{N} = 2$ super Yang-Mills

Conifold Singularity

◇ As ϵ goes round 0, $X_1 \longrightarrow X_1$ and $F_1 \longrightarrow F_1 + X_1 \Rightarrow$

$$X_1 \sim \epsilon \qquad \text{and} \qquad F_1 \sim \frac{\epsilon}{2\pi i} \log \epsilon$$

$$\Rightarrow K = \bar{\epsilon}\epsilon \log|\epsilon| \qquad \Rightarrow g_{\epsilon\bar{\epsilon}} = \partial\bar{\partial}K = \log|\epsilon|$$

$$\Rightarrow R_{\epsilon\bar{\epsilon}} = \partial_\epsilon\, g^{\cdot\bar{\cdot}}\, \bar{\partial}_{\bar{\epsilon}}\, g_{\cdot\bar{\cdot}} = \frac{1}{|\epsilon|^2(\log|\epsilon|)^2} \qquad \Rightarrow \int_{\epsilon\sim 0} \frac{d\epsilon d\bar{\epsilon}}{|\epsilon^2|(\log|\epsilon|)^2} < \infty$$

◇ Density $\det(R+g)$ strongly peaked near $\epsilon \sim 0$,

◇ Integral is finite.

What about other singularities ?

◇ Many other kinds of singularity in Calabi-Yau :

- Geometric Engineering
- Argyres-Douglas

etc.

◇ Is the enhancement always finite ?

- If it's infinite $\Rightarrow$ we might claim the vacuum will be always there.

Our result:

It's always finite for any co-dimension one singularities.

◇ Codimension d singularity

$\Leftarrow$ Need to tune d complex parameters to get to the singularity

Sketch of the derivation

◇ Possible Monodromy : constrained by a mathematical theorem

$\Rightarrow$ X and F $\Rightarrow$ Kähler form $\Rightarrow$ Metric $\Rightarrow$ Curvature

◇ Need upper bounds for each term in curvature

- upper bound for $g_{i\bar{j}}$ $\Leftarrow$ Easy
- upper bound for $g^{\bar{j}i}$ $\Leftarrow$ lower bound for $g_{i\bar{j}}$

$\Leftarrow$ Polarization of the mixed Hodge structure of the singularity

A bit more detail

◇ $(X_i, F_i) \longrightarrow M(X_i, F_i)$ for $\epsilon \longrightarrow e^{2\pi i}\epsilon$

- Eigenvalues of M = roots of unity,
- size of Jordan block ≤ 4

◇ Take k s.t. eigenvalues of M^k = 1, and change the parameter $a = \epsilon^k$.

◇ $N = M^k - 1$ satisfies $N^4 = 0$ $\Rightarrow$

$$\begin{pmatrix} X_i \\ F_i \end{pmatrix} = e^{\frac{N}{2\pi i}\log a}\left(\begin{pmatrix} X_{i(0)} \\ F_{i(0)} \end{pmatrix} + \begin{pmatrix} X_{i(1)} \\ F_{i(1)} \end{pmatrix} a + \begin{pmatrix} X_{i(2)} \\ F_{i(2)} \end{pmatrix} a^2 + \cdots\right)$$

◇ Take p s.t. $N^p(X_{i(0)}, F_{i(0)})^T \neq 0$ but $N^{p+1}(X_{i(0)}, F_{i(0)})^T = 0$.

⇒ $(X_i, F_i) \lesssim (\log a)^p$

◇ many $e^{-K} = \bar{X}_i F_i - \bar{F}_i X_i$ in the denominator in the expansion

⇒ Needs lower bound for $\bar{X}_i F_i - \bar{F}_i X_i$

◇ Leading behavior

$$\bar{X}_i F_i - \bar{F}_i X_i \sim (\bar{X}_{i(0)} N^p F_{i(0)} - \bar{F}_{i(0)} N^p X_{i(0)})(\log a)^p + \cdots$$

◇ A deep mathematical fact ensures $(\bar{X}_i N^p F_i - \bar{F}_i N^p X_i)_{(0)} \neq 0$

⇒ $e^K = (\bar{X}_i F_i - \bar{F}_i X_i)^{-1} \lesssim (\log a)^{-p}$

⇒ $\cdots$ ⇒ Integral converges !

◇ **Explicitly studied two cases:**

- **Argyres-Douglas singularity**

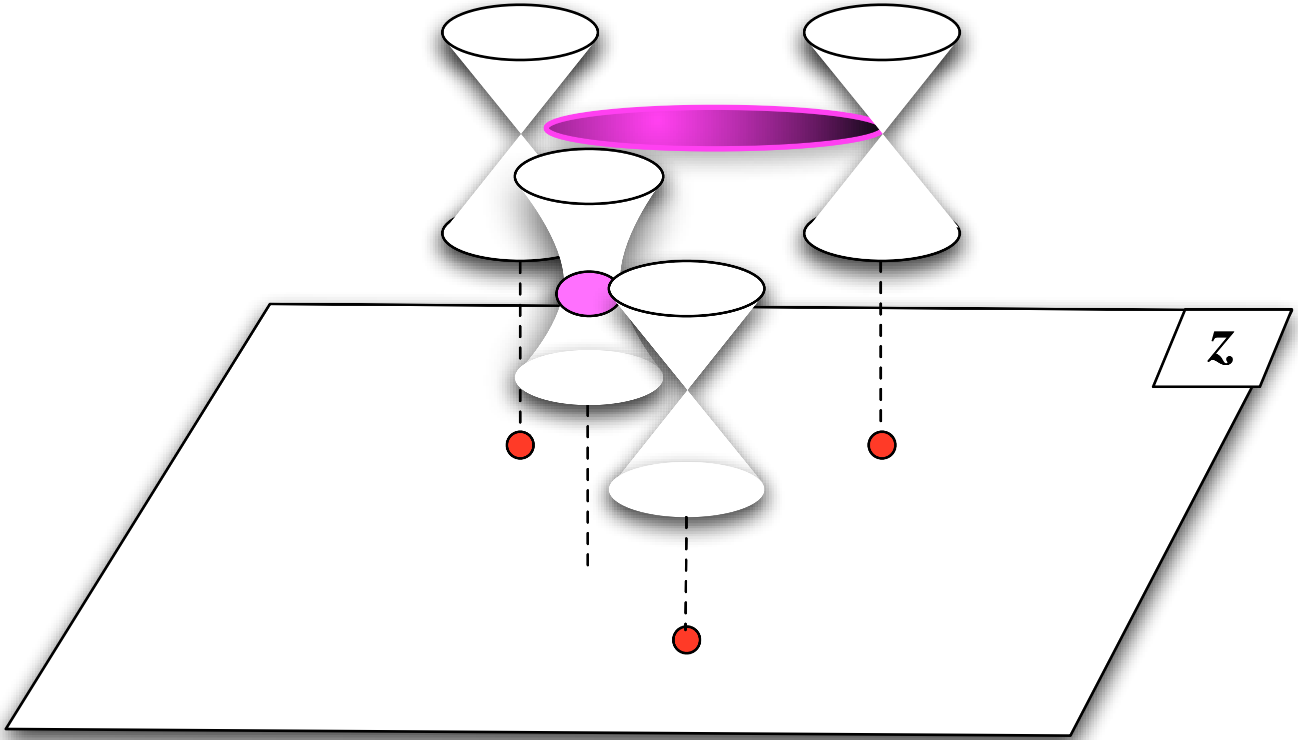
$\Leftarrow$ **Electron and Monopole become simultaneously massless**

- **Geometric-Engineering singularity**

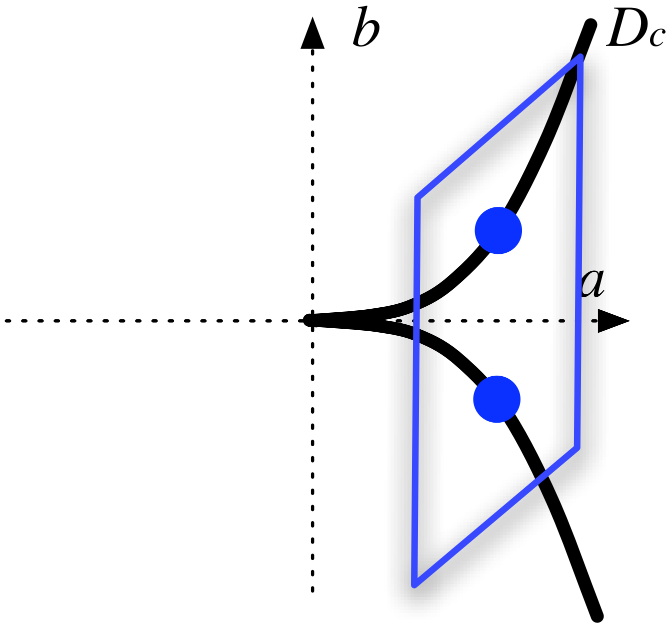
$\Leftarrow$ **Yang-Mills theory decouples from gravity**

Argyres-Douglas singularity

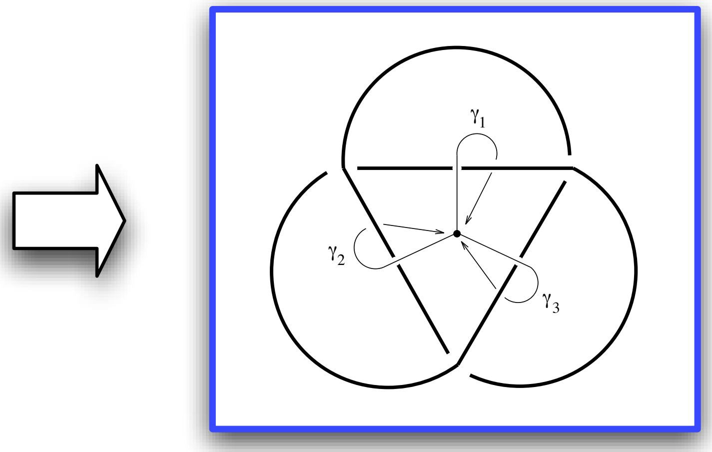
◇ **Local form** $x^2 + y^2 + w^2 = z^3 - 3az - 2b$ **with moduli** a, b



- Roots of $z^3 - 3az - 2b = 0$ determines the singularity
  - Conifold singularity $\Leftarrow$ Double root $a^2 = b^3$
  - Argyres-Douglas singularity $\Leftarrow$ Triple root $a = b = 0$

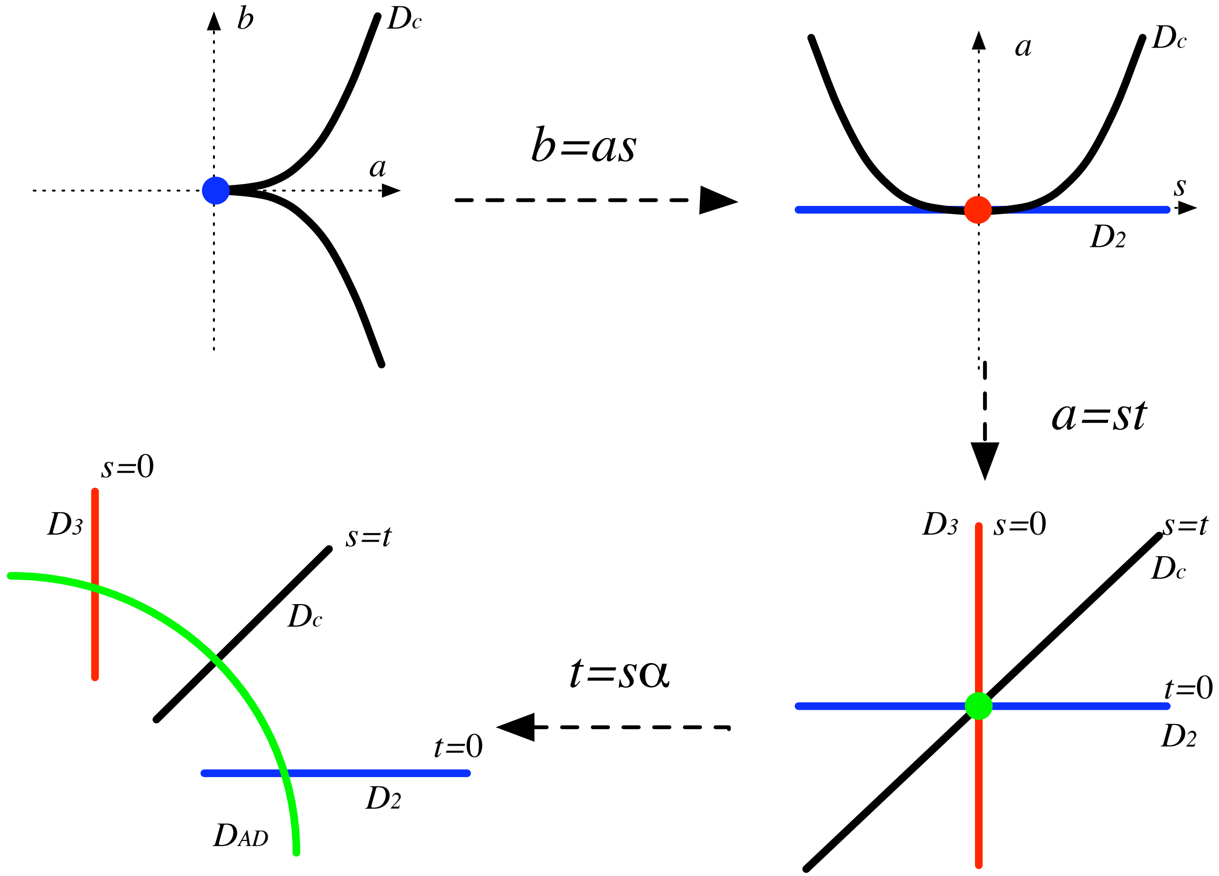


a, b : real



Constant |a|

◇ **What happens near $a \sim b \sim 0$?**



◇ **Nothing in particular !**

CONTENTS

5. Summary & Comments

- ✓ Landscape & Swampland problem in string theory.
- ✓ Moduli fixing.
- ✓ Statistics of Vacua.
- ✓ Conifold Singularities favored, but not infinitely.
- ✓ Extension to other kinds of singularities.