Seminar at YITP

Quantum toroidal algebra

Relation with integrability and application to AGT conjecture

Yutaka Matsuo

The University of Tokyo

April 5, 2019

based on works arXiv:1110.5255 (Zhang), 1306.1523 (Kanno, Zhang), 1405.3141 (Rim, Zhang), 1504.04150 (Zhu), 1512.02492 (Bourgine, Zhang), 1606.08020 (Bourine, Fukuda, Zhang, Zhu), 1810.08512 (Harada)

Introduction

- Virasoro algebra: 2D conformal symmetry, gauge symmetry of string theory, critical phenomena in 2D statistical models:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

It was very important subject during 80s and still remains actively studied.

- $W_{1+\infty}$ and quantum toroidal algebra: Universal symmetry that contains Virasoro and many other chiral algebras and their q-deformed versions. They are essential to understand subjects related to higher dimensional physics, such as Alday-Gaiotto-Tachikawa conjecture for $N=2$ $D=4$ SYM, (quantum) Seiberg-Witten curve, topological strings and so on.

Classical limit

- Virasoro: 2D conformal algebra $\delta z = -z^{n+1}$. Infinitesimal transformation is expressed as the first order derivative $\delta_n = z^{n+1}\partial_z$.
- $W_{1+\infty}$: defined by higher order derivatives

$$\delta_{n,m} = z^n D^m, \quad D = z\partial_z, \ (n \in \mathbb{Z}, m \geq 0)$$

  It is related to area preserving diffeomorphism in two dimensions.
- Toroidal algebra:

$$\delta_{n,m} = U^n V^m, \quad n, m \in \mathbb{Z}, \quad U = q^D, V = z, D = z\partial_z$$

  U, V satisfy quantum torus algebra $UV = qVU$. There is $SL(2, \mathbb{Z})$ automorphism $U' = U^a V^b, V' = U^c V^d$ with $ad - bc = 1$.

Quantum deformation

- Quantum deformation of $W_{1+\infty}$ and toroidal algebra was developed, not in the context of string theory, but in the algebraic studies of integrable models such as Calogero-Sutherland and its q-deformed version (Ruijsenaar-Schneider) with various motivations by mathematicians.
- Because of such historical reason, the quantum symmetries has many different names.
- $W_{1+\infty}$: SH^c (from shuffle algebra), Affine Yangian (Yangian symmetry associated with affine algebra $\widehat{gl}(1)$)
- Quantum toroidal algebra: also referred to as "Ding-Iohara-Miki (DIM)"
- Further deformation of quantum toroidal algebra is called "elliptic DIM" and so on.
- Since $W_{1+\infty}$ and quantum toroidal algebra has many names, I will sometimes use $\mathcal{U}$ to refer these universal symmetries in general.

Deformation parameters and reductions

- Virasoro algebra has a deformation parameter c
- $W_{1+\infty}$ has one parameter β. Sometimes it is convenient to express it by $\epsilon_1, \epsilon_2, \epsilon_3$ with $\sum_{i=1}^{3} \epsilon_i = 0$ with $\beta = -\epsilon_1/\epsilon_2$. By using scale degree of freedom, it may contain additional reduction parameter n. When it is positive integer, the algebra reduces to W_n-algebra$+U(1)$ symmetry with central charge,

$$c = (n-1)(1 - Q^2 n(n+1)) + 1, \quad Q = \sqrt{\beta} - 1/\sqrt{\beta}$$

- DIM has additional parameter t. One may sometimes use $q_1 = q$, $q_2 = t^{-1}$, $q_3 = (q_1 q_2)^{-1}$. $W_{1+\infty}$ is obtained from DIM by taking a limit $q = e^{\hbar\beta}, t = e^{\hbar}$ in $\hbar \to 0$ limit. It may have n reduction also. In that case, we obtain quantum W_n algebra with U(1) factor.
- Recently, more general reduction (p, q) reduction was found (corner VOA).

Plan of the talk

- Proof of Alday-Gaiotto-Tachikawa conjecture by $\mathcal{U}$
  - Limitation of conventional CFT approach
  - Relation with integrable models
  - Proof of AGT conjecture by $\mathcal{U}$
- Derivation of second quantized SW curve (qq-character)
- New reduction of $\mathcal{U}$: corner VOA and Web of W-algebra

AGT conjecture

Original form of AGT conjecture (2009)

- Instanton partition function of $D = 4$, $N = 2$, $N_f = 4$ super Yang-Mills with $SU(2)$ gauge group = four point function of Virasoro conformal block with $U(1)$ factor.

$$\sum_{\vec{\lambda}} q^{|\vec{\lambda}|} Z_{\mathrm{vec}}(\vec{\lambda}, \vec{a}) Z_{\mathrm{fd}}(\vec{\lambda}, \vec{a}, \vec{m}) = \langle V_{\alpha_1}(0) V_{\alpha_2}(q) V_{\alpha_e}(1) V_{\alpha_4}(\infty) \rangle$$

- Instanton partition function of $D = 4$, $N = 2^*$ super Yang-Mills $SU(2)$ gauge group= one point torus conformal block function

- $\vec{\lambda} = (\lambda_1, \lambda_2)$ is a pair of Young diagrams which label the localization fixed points in the moduli space of $SU(2)$ gauge theory.
  $|\vec{\lambda}| = |\lambda_1| + |\lambda_2|$ gives the number of boxes of two Young diagrams which is identical to the number of instantons.
- The contribution of instanton partition function is computed (regularized) by omega background with the deformation parameters ϵ_1, ϵ_2.
- $\vec{a} \in \mathbb{C}^{\otimes 2}$: VEV of scalar component of vector multiplet, $\vec{m} \in \mathbb{C}^{\otimes 4}$: mass of four hyper multiplets.
- CFT side is described $c = 1 + 6(\sqrt{b} + \sqrt{1/b})^2$ Virasoro algebra with $b = \epsilon_1/\epsilon_2$.
- $q \in \mathbb{C}$ is the instanton expansion parameter $q = e^{2\pi i \tau}$, $\tau = \theta + \frac{i}{2\pi g}$.
- $V_\alpha(z)$ is the primary field of CFT with conformal dimension related to $\alpha \in \mathbb{C}$. α_i is linearly related to $\vec{a}, \vec{m}$.

Higher rank and quiver gauge theories

Higher rank generalization (Wyllard, Morozon-Mironov):

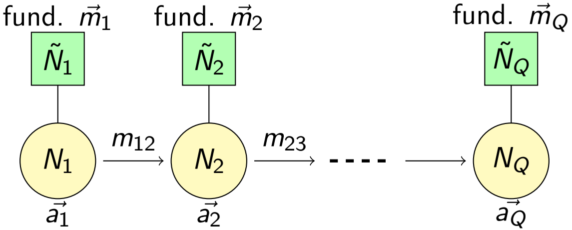
- AGT for $SU(n)$ gauge group $= W_n$ algebra $+$ U(1) factor conformal block
- W_n algebra: chiral algebra generated by spin $2, 3, \cdots, n$ (Zamolodchikov, Fateev, Lukyanov)
- CFT side is computed by Toda field theory of rank n. Correlation function was given by Fateev and Litvinov.

Linear quiver gauge theories:

- gauge group $SU(n_1) \otimes \cdots \otimes SU(n_l)$: $l+3$ point functions
- Quiver diagram (Figure in the next slide)
- When $n_1 = \cdots = n_\ell = n$, the conformal block function is given by $l+3$ point function $\langle V_{\alpha_1} \cdots V_{\alpha_{l+3}} \rangle$ of W_n-algebra (Toda field theory)

Quiver diagram and Nekrasov partition function

We consider a linear quiver gauge theories of the following type.



Partition function is written as "matrix multiplication" with index $\vec{Y}$:

$$\mathcal{Z}_{\text{inst.}} = \sum_{\vec{Y}_1,\cdots\vec{Y}_Q} \prod_{i=1}^{Q} q_i^{|\vec{Y}_i|} \mathcal{Z}_{\text{vect.}}(\vec{a}_i, \vec{Y}_i)\mathcal{Z}_{\text{fund.}}(\vec{a}_i, \vec{Y}_i; \vec{m}_i)$$

$$\cdot \prod_{i\to j\in E_Q} \mathcal{Z}_{\text{bfd.}}(\vec{a}_i, \vec{Y}_i; \vec{a}_j, \vec{Y}_j | m_{ij}) \prod_{i=1}^{Q} (\mathcal{Z}_{CS}(\vec{Y}_i))^{\kappa_i}$$

Nekrasov factor and its recursive properties

The explicit form of partition function: bifundamental rep.

$$\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m_{12})=\prod_{\ell=1}^{N_1}\prod_{\ell'=1}^{N_2}N_{Y_\ell,W_{\ell'}}(a_\ell-b_{\ell'}-m_{12})$$

$$N_{Y,W}(t)=\prod_{(i,j)\in Y}F(t,W'_j-i,Y_i-j+1)\prod_{(i,j)\in W}F(t,-Y'_j+i-1,-W_i+j)\,,$$

$$F(t,n,m)=t+\epsilon_1 m-\epsilon_2 n\ \text{(4d)},\quad 1-tq_1^{-n}q_2^{m}\ \text{(5d)}$$

The contribution from the other representations are derived from it.

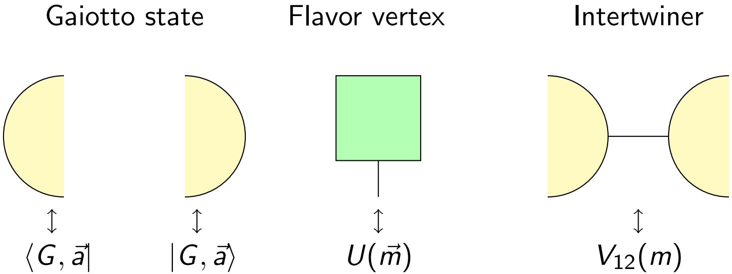
Nekrasov factor satisfies simple recursion formulae

$$\frac{N_{Y+b,W}(t)}{N_{Y,W}(t)}=\frac{\prod_{c\in A(W)}F(t,y(c)-y(b),x(b)-x(c))}{\prod_{y\in R(W)}F(t,y(c)-y(b)+1,x(b)-x(c)+1)}\,.$$

and so on. Namely, the addition/subtraction of a box is represented by product of factors on the surface. It is the most important lemma which gives our main theorem on the characterization of Gaiotto state and intertwiner which will appear later.

Correspondences between diagram and building blocks

Such partition function is written by combining "building block" written in the states/operators in SH^c



Comparison of both sides

- Conformal block function

$$\langle V(z_1)V(z_2)\cdots V(z_{l+3})\rangle = \bar{U}\cdot Z_1\cdots Z_l\cdot U$$

We use the decomposition of unity in the intermediate channel $1=\sum_{\vec{\lambda}}|\vec{a},\vec{\lambda}\rangle\langle\vec{a},\vec{\lambda}|$ where $|\vec{a},\vec{\lambda}\rangle$ is an orthonormal basis of the $W_n+U(1)$ module.

$$Z_{\vec{\lambda}_1,\vec{\lambda}_2}=\langle\vec{a}_1,\vec{\lambda}_1|V_\mu(1)|\vec{a}_2,\vec{\lambda}_2\rangle,\quad U_{\vec{\lambda}}=Z_{\vec{\lambda},\emptyset},\quad \bar{U}_{\vec{\lambda}}=Z_{\emptyset,\vec{\lambda}}$$

- Nekrasov partition function has very similar structure where

$$Z_{\vec{\lambda}_1,\vec{\lambda}_2}=Z_{bf}(\vec{a}_1,\vec{\lambda}_1;\vec{a}_2,\vec{\lambda}_2|\mu)Z_{vec}(\vec{a}_1,\vec{\lambda}_1)^{1/2}Z_{vec}(\vec{a}_2,\vec{\lambda}_2)^{1/2}$$

What should be proved

Proof of AGT conjecture is boiled down to

- Find a proper orthonormal basis of $W_n + U(1)$ module $|\vec{\lambda}\rangle$
- Show the identity

$$\langle\vec{\lambda}_1|V_\mu(1)|\vec{\lambda}_2\rangle = Z_{bf}(\vec{a}_1, \vec{\lambda}_1; \vec{a}_2\vec{\lambda}_2|\mu)Z_{vec}(\vec{a}_1, \vec{\lambda}_1)^{1/2}Z_{vec}(\vec{a}_2, \vec{\lambda}_2)^{1/2}$$

CFT approach to AGT conjecture

W-algebra: a nonlinear algebra that contains Virasoro

$$\begin{aligned} TT &\sim c+T \\ TW &= W \\ WW &= c+T+T^2 \end{aligned}$$

Algebra is explicitly written for $W^{(3)}$. For higher $W^{(n)}$, it is too complicated to express all the OPE. Instead of writing OPE among currents, we use free field realization (Fateev-Lukyanov) by quantum Miura transformation,

$$R = (Q\partial - \partial(\vec{h}_1\vec{\phi}))\cdots(Q\partial - \partial(\vec{h}_n\vec{\phi})) = \sum_{\ell=0}^{n} Q^{n-\ell} U^{(\ell)}(z)(\partial_z)^{n-l}$$

The structure of the Hilbert space is described by free field Fock space modulo the screening currents

$$Q_j^{(\pm)} = \int \frac{dz}{2\pi i} V_j^{(\pm)}(z), \quad V_j^{(\pm)} =: e^{\alpha_\pm(\vec{e}_j \vec{\phi})}$$

where $\vec{e}_j$ is jth simple root and $\alpha_+ = \beta$, $\alpha_- = 1/\beta$.
A standard method to evaluate the conformal block function is to use the Dotsenko-Fateev integral,

$$\langle V(z_1) \cdots V(z_l) \rangle = \langle V(z_1) \cdots V(z_l) \prod_{a=1}^{n-1} (Q_a^+)^{N_a} \rangle_{Fock}$$

It reduces to Selberg integral of Jack symmetric polynomial (eigenfunction of Calogero-Sutherland model). For $n = 2$, the integration can be managed (derived by Kadell) to give AGT conjecture (Morozov-Mironov). For higher n

Shapovalov matrix

The other way to check AGT conjecture is to construct the decomposition of unity,

$$1=\sum_{\vec{\lambda}_1,\vec{\lambda}_2}|\vec{\lambda}_1\rangle(S^{-1})^{\vec{\lambda}_1\vec{\lambda}_2}\langle\vec{\lambda}_2|,\qquad S_{\vec{\lambda}_1\vec{\lambda}_2}=\langle\vec{\lambda}_1|\vec{\lambda}_2\rangle.$$

The state $|\vec{\lambda}\rangle$ is a basis which is constructed as

$$J_{-n_1}\cdots J_{-n_{\ell_1}}L_{-m_1}\cdots L_{-m_{\ell_2}}\cdots|\vec{a}\rangle$$

The inner product $\langle\vec{\lambda}_1|\vec{\lambda}_2\rangle$ is called "Shapovalov matrix". It is block diagonal and we need inner product between the basis with the fixed levels. There is only a finite number of state for each level. So this is well-defined problem to compute order by order by computer. It was used by many authors to check AGT at lower levels. However, it does not give a systematic analytic proof which holds to all orders.

Connection between CFT and integrable model

The integrable model which is relevant to study CFT is Calogero-Sutherland model whose Hamiltonian is,

$$H = \sum_{i=1}^{N} (D_i)^2 + \beta \sum_{i<j} \frac{x_i + x_j}{(x_i - x_j)}(D_i - D_j), \quad D_i = x_i \partial_{x_i}$$

This Hamiltonian is known to be integrable, namely we have infinite number of commuting operators $H_2 = H, H_3, H_4, \cdots$. Their eigenfunctions are called Jack polynomial which is labeled by Young diagram.
In the large N limit, one may rewrite the coordinates x_i by the bosonic oscillators defined by, (for $n > 0$)

$$a_{-n} \leftrightarrow p_n = \sum_{i=1}^{N} (x_i)^n, \quad a_n = \frac{n}{\beta} \frac{\partial}{\partial p_n}$$

Through this correspondence, one may map Hilbert space of CS system to the Fock space of free boson theory.

AFLT proof

AFLT managed to give an algorithm to construct orthogonal basis $|\vec{\lambda}\rangle$ for $SU(2)$ gauge group.

- When one of the component is empty, $|(\lambda, \emptyset)\rangle = J_\lambda(a)|0\rangle \otimes |0\rangle$, where $J_\lambda(a)$ is Jack polynomial (eigenfunction of Calogero-Sutherland)
- For generic $\vec{\lambda}$, they proposed a recursive method which indirectly construct the basis. They show that it satisfies,

$$\langle \vec{\lambda}_1 | V_\mu(1) | \vec{\lambda}_2 \rangle = Z_{bf}(\vec{\lambda}_1, \vec{\lambda}_2 | \mu) Z_{vec}(\vec{\lambda}_1)^{1/2} Z_{vec}(\vec{\lambda}_2)^{1/2}$$

This is a proof of AGT conjecture. There remained some questions

- Why Calogero-Sutherland is relevant?
- Construction of general basis remains non-direct.

Calogero-Sutherland and CFT

Rewriting Calogero Hamiltonian

$$H=\sum_{i=1}^{N}(y_i)^2,\qquad y_i=D_i+\beta(\sum_{j>i}x_j\sigma_{ij}+\sum_{j<i}x_i\sigma_{ij})\partial_j$$

- The operator y_i is called as Dunkl operators and commute with each other $[y_i,y_j]=0$. The algebra generated by x_i, y_i, and permutation $\sigma\in S_N$ is called degenerate double affine Hecke algebra. (DDAHA)
- Higher charges of Calogero system are written as higher power sum of y_i. The eigenstate (Jack polynomial) is also written in terms of x_i, y_i as well.
- One may use different set of operator $Y_i=q^{D_i}\cdots$ (difference operator) and consider the algebra generated by $(X_i)^{\pm1},(Y_i)^{\pm1},R_i$. This is a q-deformation of DDAHA and referred to as double affine Hecke algebra (DAHA). It is a symmetry behind q-deformed Calogero (Ruijsnaar-Schneider model)

SH^c ($W_{1+\infty}$, affine Yangian)

The algebra generated by S_N symmetric combinations of DDAHA is called SH^c. We consider a combination (roughly), $D_{1,n} \sim \sum_i x_i y_i^n$, $D_{-1,n} \sim \sum_i (x_i)^{-1} y_i^n$, $D_{0,n} \sim \sum_i (y_i)^n$. One may combine them in the form of Drinfeld current,

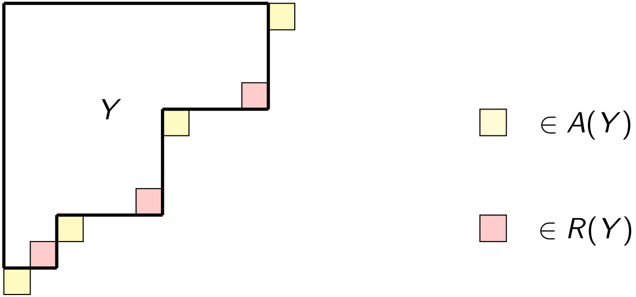
$$D_{\pm 1}(\zeta) = \sum_{n=0}^{\infty} D_{\pm 1,n}\, \zeta^{-n-1},\ D_0(\zeta) = \sum_{n=0}^{\infty} D_{0,n} \zeta^{-n-1}.$$

These generators satisfy a simple action on the Jack polynomials,

$$D_{\pm 1}(\zeta) J_\lambda = \sum_{x \in A/R(\lambda)} \frac{\Lambda_x(\lambda)}{\zeta - \phi_x} J_{\lambda + x}, \quad D_0(\zeta) J_\lambda = \sum_{x \in \lambda} \frac{1}{\zeta - \phi_x} J_\lambda,$$

where $\phi_x = \epsilon_1 (i-1) + \epsilon_2 (j-1)$.

$A(Y)$ and $R(Y)$: Figure



SH^c continued

SH^c contains two parameters $\epsilon_{1,2}$ ($\epsilon_3 = -\epsilon_+$) of omega background. Indeed only the ratio $\beta = -\epsilon_1/\epsilon_2$ is relevant. In holomorphic realization, the algebra is written compactly,

$$[D_0(u), D_0(v)] = 0 \quad [D_0(u), D_{\pm 1}(v)] = \pm \frac{D_{\pm 1}(v) - D_{\pm 1}(u)}{u - v},$$

$$h(u-v)D_1(u)D_1(v) \sim D_1(v)D_1(u)h(v-u), \quad h(u) = \prod_{i=1}^{3}(u+\epsilon_i),$$

$$[D_{-1}(u), D_1(v)] = \frac{E(v) - E(u)}{u - v}\epsilon_+^{-1}, \quad E(u) := \mathcal{Y}(u+\epsilon_+)\mathcal{Y}(u)^{-1}$$

$$\mathcal{Y}(u) := e^{c(u)} e^{\Phi(u-\epsilon_1)} e^{\Phi(u-\epsilon_2)} e^{-\Phi(u)} e^{-\Phi(u-\epsilon_+)}$$

where $\epsilon_+ = \epsilon_1 + \epsilon_2$, $c(u) = c_0 \log(u) - \sum_{n=1}^{\infty} \frac{c_n}{n u^n}$ are "central charges" of SH^c.

Relation between SH^c and W-algebra

Figure

- Horizontal line: U(1) current, Virasoro, W
- Vertical line: Drinfeld currents

$$
\begin{aligned}
D_{0,2} \sim H &= \beta^2 \sum_{n,m} (a_{-n} a_{-m} a_{n+m} + a_{-n-m} a_n a_m) + \beta(1-\beta) \sum_{n=1}^{\infty} n a_{-n} a_n \\
&= \sqrt{2\beta} \sum_{n>0} a_{-n} L_n^{(\beta)} + \text{diagonal} \\
e_0 &\sim a_{-1}, \qquad f_0 \sim a_1
\end{aligned}
$$

$D_{0,2}$, $a_{\pm 1}$ may be used to generate all the AY currents.

coproduct

Important property of SH^c: allows to take co-product (= analog of tensor product representation). While the original representation is described by one Fock space $\mathcal{F}$, one may construct more general representation $\mathcal{F}^{\otimes n}$. Since SH^c is generated by $D_{0,2}$ and $D_{\pm 1,0}$, what we need is the expression form them,

$$
\begin{aligned}
\Delta(D_{\pm 1,0}) &= D_{\pm 1,0} \otimes 1 + 1 \otimes D_{\pm 1,0} = \delta(D_{\pm 1,0}) \\
\Delta(D_{0,2}) &= \delta(D_{0,2}) + (\beta - 1) \sum_{n=1}^{\infty} n \beta^{n-1} a_{-n} \otimes a_n
\end{aligned}
$$

The last term represents the mixing due to the deformation. Coproduct of AY has similar representation

$$
D_{\pm 1}(\zeta)|\vec{\lambda}\rangle = \sum_{x \in A/R(\vec{\lambda})} \frac{\Lambda_x(\vec{\lambda})}{\zeta - \phi_x} |\vec{\lambda} \pm x\rangle, \quad D_0(\zeta)|\vec{\lambda}\rangle = \sum_{x \in \vec{\lambda}} \frac{1}{\zeta - \phi_x} |\vec{\lambda}\rangle
$$

where $\phi_x = a_\alpha + \epsilon_1(i-1) + \epsilon_2(j-1)$, $\alpha \in (1, \cdots, n)$.

Equivalence of $W_n + U(1)$ module and SH^c

- By taking coproduct $n-1$ times, $D_{0,2} \sim H$ and $D_{\pm n,0}$ are written in terms of n-bosons.
- W_n algebra $+$ $U(1)$ currents are written in terms of n-bosons,

$$(Q\partial_z - \partial_z\phi_1)\cdots(Q\partial_z - \partial_z\phi_n) = \sum_{i=0}^{n} Q^{n-i} W^{(i)}(z)\partial_z^{n-i}\,.$$

- Schiffmann and Vasserot showed that these two Hilbert space are equivalent. This identification remains true even when the Hilbert space is degenerate (such as minimal model) (Fukuda-Nakamura-M-Zhu).
- Compared with W-algebra basis, SH^c description has a definite advantage that it is expressed in terms of orthogonal basis labeled by n-tuple Young diagrams. Indeed, $|\vec{\lambda}\rangle$ can be identified with AFLT basis.

Construction of Gaiotto state

AGT conjecture for pure super-$SU(n)$ Yang-Mills is given by,

$$\langle G|\Lambda^{2L_0}|G\rangle = \sum_{\vec{\lambda}} Z_{\rm vect}(\vec{a},\vec{\lambda})\Lambda^{2|\vec{\lambda}|}$$

where $|G\rangle$ is called as Gaiotto state satisfying conherent state condition for W algebra,

$$W_1^{(d)}|G\rangle = \lambda_1^{(d)}|G\rangle, \quad 1 \le d \le N,$$

Shiffmann and Vasserot claimed such $|G\rangle$ is written as,

$$|G\rangle = \sum_{\vec{\lambda}} |\vec{a},\vec{\lambda}\rangle,$$

where we used the normalization $\langle \vec{a},\vec{\lambda}|\vec{a},\vec{\mu}\rangle = Z_{\rm vect}(\vec{a},\vec{\lambda})\delta_{\vec{\lambda},\vec{\mu}}$. It is easy to see that this state satisfies reproduces the instanton partition function.

Proof of other gauge theories

- Inclusion of fundamental matter [M-Rim-Zhang]

$$|G,\vec{m}\rangle = \sum_{\vec{\lambda}} Z_{\rm fd}(\vec{m},\vec{\lambda})|\vec{\lambda}\rangle$$

  satisfies (generalized) coherent state condition.

- Description of quiver gauge theory [Kanno-M-Zhang, Negut]
  The vertex operator in CFT is expressed as,

$$V_\mu = \sum_{\vec{\lambda},\vec{\nu}} |\vec{a},\vec{\lambda}\rangle\langle\vec{b},\vec{\nu}|Z_{\rm bf}(\vec{a},\vec{\lambda};\vec{b},\vec{\nu}|\mu)$$

  The right hand side behaves as a “primary field” under conformal transformation. (cf. Carlsson and Okounkov)

Seiberg-Witten curve and qq-character

- Seiberg-Witten curve for $SU(N)$ gauge theory:

$$y + \frac{1}{y} = u(z), \quad u(z) := \prod_{i=1}^{N} (z - a_i), \quad y \in \mathbb{C}^*, z \in \mathbb{C} \text{ or } \mathbb{C}^* .$$

- Can we derive it from second quantized system?
- Yes! It reduces to a simple property of Gaiotto state! [Bourgine-Matsuo-Zhang 2015]

$$D_{-1}(z)|G, \vec{a}\rangle \propto \frac{1}{\mathcal{Y}(z)}|G, \vec{a}\rangle, \quad D_1(z)|G, \vec{a}\rangle \propto \mathrm{P}_z^- \mathcal{Y}(z + \epsilon_+)|G, \vec{a}\rangle$$

where $\mathcal{Y}(z)$ is an operator which is diagonal w.r.t. $|\vec{\lambda}\rangle$, $\mathcal{Y}(z)|\vec{\lambda}\rangle = \mathcal{Y}(z, \vec{\lambda})|\vec{\lambda}\rangle$.

Derivation of qq-character

For gauge theory with the coupling to fundamental matter, by evaluating

$$\langle G|D_{-1}(z)q^D|G\rangle,$$

in two ways, one obtains a generating function of the correlation function of the form:

$$\mathrm{P}_z^- \left\langle \mathcal{Y}(z+\epsilon_+) + \frac{q}{\mathcal{Y}(z)} \right\rangle = 0,$$

where

$$\langle \mathcal{O} \rangle := \frac{\langle G, \vec{a}|\mathcal{O}q^D|G, \vec{a}\rangle}{\langle G, \vec{a}|q^D|G, \vec{a}\rangle}$$

It implies

$$\left\langle \mathcal{Y}(z+\epsilon_+) + q\mathcal{Y}(z)^{-1} \right\rangle = \chi(z)$$

$\chi(z)$ is N-th order polynomial which is a deformation of $u(z)$. Nekrasov called it qq-character.

First and second quantization of SW curve

- Classical: Seiberg-Witten curve for $SU(N)$ gauge theory:

$$y + \frac{1}{y} = u(z), \quad u(z) := \prod_{i=1}^{N}(z - a_i), \quad y \in \mathbb{C}^*, z \in \mathbb{C} \text{ or } \mathbb{C}^* .$$

- Quantum: replace $y = e^{\hbar\partial_z}$, Schrödinger eq. gives quantum curve

$$(e^{\hbar\partial_z} + e^{-\hbar\partial_z} + u(z))\psi(z) = \psi(z+\hbar) + \psi(z-\hbar) + u(z)\psi(z) = 0$$

  which looks like Baxter TQ relation (NS limit $\epsilon_1 = \hbar, \epsilon_2 = 0$).

- Double quantum:

$$\left\langle \mathcal{Y}(z+\epsilon_+) + q\mathcal{Y}(z)^{-1} \right\rangle = \chi(z)$$

SH^c and quantum toroidal algebra knows how to describe quantized geometry from the representation theory. It also applies to toric Calabi-Yau which is described by topological vertex.

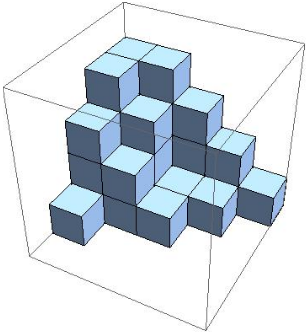
Plane partition realization of Affine Yangian

Plane partition realization: $|\Lambda\rangle$ the orthogonal basis labelled by a plane partition Λ, (recursion formula for Nekrasov factor)

$$
\begin{aligned}
\psi(u)\,|\Lambda\rangle &= \psi_\Lambda(u)\,|\Lambda\rangle\,, \\
e(u)\,|\Lambda\rangle &= \sum_{\square\in\Lambda^+}\frac{1}{u-q-h_\square}\sqrt{-\frac{1}{\sigma_3}\mathrm{res}_{u\to q+h_\square}\psi_\Lambda(u)}\,|\Lambda+\square\rangle\,, \\
f(u)\,|\Lambda\rangle &= \sum_{\square\in\Lambda^-}\frac{1}{u-q-h_\square}\sqrt{-\frac{1}{\sigma_3}\mathrm{res}_{u\to q+h_\square}\psi_{\Lambda-\square}(u)}\,|\Lambda-\square\rangle\,, \\
\psi_\Lambda(u) &= \psi_0(u-q)\prod_{\square\in\Lambda}\varphi(u-q-h_\square)
\end{aligned}
$$

with $h_\square = h_1x+h_2y+h_3z$ when the box is located at (x,y,z). Plane partition appears in the context of Calabi-Yau geometry through melting crystal picture.

Illustration of plane partition



When it is sliced in one direction, it is decomposed into a set of partitions $\lambda_1 \succeq \cdots \succeq \lambda_n$. Plane partition may represent random surface or "melting crystal".

Corner VOA: more general truncation of AY

- Corner VOA Y_{LMN} appears as a truncation of the affine Yangian through constraints on its parameters. (Prochazka and Rapcak)

$$\frac{L}{\lambda_1} + \frac{M}{\lambda_2} + \frac{N}{\lambda_3} = 1$$

- With this constraint, the plane partition contains a null state (pit) at $(L+1, M+1, N+1)$. Note: there is a shift symmetry $(L, M, N) \to (L+k, M+k, N+k)$ and one may set one of L, M, N to be zero.
- When $(L, M, N) = (0, 0, N)$, the height of the plane partition is limited by N. Such diagram can be sliced horizontally to describe N Young diagrams $\lambda_1 \succeq \lambda_2 \succeq \cdots \succeq \lambda_N$, which can be identified with the Hilbert space of W_N algebra (plus $U(1)$ boson). This agrees with the horizontal picture.

Plane partition with infinite legs

Plane partition may have nonvanishing asymptotic Young diagrams in three directions, say λ, μ, ρ. The partition function for original plane partition is MacMahon function,

$$\prod_{n=1}^{\infty}(1-q^n)^{-n} = M(q)$$

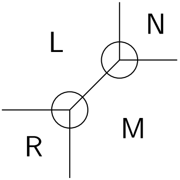
With the infinite legs attached, the partition function is proportional to,

$$C_{\mu\nu\rho}(q)M(q)$$

where $C_{\mu\nu\rho}(q)$ is the (unrefined) topological vertex. It describes a nontrivial representation of AY. The conformal weight and $U(1)$ charge becomes also nonvanishing. They are computed from the coefficients of $\psi(u)$. This function remains finite since the contributions of infinite boxes cancel with each other and only finite factor remains.

Connecting two Y-algebras: Web of W (WoW)

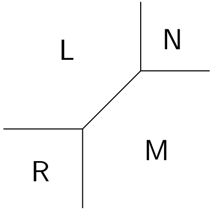
One may connect two corner VOAs by sharing one asymptotic Young diagram with another one.



We have two AYs acting on two plane partitions depicted as two circles. In addition to them, we need to introduce extra generators which changes the shape of the asymptotic Young diagram. Thus, WoW consists of tensor product of two $W_{1+\infty}$ with extra generators.

WoW: generalization

By changing L, M, N, R, we obtain different types of VOA.



- $(L, M, N, R) = (0, 1, 1, 0)$ $bc\beta\gamma$ system.
- $(L, M, N, R) = (0, 1, 2, 0)$ $N = 2$ SCA
- $(L, M, N, R) = (0, 1, 3, 0)$ Polyakov-Bershadsky VOA

One may connect the third vertex (WoWoW) and so on, and it describes infinitely many unknown algebras. The treatment of negative Young diagrams may be more complicated for the general models.

Double truncation

In the following, we consider a special case where we put the second constraint on the parameters.

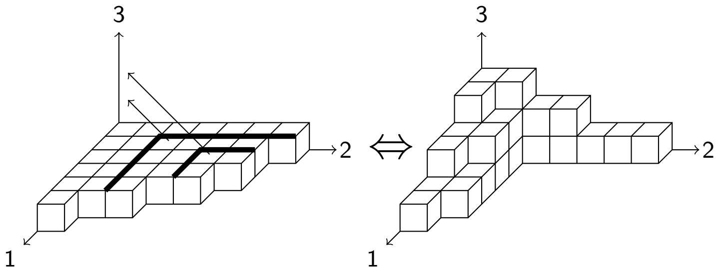
$$\frac{L_1}{\lambda_1} + \frac{M_1}{\lambda_2} + \frac{N_1}{\lambda_3} = 1\,, \qquad \frac{L_2}{\lambda_1} + \frac{M_2}{\lambda_2} + \frac{N_2}{\lambda_3} = 1\,.$$

In this case, we seem to have two pits at (L_i, M_i, N_i) $(i = 1, 2)$. This is, however, too naive. With such parametrization, we have periodicity in h function,

$$h_{x+L_1,y+M_1,z+N_1} = h_{x+L_2,y+M_2,z+N_2}$$

Since h governs the representation of the plane partition, above relation implies that the plane partition becomes degenerate, namely the boxes in the different locations should be identified. We will denote $Y_{L_1M_1N_1:L_2M_2N_2}$ to describe such algebra. With two constraints, there is no free parameter.

We illustrate the degeneration phenomena in the simplest case $Y_{120:001}$.



The algebra may be identified as Y_{001} described by one Young diagram or Y_{120} by plane partition with a pit at $(2,3,1)$. One may construct the latter diagram by cutting Young diagram into hooks. One needs to impose that both diagrams are consistent as plane partition which imposes tight constraints on both.

Double truncation and minimal models [Harada-M]

Theorem (Condition for Minimal models in corner VOA and WoW)

Double truncation of corner VOA and WoW describes their minimal models. The primary fields are labeled by asymptotic Young diagrams in the free legs.

- While we use "theorem", it remains a conjecture.
- We checked it for W_n algebra and $N = 2$ SCA. It works perfectly.
- In the conventional method, minimal models of VOA are obtained by examining the null states and modular duality. Here we obtained them from a simple geometrical requirement.

Summary

- The algebra $\mathcal{U}$ (quantum toroidal algebra, DIM, SH^c, Affine Yangian ..) provide a universal picture of conformal field theory and integrable models.
- It manages to describe higher dimensional physics such as super Yang-Mills in 4D and 5D and topological strings.
- It also gives the second quantized picture of SW curve and Calabi-Yau geometry.
- There appears a new truncation of $\mathcal{U}$ which gives a new family of VOA.
- Double truncation of $\mathcal{U}$ gives a geometrical picture of minimal models of VOA.