Triangle Anomaly
from
Einstein Manifolds

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based on hep-th/0512nnn
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December, 2005 @ Princeton

1. Introduction & Summary

AdS/CFT correspondence

Consider large # of D3 branes put on the tip of the CY cone Y

$$ds_Y^2 = dr^2 + r^2 ds_X^2$$

- X_5 is a 5d base called Sasaki-Einstein.
- $d = 4, \mathcal{N} = 1$

◇ Field theory on the D3 $\Rightarrow$ Some quiver gauge theory

◇ Near Horizon Limit of the D3 $\Rightarrow$ $\mathrm{AdS}_5 \times X_5$

Both should describe the same physics

Examples

◇ Many explicit metrics for X_5 : S^5, $T^{1,1}$, $Y^{p,q}$ and $L^{p,q,r}$ (Gauntlett et al.)

⇒ Central charge a = (volume)$^{-1}$

◇ Quiver theories known through tiling etc.(Hanany et al.)

⇒ Central charge a from a-maximization (Intriligator-Wecht)

They agree and are, in general, irrational.

Generalization

◇ Toric Sasaki-Einsteins

⇒ Z-minimization (Martelli-Sparks-Yau) gives the volume

◇ Corresponding quivers

⇒ #(chiral superfields) and # (gauge groups) known

⇒ a-maximization

They agree (Butti-Zaffaroni)!

Recall

the correspondence of CS terms with Triangle Anomalies

AdS		CFT
ϕ		$\mathcal{O}$
$Z[\phi(x)\big\vert_{x_5=\infty} = \hat{\phi}(x)]$	$=$	$\langle e^{-\int \hat{\phi}(x)\mathcal{O}(x)d^4x}\rangle$
A^I_μ	$\leftrightarrow$	J^μ_I : current for Q_I
$Z[A^I_\mu(x)\big\vert_{x_5=\infty} = \hat{A}^I_\mu(x)]$		$\langle e^{-\int \hat{A}^I_\mu J^\mu_I}\rangle_{SCFT}$

$$S_{CS} = \int \frac{1}{24\pi^2} c_{IJK} A^I \wedge F^J \wedge F^K$$

$$\Downarrow$$

Gauge dependence

$$\delta_\chi \langle e^{-\int A^I_\mu J^\mu_I} \rangle_{SCFT} = \int_{AdS_5} \frac{1}{24\pi^2} c_{IJK}\, \delta_\chi(A^I) \wedge F^J \wedge F^K$$

$$= \int_{M^4} \frac{1}{24\pi^2} c_{IJK}\, \chi^I F^J \wedge F^K$$

$$\Downarrow$$

$$c_{IJK} = \mathrm{tr}\, Q_I Q_J Q_K .$$

Our goal today

Calculate both sides and check that they match !

- **CS terms from Kaluza-Klein reduction on X**

$$\Rightarrow c_{IJK} = \frac{N^2}{2} \int_X \omega_{\{I} \wedge \iota_{k_J} \omega_{K\}} \qquad \text{(general)}$$

$$\Rightarrow c_{IJK} = \frac{N^2}{2} |\det(k_I, k_J, k_K)| \qquad \text{(toric)}$$

- **Triangle anomaly from the structure of the quiver**

$$\Rightarrow \text{tr}\, Q_I Q_J Q_K = \frac{N^2}{2} |\det(k_I, k_J, k_K)|$$

◇ c_{IJK} independent of $k_{L\neq I,J,K}$?

⇒ Higgsing through **Dibaryon condensation.**

◇ Intricate mixing of angular momenta and baryonic charges

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & HG_3(X) & \longrightarrow & H_3(X) & \longrightarrow & 0 \\ 0 & \longrightarrow & H^3(X) & \longrightarrow & HG^3(X) & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

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2. KK reduction

cf. M on CY_3

- M theory on Calabi-Yau threefold X

$$C = \underbrace{A^I}_{M^5} \wedge \underbrace{\omega_I}_{CY}$$

- $b^2 = \dim H^2(X)$ of gauge fields

- $\int_{M^{11}} C \wedge dC \wedge dC \Rightarrow$

$$c_{IJK} \int_{M^5} A^I \wedge F^J \wedge F^K \qquad \text{where} \qquad c_{IJK} = \int_{CY} \omega_I \wedge \omega_J \wedge \omega_K$$

IIB on X_5

- Type IIB on Einstein mfd X
- $X \curvearrowleft U(1)^{\ell}$ with $b^3 = \dim H^3(X)$
- Expected: ℓ gauge fields from $g_{\mu\nu}$, b^3 from F_5.
- Ansatz ? c_{IJK} ?

Metric

$$ds^2_{\mathrm{AdS}_5} = \eta_{\mu\nu} f^\mu f^\nu$$
$$ds^2_{X^5} = \sum (e^i)^2 \quad \Rightarrow \quad \sum (\hat{e}^i)^2 = \sum (e^i + k^i_a A^a)^2$$

N.B. The Hodge star $*$ exchanges

$$f^1, \ldots, f^5 \quad \Leftrightarrow \quad \hat{e}^1, \ldots, \hat{e}^5$$

Thus

$$*\mathrm{vol}_{AdS_5} = \hat{e}^1 \cdots \hat{e}^5 = \mathrm{vol}_X + A^a \wedge \iota_{k_a} \mathrm{vol}_X + \cdots .$$

Five-form

$$F_5 = F_{0,5} + F_{1,4} + F_{2,3} + F_{3,2} + F_{4,1} + F_{5,0}.$$

where

$$F_{0,5} = \frac{2\pi N}{V}\mathrm{vol}_X, \qquad F_{5,0} = \frac{2\pi N}{V}\mathrm{vol}_{\mathrm{AdS}},$$
$$F_{1,4} = \frac{2\pi N}{V} A^a \wedge \iota_{k_a}\mathrm{vol}_X, \qquad F_{4,1} = 0,$$
$$F_{2,3} = N F^I \wedge \omega_I, \qquad F_{3,2} = N(*F^I) \wedge *\omega_I.$$

which satisfies $F_5 = *F_5$ up to first order.

N.B. Other fields $= 0$; assume all internal w.f. are $U(1)^\ell$ inv.

Impose $dF_5 = 0 \Rightarrow d_{\rm AdS} F_{p,q+1} + d_X F_{p+1,q} = 0$:

$$
\begin{aligned}
0 &= (d_{\rm AdS} * F^I) \wedge *\omega_I, \\
0 &= (d_{\rm AdS} F^I) \wedge \omega_I + (*F^I) \wedge d_X * \omega_I, \\
0 &= 2\pi (d_{\rm AdS} A^a) \wedge \iota_{k_a} {\rm vol}_X + V F^I \wedge d_X \omega_I .
\end{aligned}
$$

$$\Downarrow$$

$dF^I = d * F^I = 0$ iff $\exists c_I^a$ s.t.

$$
\begin{aligned}
d * \omega_I &= 0 \\
-V d\omega_I &= c_I^a \, 2\pi \, \iota_{k^a} {\rm vol}_X \\
dA^a &= c_I^a F^I
\end{aligned}
$$

Recap.

$$d * \omega_I = 0$$
$$d\omega_I + \iota_{k_I}\mathrm{vol}^\circ = 0 \qquad \text{where} \quad k_I = 2\pi c_I^a k_a, \quad \mathrm{vol}^\circ = \mathrm{vol}/V.$$

$$\delta F_5 = N d(A^I \wedge \omega_I) \qquad \text{where} \quad A^a = c_I^a A^I.$$

- ω_I closed and co-closed $\Rightarrow b^3$ of them
- ω_I closed-up-to-isom. and co-closed $\Rightarrow$ $d = b^3 + \ell$ of them

OK, how about c_{IJK}?

One contribution

$F_5 = *F_5$ forces at 2nd order

$$F_{2,3} = F^I \wedge \omega_I$$
$$F_{3,2} = (*F^I) \wedge (*\omega_I) + A^a \wedge F^I \wedge \iota_{k_a}\omega_I$$

$$\Downarrow$$

$d_{\mathrm{AdS}} F_{4,1} + d_X F_{3,2} = 0$ gives

$$(d * F_I) \wedge (*\omega_I) = F^a \wedge F^I \wedge (\iota_{k_a}\omega_I)$$

Result

$$d * F^I \int_X (\omega_K \wedge *\omega_I + \frac{1}{16V^2} k_K k_I \mathrm{vol}) = \frac{1}{3\pi} F^I \wedge F^J \int_X \omega_{\{I} \wedge \iota_{k_J} \omega_{K\}}$$

$$\Downarrow$$

$$\tau_{IJ} = \frac{N^2}{2\pi} \int_X (\omega_J \wedge *\omega_I + \frac{1}{16V^2} k_J k_I \mathrm{vol})$$

$$c_{IJK} = \frac{N^2}{2} \int_X \omega_{\{I} \wedge \iota_{k_J} \omega_{K\}}$$

where $\quad d\omega_I + \iota_{k_I} \mathrm{vol}^\circ = 0$

cf. τ_{IJ} obtained in (Barnes-Gorbatov-Intriligator-Wright)

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3. Giant Gravitons and Normalization of Charges

A^I from:

- $g_{\mu\nu} \Rightarrow$ KK Angular Momenta : ℓ of them
- $F_5 \Rightarrow$ Baryonic Charges : b^3 of them

Total: $d = \ell + b^3$ of gauge fields

What's the integral basis ?

◇ $\delta F_5 = N d(A^I \wedge \omega_I) \Rightarrow$ D3-brane wrapping on C has charge $N \int_C \omega_I$

◇ ω_I is not closed ! $d\omega_I + \iota_{k_I} \mathrm{vol}^\circ = 0$

$\Rightarrow$ Depends more than just its homology class.

◇ Consider branes 'at rest' which is charge eigenstate

$\Rightarrow$ C : invariant under $U(1)^\ell$

◇ E.g. $S^5 \curvearrowleft SO(6) \supset U(1)^3$

$\Rightarrow$ Brane wrapping the S^3 at equator : Maximal Giant Gravitons !

◇ **Define $C \sim C'$ if $C - C' = \partial D$, D: invariant**

$$\Rightarrow \quad \left(\int_C - \int_{C'}\right)\omega_I = \int_{\partial D} \omega_I = \int_D \iota_{k_I} \mathrm{vol} = 0$$

◇ **Assume there are $d = \ell + b^3$ such three-cycles $\Rightarrow$ Normalize**

$$\int_{C^I} \omega_J = \delta^I_J .$$

N.B. As we will see, assumption has been checked for toric SEs.

Robustness of $c_{IJK} \propto \int \omega_{\{I} \wedge \iota_{k_J} \omega_{K\}}$.

- Shift $\omega_I \to \omega_I + d\alpha_I$

$$\Rightarrow \quad \delta c_{IJK} \propto \int d\alpha_{\{I} \wedge \iota_{k_J} \omega_{K\}} = \int \alpha_{\{I} \wedge \iota_{k_J} \iota_{k_K\}} \mathrm{vol}^\circ = 0$$

- Shift $\mathrm{vol}^\circ \to \mathrm{vol}^\circ + d\alpha$. $\quad d\omega_I + \iota_{k_I} \mathrm{vol}^\circ = 0 \Rightarrow \omega_I \to \omega_I + \iota_{k_I} \alpha$

$$\Rightarrow \quad \delta c_{IJK} \propto \int \iota_{k_{\{I}} \alpha \wedge \iota_{k_J} \omega_{K\}} = - \int \alpha \wedge \iota_{\{I} \iota_{k_J} \iota_{k_K\}} \mathrm{vol}^\circ = 0$$

Also

$$\delta \int_{C^I} \omega_J = \int_{C^I} \iota_{k_J} \alpha = 0.$$

Method to obtain c_{IJK}

Step1. Find any five-form vol° s.t. $\int \mathrm{vol}^\circ = 1$.

Step2. Find invariant three-cycles C^I.

Step3. Find the dual ω_I by $\int_{C^I} \omega_J = \delta^I_J$ so that $d\omega_I + \iota_{k_I}\mathrm{vol}^\circ = 0$.

Step4. Plug in to $c_{IJK} = \frac{N^2}{2} \int \omega_{\{I} \wedge \iota_{k_J} \omega_{K\}}$.

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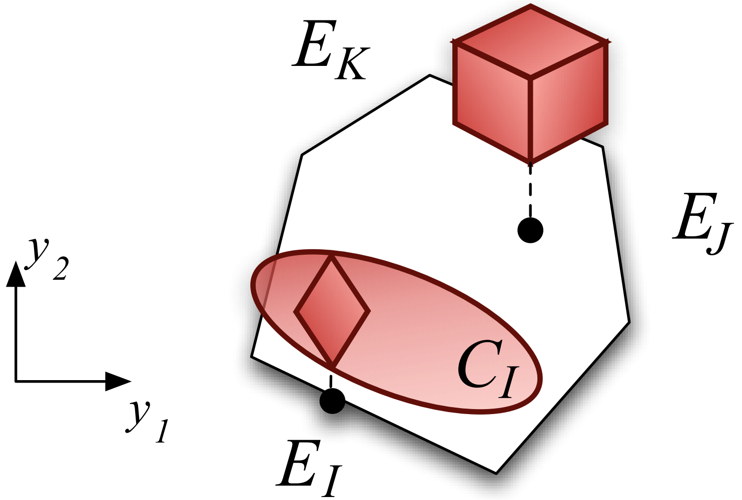
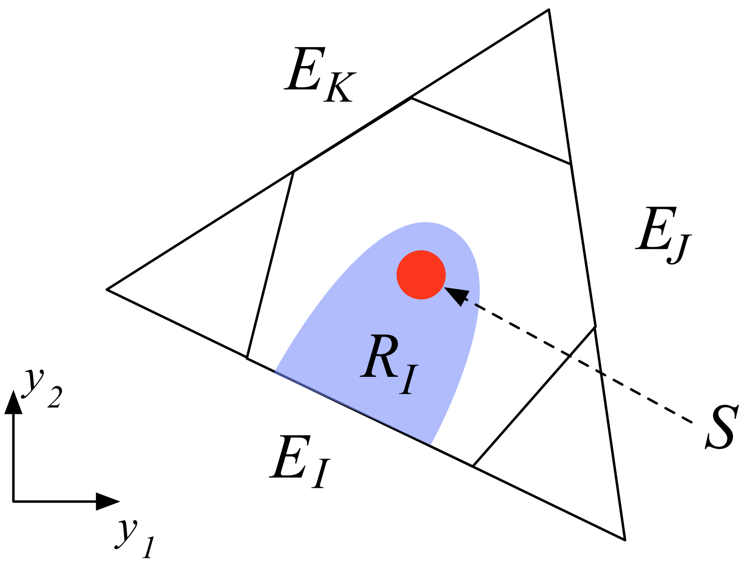
4. Evaluation for toric SEs

- The cone over a toric SE $X \curvearrowleft U(1)^3$ is a toric Calabi-Yau
- $X \xrightarrow{\text{mom. map}}$ the interior of d-gon B with T^3 fiber:

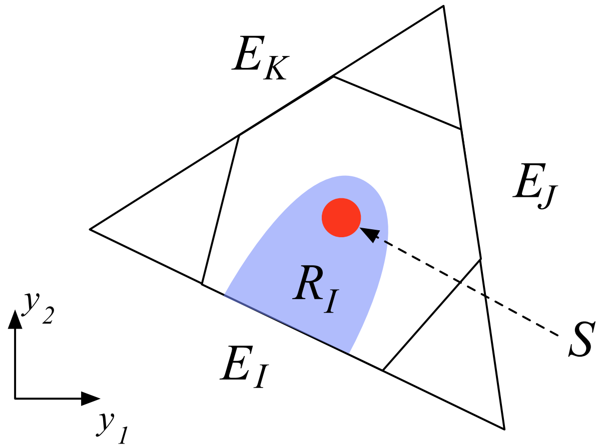


Coordinate along T^3 : $\theta^{1,2,3}$ with $\theta^i \sim \theta^i + 1$

- $k_I = k_{I^i}\partial_{\theta^i}$ degenerates on E_I.



- Take $\mathcal{F}$ supported on S s.t. $\int_B \mathcal{F} = 1$
- Take $\mathcal{A}_I$ supported on R_I s.t. $d\mathcal{A}_I = \mathcal{F}$



$$\mathrm{vol}^\circ = \mathcal{F} \wedge d\theta^1 d\theta^2 d\theta^3$$
$$\omega_I = -\mathcal{A}_I \wedge \iota_{k_I} d\theta^1 d\theta^2 d\theta^3$$

$$d\omega_I + \iota_{k_I}\mathrm{vol}^\circ = 0$$

◇ ω_I independent of other edges $\Rightarrow c_{IJK}$ independent of $k_{L\neq I,J,K}$

◇ X: toric SE for the triangle $k_{I,J,K}$

◇ Y: Periodicity changed $\theta^i \sim \theta^i + k^i_{I,J,K}$ = S^5

⇒ $X = Y/\Gamma$ where $\Gamma = \mathbb{Z}^3/(\mathbb{Z}k_I + \mathbb{Z}k_J + \mathbb{Z}k_K)$ with

$$\#\Gamma = |\det(k_I, k_J, k_K)| \,.$$

⇒ c_{IJK} is $\#\Gamma$ times that of S^5, which is $N^2/2$.

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5. Matching to Quivers

Toric Data $\Longleftrightarrow$ Quivers

- ◇ Tiling $\Rightarrow$ Toric Data: Forward Algorithm
- ◇ Toric Data $\Rightarrow$ Tiling: Inverse Algorithm, Rhombi, ...

- ◇ Many works

Known Properties

Given the toric data sorted counterclockwise

$$k_I = (1, \vec{k}_I), \; (I = 1, \ldots, d)$$

As (p,q)-web: $v_i = \vec{k}_I - \vec{k}_{I-1}$

- ◇ Gauge groups : all $SU(N)$
- ◇ # (Gauge groups) = Twice the area of the toric diagram

- ◇ d global nonanomalous $U(1)$ symmetries Q_I. Consider $Q = a^I Q_I$.

- ◇ Chiral superfields classified into $\mathcal{B}_{ij}$ s.t.
  - Net # $\mathcal{B}_{ij}$ = $\det(\vec{v}_i, \vec{v}_j)$
  - The charge under Q is $a_I + a_{I+1} + \cdots + a_{J-1}$

- ◇ Charge of the superpotential = $\sum a_I$

$$\Downarrow$$

$$c_{IJK}a^I a^J a^K = \overbrace{n_V(\frac{1}{2}\sum a^I)^3}^{\text{gaugini}} + \underbrace{\sum_{I<J} n_{IJ}\left(\sum_{K=I}^{J-1} a^K - \frac{1}{2}\sum a^I\right)^3}_{\text{fermions in chiral multiplet}}$$

where

$$n_V = \sum \det(\vec{k}_I - \vec{k}_1, \vec{k}_{I+1} - \vec{k}_1),$$
$$n_{IJ} = \det(\vec{k}_J - \vec{k}_{J-1}, \vec{k}_I - \vec{k}_{I-1})$$

(Butti-Zaffaroni)

How can we simplify?

◇ Denote the quantity for toric data $\{k_1, \ldots, k_{N-1}\}$ by adding $\tilde{}$.

$$\Rightarrow \quad n_{I,N-1} + n_{I,N} = \tilde{n}_{I,N-1} \quad \text{and} \quad n_V - n_{N-1,N} = \tilde{n}_V.$$

That is,

$$\mathcal{B}_{I,N-1} \cup \mathcal{B}_{I,N} \to \tilde{\mathcal{B}}_{I,N-1} \qquad \text{and} \qquad \mathcal{G} - \mathcal{B}_{N-1,N} \to \tilde{\mathcal{G}}.$$

$$\Rightarrow \quad c_{IJK} a^I a^J a^K \big|_{a^N=0} = \tilde{c}_{IJK} a^I a^J a^K.$$

◇ c_{IJK} is independent of $k_{L \neq I,J,K}$.

◇ One can check $c_{IJK} = \frac{N^2}{2} |\det(k_I, k_J, k_K)|$.

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6. Comments and Outlook

- We obtained the formula

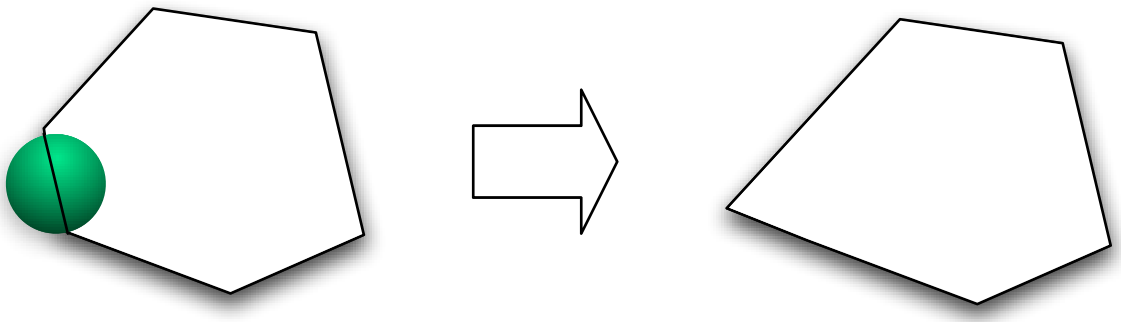
$$c_{IJK} = \frac{N^2}{2} \int \omega_{\{I} \wedge \iota_{k_J} \omega_{K\}}$$

- On both sides of the duality for toric SEs, we obtained

$$c_{IJK} = \frac{N^2}{2} |\det(k_I, k_J, k_K)|$$

- So far, so good.

Why is it independent of $k_{L \neq I,J,K}$?



- Consider D3-branes wrapping on L-th edge
- Dibaryons charged only w.r.t. L-th global symmetry
- **Condense** !

$\Rightarrow$ L-th global symmetry **broken** ; L-th edge **shrunk**

- c_{IJK} can be evaluated **before** or **after** the Higgsing

$\Rightarrow$ Independent of k_L !

Outlook

- ◇ Study more about dibaryon condensation & topology change
  - Reminiscent of blackhole condensation & topology change in CY
- ◇ Study more about the charge lattice. $K_G(X)$?
- ◇ Study the SUSY on 5d, extend the reduction to scalars.

Much to be done !

7. On charge lattice

String theory : assigns to X a lattice of rank $\ell + b^3$.

- What is it mathematically ? $\Rightarrow \{\omega_I\}$ at the level of 'de Rham'
- Integral structure: not the direct sum of KK + D3

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \stackrel{\iota}{\longrightarrow} & HG_3(X) & \longrightarrow & H_3(X) & \longrightarrow & 0 \\
0 & \longrightarrow & H^3(X) & \longrightarrow & HG^3(X) & \stackrel{p}{\longrightarrow} & N & \longrightarrow & 0
\end{array}
$$

N : the space of Killing vectors. $N_{\mathbb{Z}}$: with periodicity 2π

M : dual of N. $M_{\mathbb{Z}}$: irreps of $U(1)^\ell$

- p: defined by $d\omega_I + \iota_{k_I}\mathrm{vol}^\circ = 0$
- ι: See blackboard ...

Your help appreciated.