# Clebsch-Gordan係数

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$$|\, j_1, j_2; J, M\rangle\rangle \;=\; \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1, j_2; m_1, m_2 \mid j_1, j_2; J, M\rangle\rangle \cdot |\, j_1, j_2; m_1, m_2\rangle$$

**漸化式**

$$\begin{aligned} & \sqrt{(J \mp 1)(J \pm M + 1)}\langle j_1, j_2; m_1, m_2 \mid j_1, j_2; J, M \pm 1\rangle\rangle \\ = \; & \sqrt{(j_1 \pm 1)(j_1 \mp m_1 + 1)}\langle j_1, j_2; m_1 \mp 1, m_2 \mid j_1, j_2; J, M\rangle\rangle \\ & + \sqrt{(j_2 \pm 1)(j_2 \mp m_2 + 1)}\langle j_1, j_2; m_1, m_2 \mp 1 \mid j_1, j_2; J, M\rangle\rangle \end{aligned}$$

**一般式（Racah の公式）**

$$\begin{aligned} & \langle j_1, j_2; m_1, m_2 \mid j_1, j_2; J, M\rangle\rangle \\ & = \delta_{m_1+m_2,m} \\ & \times \left[\frac{(2J+1)(j_1+j_2-J)!(j_1-j_2+J)!(-j_1+j_2+J)!}{(j_1+j_2+J+1)!}\right]^{1/2} \\ & \times \left[(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(J+M)!(J-M)!\right]^{1/2} \\ & \times \sum_{k \in \mathbf{Z}} (-1)^k \frac{1}{k!(j_1+j_2-J-k)!(j_1-m_1-k)!(j_2+m_2-k)!(J-j_2+m_1+k)!(J-j_1-m_2+k)!} \end{aligned}$$

**例：** $j_1 = any\ (\geq 1/2)$, $j_2 = 1/2$

$$|j_1, \frac{1}{2}; J, M\rangle\rangle \;=\; \sum_{m_2=\pm 1/2} \langle j_1, \frac{1}{2}; M - m_2, m_2 \mid j_1, \frac{1}{2}; J, M\rangle\rangle \cdot |j_1, \frac{1}{2}; M - m_2, m_2\rangle$$

| $\langle j_1, \frac{1}{2}; M-m_2, m_2 \mid j_1, \frac{1}{2}; J, M\rangle\rangle$ | $m_2 = 1/2$ | $m_2 = -1/2$ |
|---|---|---|
| $J = j_1 + \frac{1}{2}$ | $\sqrt{\frac{j_1+M+1/2}{2j_1+1}}$ | $\sqrt{\frac{j_1-M+1/2}{2j_1+1}}$ |
| $J = j_1 - \frac{1}{2}$ | $-\sqrt{\frac{j_1-M+1/2}{2j_1+1}}$ | $\sqrt{\frac{j_1+M+1/2}{2j_1+1}}$ |

**例：** $j_1 = any\ (\geq 1)$, $j_2 = 1$

$$|j_1, 1; J, M\rangle\rangle \;=\; \sum_{m_2=1,0,-1} \langle j_1, 1; M - m_2, m_2 \mid j_1, 1; J, M\rangle\rangle \cdot |j_1, 1; M - m_2, m_2\rangle$$

| $\langle j_1, 1; M-m_2, m_2 \mid j_1, 1; J, M\rangle\rangle$ | $m_2 = 1$ | $m_2 = 0$ | $m_2 = -1$ |
|---|---|---|---|
| $J = j_1 + 1$ | $\sqrt{\frac{(j_1+M)(j_1+M+1)}{(2j_1+1)(2j_1+2)}}$ | $\sqrt{\frac{(j_1-M+1)(j_1+M+1)}{(2j_1+1)(j_1+1)}}$ | $\sqrt{\frac{(j_1-M)(j_1-M+1)}{(2j_1+1)(2j_1+2)}}$ |
| $J = j_1$ | $-\sqrt{\frac{(j_1+M)(j_1-M+1)}{2j_1(j_1+1)}}$ | $\frac{M}{\sqrt{j_1(j_1+1)}}$ | $\sqrt{\frac{(j_1-M)(j_1+M+1)}{2j_1(j_1+1)}}$ |
| $J = j_1 - 1$ | $\sqrt{\frac{(j_1-M)(j_1-M+1)}{2j_1(2j_1+1)}}$ | $-\sqrt{\frac{(j_1-M)(j_1+M)}{j_1(2j_1+1)}}$ | $\sqrt{\frac{(j_1+M)(j_1+M+1)}{2j_1(2j_1+1)}}$ |

訂正がある場合、http://www-hep.phys.s.u-tokyo.ac.jp/~hama/lectures/2012_QM2.html に掲載する。