Off-shell $d = 5$ Supergravity

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In this talk, we review an off-shell formulation of matter-Yang-Mills system coupled to supergravity in five dimensional space-time. 5D supergravity tensor calculus is first obtained from 6D superconformal tensor calculus by Bergshoeff, Sezgin and Van Proeyen. Based on the derived tensor calculus, we obtain an invariant action for the general system of vector multiplets and hypermultiplets coupled to supergravity as well as the supersymmetry transformation rules. All the auxiliary fields are kept un-eliminated so that the supersymmetry transformation rules remain unchanged even when the action is changed.

§1. Introduction

In this lecture, I would like to review an off-shell formulation of matter-Yang-Mills system coupled to supergravity in five dimensional space-time. This is based on the work in collaboration with Keisuke Ohashi.

It is really a revolutionary and interesting idea that our four-dimensional world may be a '3-brane' embedded in a higher dimensional spacetime. In order to investigate various problems seriously in such brane world scenarios, however, we need to know supergravity theory in five dimensions.[1),2)] We are interested in five dimensional space-time since it provides us with the simplest case in which we have a single extra dimension. Also in a more realistic situation, M-theory, whose low energy effective theory is described by eleven dimensional supergravity, is expected to be compactified on Calabi-Yau 3-folds and becomes to be described by effective five dimensional supergravity theories.[3),4)]

In various problems, we need an off-shell formulation of supergravity containing the auxiliary fields with which the supersymmetry transformation laws are made system independent and the algebra closes without using equations of motion. For instance, Mirabelli and Peskin[5)] were able to give a simple algorithm based on an off-shell formulation for finding how to couple bulk fields to boundary fields, in the work in which they considered a five-dimensional super Yang-Mills theory compactified on S^1/Z_2 and clarified how supersymmetry breaking occurring on one boundary is communicated to another. Moreover, if one wish to study some problems by adding D-branes to such a system, then, without an off-shell formulation, one must find a new supersymmetry transformation rule of the bulk fields at each time one adds new branes, since the supersymmetry transformation law in the on-shell formulation depends on the Lagrangian of the system.

A 5D supergravity tensor calculus for constructing off-shell formulation has been given by Zucker.[6)] In our first paper,[7)] which we refer to as I henceforth, we have

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derived a more complete tensor calculus by dimensional reduction from 6D superconformal tensor calculus.[8] Tensor calculus gives a set of rules in the off-shell supergravity: that is, i) transformation laws of the various type of supermultiplets, ii) composition laws of multiplets from multiplets, and iii) invariant action formulas. In the second paper,[9] refered to as II, we have constructed an action for a general system of vector multiplets and hypermultiplets coupled to supergravity based on the tensor calculus presented in I. In the summer institute 2000, the result in II was not yet fully available and was only partially reported.

§2. Dimensional reduction

The superconformal tensor calculus in 6D was given in an excellent and elaborate paper by Bergshoeff, Sezgin and Van Proeyen,[8] referred to as BSVP henceforth.

We perform a 'trivial' dimensional reduction from 6 to 5 dimensions, by simply letting all the fields and local transformation parameters be independent of the fifth spatial coordinate z. As in the usual procedure,[10] we fix the off-diagonal local Lorentz $\boldsymbol{M}_{a5}$ gauge by setting $\underline{e}_{\,z}^{\,a}=0$, and then the sechsbein $\underline{e}_{\underline{\mu}}^{\,\underline{a}}$ and its inverse take the forms

$$\underline{e}_{\underline{\mu}}^{\,\underline{a}}=\begin{pmatrix} e_\mu^a & e_\mu^5=\alpha^{-1}A_\mu \\ e_z^a=0 & e_z^5=\alpha^{-1}\end{pmatrix},\qquad \underline{e}_{\underline{a}}^{\,\underline{\mu}}=\begin{pmatrix} e_a^\mu & e_a^z=-e_a^\mu A_\mu \\ e_5^\mu=0 & e_5^z=\alpha\end{pmatrix}. \tag{2·1}$$

One should note, as usual in the dimensional reduction, that the field *with flat (local Lorentz) indices alone* is the same in 6D and 5D. Thus, for instance, the vector field h_μ in 5D is given by

$$h_\mu\equiv\underline{h}_{\,\mu}-A_\mu\underline{h}_{\,z}, \tag{2·2}$$

in terms of that $\underline{h}_{\,\mu}$ in 6D.

In reducing to 5 dimensions by setting $e_z^a=0$, it turns out to be simpler to also fix the conformal $\boldsymbol{S}$ and $\boldsymbol{K}$ gauge symmetries by using the following gauge-fixing conditions:

$$\begin{aligned}&\boldsymbol{M}_{a5}:\ \ e_z^a=0, &&\boldsymbol{S}^i:\ \ \psi_5^i=0,\\ &\boldsymbol{K}_a:\ \ b_\mu-\alpha^{-1}\partial_\mu\alpha=0, &&\boldsymbol{K}_5:\ \ b_5=0.\end{aligned} \tag{2·3}$$

The $\boldsymbol{S}$ gauge is chosen to satisfy $\psi_5^i=0$ (implying also $\psi_z=e_z^5\psi_5=0$), so as to make the condition $e_z^a=0$ invariant under the supersymmetry transformation, $\underline{\delta}_{\,Q}(\varepsilon)e_z^a=-2i\bar\varepsilon\underline{\gamma}^a\psi_z$. Note that this gauge also implies the $\boldsymbol{Q}$-invariance of the 'dilaton' field $\alpha\equiv(e_z^5)^{-1}=e_5^z$. Moreover, the $\boldsymbol{K}_a$ gauge $b_\mu=\alpha^{-1}\partial_\mu\alpha$ is chosen so as to make α also covariantly constant $\hat{\mathcal{D}}_\mu\alpha=\partial_\mu\alpha-b_\mu\alpha=0$ in the reduced 5D theory, as we see below. Thanks to these two properties, the field α carrying Weyl weight $w=1$ can be treated as if it were a constant and is freely used to adjust the Weyl weights of any quantities to arbitrary desired values. We, however, keep the dilatation gauge symmetry unfixed, since it becomes useful later when we change the Einstein-Hilbert and Rarita-Schwinger terms in the action into canonical form.

The supersymmetry transformation in 5D is found to be given by the following

Table I. Field contents of the multiplets

field	type	restrictions	$SU(2)$	Weyl-weight
		Weyl multiplet		
$e_\mu{}^a$	boson	fünfbein	$\mathbf{1}$	-1
ψ^i_μ	fermion	$SU(2)$-Majorana	$\mathbf{2}$	$-\frac{1}{2}$
V^{ij}_μ	boson	$SU(2)$ gauge field $V^{ij}_\mu = V^{ji}_\mu = -V^*_{\mu ij}$	$\mathbf{3}$	0
A_μ	boson	gravi-photon $A_\mu = e^z_5 e^5_\mu$	$\mathbf{1}$	0
α	boson	'dilaton' $\alpha = e^z_5$	$\mathbf{1}$	1
t^{ij}	boson	$t^{ij} = t^{ji} = -t^*_{ij}$ $(= -V^{ij}_5)$	$\mathbf{3}$	1
v_{ab}	boson	real tensor $v_{ab} = -v_{ba}$	$\mathbf{1}$	1
χ^i	fermion	$SU(2)$-Majorana	$\mathbf{2}$	$\frac{3}{2}$
C	boson	real scalar	$\mathbf{1}$	2
b_μ	boson	$\boldsymbol{D}$ gauge field $b_\mu = \alpha^{-1}\partial_\mu\alpha$	$\mathbf{1}$	0
$\omega_\mu{}^{ab}$	boson	spin connection	$\mathbf{1}$	0
		Vector multiplet		
W_μ	boson	real gauge field	$\mathbf{1}$	0
M	boson	real scalar, $M = -W_5$	$\mathbf{1}$	1
Ω^i	fermion	$SU(2)$-Majorana	$\mathbf{2}$	$\frac{3}{2}$
Y_{ij}	boson	$Y^{ij} = Y^{ji} = -Y^*_{ij}$	$\mathbf{3}$	2
		Linear multiplet		
L^{ij}	boson	$L^{ij} = L^{ji} = -(L_{ij})^*$	$\mathbf{3}$	3
φ^i	fermion	$SU(2)$-Majorana	$\mathbf{2}$	$\frac{7}{2}$
E_a	boson	real, constrained by (3·6)	$\mathbf{1}$	4
N	boson	real, $N = -E_5$	$\mathbf{1}$	4
		Hypermultiplet		
$\mathcal{A}^\alpha_i$	boson	$\mathcal{A}^i_\alpha = \varepsilon^{ij}\mathcal{A}^\beta_j\rho_{\beta\alpha} = -(\mathcal{A}^\alpha_i)^*$	$\mathbf{2}$	$\frac{3}{2}$
ζ^α	fermion	$\bar\zeta^\alpha \equiv (\zeta_\alpha)^\dagger\gamma_0 = \zeta^{\alpha\mathrm{T}}C$	$\mathbf{1}$	2
$\mathcal{F}^\alpha_i$	boson	$\mathcal{F}^i_\alpha = -(\mathcal{F}^\alpha_i)^*$	$\mathbf{2}$	$\frac{5}{2}$

combination of 6D transformations of $\boldsymbol{Q}$, $\boldsymbol{S}$ and $\boldsymbol{K}$:

$$
\begin{aligned}
&\delta_Q(\varepsilon) = \underline{\delta}_Q(\varepsilon) + \delta_S(\eta(\varepsilon)) + \delta_K(\xi^{\underline{a}}_K(\varepsilon))\,,\\
&\text{with}\quad \eta(\varepsilon)^i = -\tfrac{1}{16\alpha}\gamma\cdot\hat F(A)\varepsilon^i - \tfrac{1}{4}\gamma\cdot v\varepsilon^i - t^i_j\varepsilon^j\,,\\
&\qquad \xi^a_K(\varepsilon) = -i\bar\varepsilon\Big(\phi^a - \eta(\psi^a) + \gamma^a(\tfrac{1}{2}\chi - \phi_5)\Big)\,,\\
&\qquad \xi^5_K(\varepsilon) = \tfrac{1}{2}i\bar\varepsilon\chi\,.
\end{aligned}
\tag{2·4}
$$

Here, $\gamma\cdot T$ for any tensor $T_{a_1\cdots a_n}$ generally denotes the contraction $\gamma^{a_1\cdots a_n}T_{a_1\cdots a_n}$.

If we use the relations (2·4), it is straightforward (although tedious) to obtain the supersymmetry transformation laws in 5D for various multiplets, from 6D superconformal transformation rule.

We list in Table I the field contents of the Weyl multiplets, vector multiplets, linear multiplets and hypermultiplets, which we deal with in this paper. (The dilatation gauge field b_μ and spin connection ω^{ab}_μ are also listed although they are dependent fields.)

§3. Supersymmetry transformation laws

3.1. *Weyl multiplet*

The supersymmetry transformation law for our 5D Weyl multiplet is found to be ($\delta \equiv \delta_Q(\varepsilon)$)

$$\begin{aligned}
\delta e_\mu{}^a &= -2i\bar\varepsilon\gamma^a\psi_\mu, \\
\delta\psi^i_\mu &= \mathcal{D}_\mu\varepsilon^i + \tfrac{1}{2}\gamma_{\mu ab}\varepsilon^i v^{ab} + \tfrac{1}{2\alpha}\gamma^a\varepsilon^i\hat F(A)_{\mu a} + \gamma_\mu\varepsilon^j t^i{}_j, \\
\delta\alpha &= 0, \quad (\mathcal{D}_\mu\varepsilon^i \equiv \partial_\mu\varepsilon^i + \tfrac{1}{2}b_\mu\varepsilon^i - \tfrac{1}{4}\omega^{ab}_\mu\gamma_{ab}\varepsilon^i - V_\mu{}^i{}_j\varepsilon^j) \\
\delta A_\mu &= 2i\alpha\bar\varepsilon\psi_\mu, \\
\delta V^{ij}_\mu &= -4i\bar\varepsilon^{(i}\gamma_\mu\chi^{j)} - 2i\bar\varepsilon^{(i}\gamma^a\hat{\mathcal{R}}_{a\mu}{}^{j)}(Q) \\
&\quad + \tfrac{i}{\alpha}\bar\varepsilon^{(i}\gamma\cdot\hat F(A)\psi^{j)}_\mu + 4i\bar\varepsilon^{(i}\gamma\cdot v\psi^{j)}_\mu - 6i\bar\varepsilon\psi_\mu t^{ij}, \\
\delta t^{ij} &= 4i\bar\varepsilon^{(i}\chi^{j)}, \\
\delta v_{ab} &= \tfrac{i}{4}\bar\varepsilon\gamma_{abcd}\hat{\mathcal{R}}^{cd}(Q) - 2i\bar\varepsilon\gamma_{ab}\chi, \\
\delta\chi^i &= -\tfrac{1}{2}\gamma^a\varepsilon^i\hat{\mathcal{D}}^b v_{ab} + \tfrac{1}{2}\hat{\not{\mathcal{D}}}t^i{}_j\varepsilon^j - \gamma\cdot v t^i{}_j\varepsilon^j - \tfrac{1}{4\alpha}\gamma\cdot\hat F(A)t^i{}_j\varepsilon^j \\
&\quad + \tfrac{1}{2}C\varepsilon^i - \tfrac{1}{32\alpha^2}\gamma^{abcd}\varepsilon^i\hat F_{ab}(A)\hat F_{cd}(A), \\
\delta C &= -2i\bar\varepsilon\hat{\not{\mathcal{D}}}\chi + 22i\bar\varepsilon^i\chi^j t_{ij} - 3i\bar\varepsilon\gamma\cdot v\chi - i\bar\varepsilon^i\gamma^{ab}\hat{\mathcal{R}}_{ab}{}^j(Q)t_{ij}, \\
\delta\omega_\mu{}^{ab} &= -2i\bar\varepsilon\gamma^{[a}\hat{\mathcal{R}}_\mu{}^{b]}(Q) - i\bar\varepsilon\gamma_\mu\hat{\mathcal{R}}^{ab}(Q) \\
&\quad + \tfrac{2i}{\alpha}\bar\varepsilon\psi_\mu\hat F^{ab}(A) - 2i\bar\varepsilon\gamma^{abcd}\psi_\mu v_{cd} - 4i\bar\varepsilon^i\gamma^{ab}\psi^j_\mu t_{ij}.
\end{aligned} \tag{3·1}$$

Here we have also written the transformation law of the spin connection, although it is a dependent field given by

$$\begin{aligned}
\omega_\mu{}^{ab} &= \omega^{0\ ab}_\mu + i(2\bar\psi_\mu\gamma^{[a}\psi^{b]} + \bar\psi^a\gamma_\mu\psi^b) - 2e_\mu{}^{[a}b^{b]}\,, \\
\omega^{0\ ab}_\mu &\equiv -2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} + e^{\rho[a}e^{b]\sigma}e_\mu{}^c\partial_\rho e_{\sigma c}\,.
\end{aligned} \tag{3·2}$$

3.2. *Vector multiplet*

The vector multiplet in 5D consists of the fields given in Table I, where the real scalar M comes from the fifth spatial component of a 6D vector, $M \equiv -W_5 = -\alpha W_z$. Note that it carries a Weyl weight $w = 1$, since $w(\alpha) = 1$ and $w(W_{\underline{\mu}}) = 0$. As in 6D, all the component fields are Lie-algebra valued, e.g., M is a matrix $M^\alpha{}_\beta = M^A(t_A)^\alpha{}_\beta$, where the t_A are (anti-hermitian) generators of the gauge group G. The $\boldsymbol{Q}$ transformation rules are found from 6D one to be

$$\begin{aligned}
\delta W_\mu &= -2i\bar\varepsilon\gamma_\mu\Omega + 2i\bar\varepsilon\psi_\mu M\,, \\
\delta M &= 2i\bar\varepsilon\Omega\,, \\
\delta\Omega^i &= -\tfrac{1}{4}\gamma\cdot\hat G(W)\varepsilon^i - \tfrac{1}{2}\gamma^a\varepsilon^i\hat{\mathcal{D}}_a M - Y^{ij}\varepsilon_j\,, \\
\delta Y^{ij} &= 2i\bar\varepsilon^{(i}\hat{\not{\mathcal{D}}}\Omega^{j)} - i\bar\varepsilon^{(i}\gamma\cdot v\Omega^{j)} + 2i\bar\varepsilon^{(i}t^{j)}{}_k\Omega^k + 4i\bar\varepsilon\Omega t^{ij} - 2ig\bar\varepsilon^{(i}[M,\Omega^{j)}]\,,
\end{aligned} \tag{3·3}$$

where g is a gauge coupling constant and $\hat G_{ab}(W)$ is the following combination of the supercovariant field strength $\hat F_{ab}(W)$ of W_μ and that of the gravi-photon A_μ:

$$\hat G_{ab}(W) \equiv \hat F_{ab}(W) - \tfrac{1}{\alpha}M\hat F_{ab}(A), \tag{3·4}$$

which is actually the field strength $\underline{\hat{F}}_{ab}(W)$ in 6D.

3.3. *Linear multiplet*

The linear multiplet consists of the components listed in Table I and may generally carry a non-Abelian charge of the gauge Group G.

The $\boldsymbol{Q}$ transformation rules are found to be

$$
\begin{aligned}
\delta L^{ij} &= 2i\bar{\varepsilon}^{(i}\varphi^{j)}\,, \\
\delta\varphi^i &= -\hat{\not{\mathcal{D}}} L^{ij}\varepsilon_j - 4t^l{}_k L^k{}_l\varepsilon^i - 6t^{(i}{}_k L^{j)k}\varepsilon_j + gML^{ij}\varepsilon_j \\
&\quad + \tfrac{1}{2}\gamma^a\varepsilon^i E_a + \tfrac{1}{2}\varepsilon^i N + \tfrac{1}{2\alpha}\gamma\cdot\hat{F}(A)\varepsilon_j L^{ij} + 2\gamma\cdot v\varepsilon_j L^{ij}\,, \\
\delta E_a &= 2i\bar{\varepsilon}\gamma_{ab}\hat{\mathcal{D}}^b\varphi - \tfrac{i}{\alpha}\bar{\varepsilon}\gamma_{abc}\varphi\hat{F}^{bc}(A) + \tfrac{i}{\alpha}\bar{\varepsilon}\gamma^b\varphi\hat{F}_{ab}(A) - 2i\bar{\varepsilon}\gamma_{abc}\varphi v^{bc} + 6i\bar{\varepsilon}\gamma^b\varphi v_{ab} \\
&\quad - 8i\bar{\varepsilon}^i\gamma_a\varphi^j t_{ij} + 2i\bar{\varepsilon}^i\gamma_{abc}\hat{\mathcal{R}}^{bcj}(Q)L_{ij} + 2ig\bar{\varepsilon}\gamma_a M\varphi - 4ig\bar{\varepsilon}^i\gamma_a\Omega^j L_{ij}\,, \\
\delta N &= -2i\bar{\varepsilon}\hat{\not{\mathcal{D}}}\varphi - 3i\bar{\varepsilon}\gamma\cdot v\varphi - 2i\bar{\varepsilon}^i\gamma\cdot\hat{\mathcal{R}}^j(Q)L_{ij} - 10i\bar{\varepsilon}^i\varphi^j t_{ij} + 4ig\bar{\varepsilon}^i\Omega^j L_{ij}\,. \qquad (3\cdot5)
\end{aligned}
$$

The vector component E_a is subject to the following $\boldsymbol{Q}$-invariant 'divergenceless' constraint:

$$\hat{\mathcal{D}}^a E_a + i\bar{\varphi}\gamma\cdot\hat{\mathcal{R}}(Q) + gMN + 4ig\bar{\Omega}\varphi + 2gY^{ij}L_{ij} = 0. \qquad (3\cdot6)$$

When the linear multiplet carries no gauge-group charges (i.e., $g=0$ in the above transformation law), this constraint can be solved and the constrained E^a can be expressed in terms of an unconstrained antisymmetric tensor-density gauge field $E^{\mu\nu}$, which possesses an additional gauge symmetry $\delta E^{\mu\nu} = \partial_\rho(\Lambda^{\mu\nu\rho})$:

$$
\begin{aligned}
E^a &= -e^{-1}e^a_\mu\hat{\mathcal{D}}_\nu E^{\mu\nu}, \\
\hat{\mathcal{D}}_\nu E^{\mu\nu} &\equiv \partial_\nu E^{\mu\nu} + 2ie\bar{\psi}_\nu\gamma^{\mu\nu}\varphi + 2ie\bar{\psi}^i_\nu\gamma^{\mu\nu\rho}\psi^j_\rho L_{ij}. \qquad (3\cdot7)
\end{aligned}
$$

Because of the covariance of E^a, the $\boldsymbol{Q}$ transformation law of $E^{\mu\nu}$ must take the following form, which is also in accordance with a direct calculation:

$$\delta E^{\mu\nu} = -2ie\bar{\varepsilon}\gamma^{\mu\nu}\varphi - 4ie\bar{\varepsilon}^i\gamma^{\mu\nu\rho}\psi^j_\rho L_{ij}. \qquad (3\cdot8)$$

3.4. *hypermultiplet*

The supersymmetry transformation law for the 5D hypermultiplet is given by

$$
\begin{aligned}
\delta\mathcal{A}^i_\alpha &= 2i\bar{\varepsilon}^i\zeta_\alpha\,, \\
\delta\zeta_\alpha &= -\hat{\not{\mathcal{D}}}\mathcal{A}^i_\alpha\varepsilon_i + \mathcal{F}^i_\alpha\varepsilon_i + 3t_{ij}\varepsilon^i\mathcal{A}^j_\alpha + gM_\alpha{}^\beta\varepsilon^i\mathcal{A}_{i\beta} + \tfrac{1}{4\alpha}\gamma\cdot\hat{F}(A)\varepsilon_i\mathcal{A}^i_\alpha + \gamma\cdot v\varepsilon_i\mathcal{A}^i_\alpha\,, \\
\delta\mathcal{F}^i_\alpha &= -2i\bar{\varepsilon}^i\Big(\hat{\not{\mathcal{D}}}\zeta_\alpha - \tfrac{1}{4\alpha}\gamma\cdot\hat{F}(A)\zeta_\alpha + \tfrac{1}{2}\gamma\cdot v\zeta_\alpha - \tfrac{1}{2}\mathcal{A}^i_\alpha\gamma^{ab}\hat{\mathcal{R}}_{abi}(Q) \\
&\qquad + 2\mathcal{A}^i_\alpha\chi_i - gM_\alpha{}^\beta\zeta_\beta + 2g\Omega^i{}_\alpha{}^\beta\mathcal{A}_{\beta i}\Big). \qquad (3\cdot9)
\end{aligned}
$$

The central charge transformation law is

$$
\begin{aligned}
\alpha\delta_Z(\theta)\mathcal{A}^i_\alpha &= \theta\mathcal{F}^i_\alpha\,, \\
\alpha\delta_Z(\theta)\zeta_\alpha &= -\theta\Big(\hat{\not{\mathcal{D}}}\zeta_\alpha - \tfrac{1}{4\alpha}\gamma\cdot\hat{F}(A)\zeta_\alpha + \tfrac{1}{2}\gamma\cdot v\zeta_\alpha - \tfrac{1}{2}\mathcal{A}^i_\alpha\gamma^{ab}\hat{\mathcal{R}}_{abi}(Q)
\end{aligned}
$$

$$+ 2\mathcal{A}^i_\alpha\chi_i - gM_\alpha{}^\beta\zeta_\beta + 2g\Omega^i{}_\alpha{}^\beta\mathcal{A}_{\beta i}\Big) ,$$

$$\alpha\delta_Z(\theta)\mathcal{F}^i_\alpha = \theta\Big((\hat{\mathcal{D}}^a\hat{\mathcal{D}}_a - \mathcal{C})\mathcal{A}^i_\alpha + 4i\bar{\zeta}_\alpha\chi^i + 4ig\bar{\Omega}^i{}_\alpha{}^\beta\zeta_\beta - 2gY^i{}_{j\alpha}{}^\beta\mathcal{A}^j_\beta - t^i{}_jt^j{}_k\mathcal{A}^k_\alpha$$
$$+ 2gt^i{}_jM_\alpha{}^\beta\mathcal{A}^j_\beta - g^2M_\alpha{}^\beta M_\beta{}^\gamma\mathcal{A}^i_\gamma - 2t^i{}_j\mathcal{F}^j_\alpha + 2gM_\alpha{}^\beta\mathcal{F}^i_\beta\Big) . \qquad (3\cdot10)$$

One should note that the covariant derivatives $\hat{\mathcal{D}}_a$ here as well as in (3·9) contain also the covariantization term $-\delta_Z(A_a)$ with respect to the central charge, so that these definitions of $\delta_Z(\theta)\zeta_\alpha$ and $\delta_Z(\theta)\mathcal{F}^i_\alpha$ by the second and third equations of (3·10) are recursive. However, they can easily be solved algebraically; for instance, the second equation gives

$$\delta_Z(\theta)\zeta_\alpha = -\theta\frac{\alpha + \not{A}}{\alpha^2 - A^2}\Big(\hat{\not{\mathcal{D}}}'\zeta_\alpha - \tfrac{1}{4\alpha}\gamma\cdot\hat{F}(A)\zeta_\alpha + \tfrac{1}{2}\gamma\cdot v\zeta_\alpha - \tfrac{1}{2}\mathcal{A}^i_\alpha\gamma^{ab}\hat{\mathcal{R}}_{abi}(Q)$$
$$+ 2\mathcal{A}^i_\alpha\chi_i - gM_\alpha{}^\beta\zeta_\beta + 2g\Omega^i{}_\alpha{}^\beta\mathcal{A}_{\beta i}\Big), \qquad (3\cdot11)$$

where $\hat{\mathcal{D}}'_a$ denotes a covariant derivative with the $-\delta_Z(A_a)$ term omitted.

3.5. *V-L invariant action formula*

The V-L action formula gives an invariant action for any pair of an Abelian vector multiplet $\boldsymbol{V} = (M, W_\mu, \Omega^i, Y^{ij})$ and a linear multiplet $\boldsymbol{L} = (L^{ij},\ \varphi^i,\ E_a \text{ (or } E^{\mu\nu}),\ N)$:

$$e^{-1}\mathcal{L}_{\rm VL} = Y^{ij}L_{ij} + 2i\bar{\Omega}\varphi + 2i\bar{\psi}^i_a\gamma^a\Omega^jL_{ij} + \tfrac{1}{2}M(N - 2i\bar{\psi}_b\gamma^b\varphi - 2i\bar{\psi}^i_a\gamma^{ab}\psi^j_bL_{ij})$$
$$- \tfrac{1}{2}W_a(E^a - 2i\bar{\psi}_b\gamma^{ba}\varphi + 2i\bar{\psi}^i_b\gamma^{abc}\psi^j_cL_{ij}) . \qquad (3\cdot12)$$

This formula is valid only when the liner multiplet $\boldsymbol{L}$ carries no gauge group charges or is charged only under the abelian group of this vector multiplet $\boldsymbol{V}$. When the linear multiplet carries no charges, the constrained component E^a can be replaced by the unconstrained anti-symmetric tensor $E^{\mu\nu}$ and the action formula (3·12) can be rewritten in a simpler form:

$$e^{-1}\mathcal{L}_{\rm VL} = Y^{ij}L_{ij} + 2i\bar{\Omega}\varphi + 2i\bar{\psi}^i_a\gamma^a\Omega^jL_{ij} + \tfrac{1}{2}M(N - 2i\bar{\psi}_b\gamma^b\varphi - 2i\bar{\psi}^i_a\gamma^{ab}\psi^j_bL_{ij})$$
$$+ \tfrac{1}{4}e^{-1}F_{\mu\nu}(W)E^{\mu\nu}. \qquad (3\cdot13)$$

§4. Vector multiplet action

Let $\boldsymbol{V}^I \equiv (M^I,\ W^I_\mu,\ \Omega^{I\,i},\ Y^{I\,ij})$ $(I = 1, 2, \cdots, n)$ be vector multiplets of a gauge group G, which we assume to be given generally by a direct product of simple groups G_i and $U(1)$ groups:

$$G = \prod_i G_i \times \prod_x U(1)_x, \qquad (G_i:\ \text{simple}). \qquad (4\cdot1)$$

The structure constant $f_{IJ}{}^K$ of G, $[t_I,\ t_J] = -f_{IJ}{}^Kt_K$, is nonvanishing only when all I, J and K belong to a common simple factor group G_i, and then it is the same as the structure constant of the simple group G_i. In addition to these $\boldsymbol{V}^I$ $(I = 1, 2, \cdots, n)$, we have a special vector multiplet, called 'central charge vector multiplet', which consists of the dilaton $\alpha = e_5^z$ and the gravi-photon $A_\mu = e_5^z e_\mu^5$ among the Weyl multiplet:

$$\boldsymbol{V}^{I=0} = (M^{I=0},\, W_\mu^{I=0},\, \Omega^{I=0\,i},\, Y^{I=0\,ij}) \equiv (\alpha,\, A_\mu,\, 0,\, 0). \tag{4·2}$$

We henceforth extend the group index I to run from 0 to n and use $I=0$ to denote this central charge vector multiplet as already written here. Corresponding to this extension, the gauge group G should also be understood henceforth to include the central charge $\boldsymbol{Z}$ as one of the Abelian $U(1)_x$ factor groups.

We can construct a linear multiplet $\boldsymbol{L} = (L^{ij},\, \varphi^i,\, E_a,\, N)$, denoted by $f(\boldsymbol{V})$, from vector multiplets $\boldsymbol{V}^I$ using any *homogeneous quadratic* polynomial in M^I,

$$f(M) = \tfrac{1}{2} f_{IJ} M^I M^J, \tag{4·3}$$

where I, J run from 0. We can identify

$$\begin{aligned}
L^{ij} &= -2ft^{ij} + f_I Y^{I\,ij} - if_{IJ}\bar\Omega^{Ii}\Omega^{Jj}\,,\\
\varphi^i &= -4f\chi^i + f_I\Big((\hat{\not{\mathcal{D}}} - \tfrac{1}{2}\gamma\cdot v)\Omega^{Ii} + t^i{}_j\Omega^{Ij} - g[M,\Omega^i]^I\Big)\\
&\quad - f_{IJ}\Big((\tfrac{1}{4}\gamma\cdot\hat G(W)^I - \tfrac{1}{2}\hat{\not{\mathcal{D}}} M^I)\Omega^{Ji} + Y^{I\,i}{}_j\Omega^{Jj}\Big)\,,\\
E_a &= \hat{\mathcal{D}}^b(4fv_{ab} + f_I\hat G_{ab}(W)^I + if_{IJ}\bar\Omega^I\gamma_{ab}\Omega^J)\\
&\quad - if_I\bar\Omega^I\gamma_{abc}\hat{\mathcal{R}}^{bc}(Q) + \tfrac{1}{8}\epsilon_{abcde}f_{IJ}\hat F^{bc}(W)^I\hat F^{de}(W)^J\\
&\quad - 2igf_I[\bar\Omega,\gamma_a\Omega]^I - 2igf_{IJ}\bar\Omega^I\gamma_a[M,\Omega]^J + gf_I[M,\hat{\mathcal{D}}_aM]^I\,,\\
N &= -4f(C + 4t^i{}_jt^j{}_i) - \hat{\mathcal{D}}^a\hat{\mathcal{D}}_af\\
&\quad + f_I\Big(-2\hat G_{ab}(W)^Iv^{ab} + \tfrac{1}{2\alpha}\hat G_{ab}(W)^I\hat F^{ab}(A) + 4t^i{}_jY^{I\,j}{}_i\\
&\qquad\quad + i\bar\Omega^I\gamma^{ab}\hat{\mathcal{R}}_{ab}(Q) - 16i\bar\Omega^I\chi + 2ig[\bar\Omega,\Omega]^I\Big)\\
&\quad + f_{IJ}\Big(-\tfrac{1}{4}\hat G(W)^I\cdot\hat G(W)^J + \tfrac{1}{2}\hat{\mathcal{D}}^aM^I\hat{\mathcal{D}}_aM^J - Y^{I\,i}{}_jY^{J\,j}{}_i + 2i\bar\Omega^I\hat{\not{\mathcal{D}}}\Omega^J\\
&\qquad\quad + \tfrac{i}{2\alpha}\bar\Omega^I\gamma\cdot\hat F(A)\Omega^J - i\bar\Omega^I\gamma\cdot v\Omega^J + 2i\bar\Omega^{Ii}\Omega^{Jj}t_{ij}\Big),
\end{aligned} \tag{4·4}$$

where the commutator $[X,Y]^I$ represents $[X,Y]^It_I \equiv X^JX^K[t_J,\, t_K]$, and

$$f \equiv f(M), \quad f_I \equiv \frac{\partial f}{\partial M^I}, \quad f_{IJ} \equiv \frac{\partial^2 f}{\partial M^I \partial M^J}. \tag{4·5}$$

If $f(M)$ is G-invariant, this linear multiplet $f(\boldsymbol{V})$ carries no group charge. Then, it must be possible to rewrite the embedded E^a component into the form (3·7) in terms of an antisymmetric tensor gauge field $E^{\mu\nu}$. Indeed, we find

$$\begin{aligned}
E^{\mu\nu} &= -e(4fv^{\mu\nu} + f_I\hat G^{\mu\nu}(W)^I + if_{IJ}\bar\Omega^I\gamma^{\mu\nu}\Omega^J + if\bar\psi_\rho\gamma^{\mu\nu\rho\sigma}\psi_\sigma - 2if_I\bar\psi_\lambda\gamma^{\mu\nu\lambda}\Omega^I)\\
&\quad - \tfrac{1}{4}f_{IJ}\epsilon^{\mu\nu\lambda\rho\sigma}W_\lambda^I\left(F_{\rho\sigma}^J(W) + \tfrac{1}{3}g[W_\rho,\, W_\sigma]^J\right)\,.
\end{aligned} \tag{4·6}$$

Now we use the V-L invariant action formula (3·13) to construct a general action for our set of vector multiplets $\{\boldsymbol{V}^I\}$. Since it applies only to Abelian vector multiplet $\boldsymbol{V}$, we first choose all the *Abelian* vector multiplets $\{\boldsymbol{V}^A\}$ from $\{\boldsymbol{V}^I\}$, and, for each abelian index A we prepare a G-invariant quadratic polynomial $f_A(M)$ to construct a neutral linear multiplet $\boldsymbol{L}_A = f_A(\boldsymbol{V})$ using Eqs. (I 5·3) and (I 5·5). We apply the V-L action formula (3·13) to these pairs of $\boldsymbol{V}^A$ and $\boldsymbol{L}_A = f_A(\boldsymbol{V})$ and sum over all the Abelian indices A. There we rewrite the super-covariantized quantities like $\hat{G}_{ab}(W)$, $\hat{\mathcal{D}}_a M^I$, $\hat{\mathcal{D}}_a \Omega^I$ etc into the usual covariant ones with $\mathcal{D}_\mu$ which is covariant only with respect to the homogeneous transformations, $\boldsymbol{M}_{ab}$, $\boldsymbol{U}_{ij}$, $\boldsymbol{D}$ and $\boldsymbol{G}$. Then, interestingly, many cancellations occur and the resultant expression is no more complicated than that written with supercovariantized quantities. The result is given by

$$
\begin{aligned}
&Y^{Aij}L_{Aij} + 2i\bar{\Omega}^A\varphi_A + 2i\bar{\psi}^i_a\gamma^a\Omega^{Aj}L_{Aij} \\
&\quad = f_A\left(2Y^A\cdot t - 4i\bar{\psi}\cdot\gamma t\Omega^A - 8i\bar{\Omega}^A\chi\right) \\
&\qquad + f_{A,J}\begin{pmatrix} -Y^A\cdot Y^J + 2i\bar{\Omega}^A(\not{\mathcal{D}} - \frac{1}{2}\gamma\cdot v + t)\Omega^J + i\bar{\Omega}^A\gamma^a(\frac{1}{2}\gamma\cdot G + \not{\mathcal{D}}M)^J\psi_a \\ - 2(\bar{\Omega}^A\gamma^a\gamma^{bc}\psi_a)(\bar{\psi}_b\gamma_c\Omega^J) + 2(\bar{\Omega}^A\gamma^a\gamma^b\psi_a)(\bar{\psi}_b\Omega^J) \end{pmatrix} \\
&\qquad + f_{A,JK}\begin{pmatrix} -2i\bar{\Omega}^A(\frac{1}{4}\gamma\cdot G - \frac{1}{2}\not{\mathcal{D}}M + Y)^J\Omega^K - i\bar{\Omega}^J Y^A\Omega^K \\ + 2(\bar{\Omega}^A\gamma^{ab}\Omega^J)(\bar{\psi}_a\gamma_b\Omega^K) + 2(\bar{\Omega}^A\gamma^a\Omega^J)(\bar{\psi}_a\Omega^K) \\ + 2(\bar{\psi}^i\cdot\gamma\Omega^{Aj})(\bar{\Omega}^J_{(i}\Omega^K_{j)}) \end{pmatrix},
\end{aligned}
\tag{4·7}
$$

$$
\begin{aligned}
&\tfrac{1}{2}M^A(N_A - 2i\bar{\psi}_b\gamma^b\varphi_A - 2i\bar{\psi}^i_a\gamma^{ab}\psi^j_b L_{Aij}) + \tfrac{1}{4}e^{-1}F^A_{\mu\nu}(W)E^{\mu\nu}_A \\
&\quad = \tfrac{1}{2}f_{A,J}\mathcal{D}^a M^A(\mathcal{D}_a M^J - 2i\bar{\psi}_b\gamma^a\gamma^b\Omega^J) \\
&\qquad + \tfrac{1}{2}f_A M^A\begin{pmatrix} -4C - 16t\cdot t - \frac{1}{2\alpha}F_{ab}(A)(4v^{ab} + i\bar{\psi}_c\gamma^{abcd}\psi_d) \\ + 8i\bar{\psi}\cdot\gamma\chi + 4i\bar{\psi}_a\gamma^{ab}t\psi_b \end{pmatrix} \\
&\qquad + \tfrac{1}{2}f_{A,J}M^A\begin{pmatrix} 4t\cdot Y^J - 16i\bar{\Omega}^J\chi - 8i\bar{\psi}\cdot\gamma t\Omega^J + 2ig[\bar{\Omega},\,\Omega]^J \\ - \frac{1}{2}G^{Jab}(W)(4v^{ab} + i\bar{\psi}_c\gamma^{abcd}\psi_d) \end{pmatrix} \\
&\qquad + \tfrac{1}{2}f_{A,JK}M^A\begin{pmatrix} -\frac{1}{4}G^J(W)\cdot G^K(W) + \frac{1}{2}\mathcal{D}_a M^J\mathcal{D}^a M^K - Y^J\cdot Y^K \\ +2i\bar{\Omega}^J(\not{\mathcal{D}} - \frac{1}{2}\gamma\cdot v + t)\Omega^K + i\bar{\psi}_a(\gamma\cdot G - 2\not{\mathcal{D}}M)^J\gamma^a\Omega^K \\ +(\bar{\Omega}^J\gamma^{ab}\Omega^K)(\bar{\psi}_a\psi_b) - 2(\bar{\psi}^i_a\gamma^{ab}\psi^j_b)(\bar{\Omega}^J_i\Omega^K_j) \\ -4(\bar{\psi}_a\gamma^{abc}\Omega^K)(\bar{\psi}_b\gamma_c\Omega^K) + 4(\bar{\psi}_{[a}\gamma_{b]}\Omega^J)(\bar{\psi}^a\gamma^b\Omega^K) \\ -2(\bar{\psi}_a\Omega^J)(\bar{\psi}^a\Omega^K) \end{pmatrix} \\
&\qquad -\tfrac{1}{4}G^A_{\mu\nu}(W)\Big(f_A(4v^{\mu\nu} + i\bar{\psi}_\rho\gamma^{\mu\nu\rho\sigma}\psi_\sigma) + if_{A,JK}\bar{\Omega}^J\gamma^{\mu\nu}\Omega^K \\
&\qquad\qquad + f_{A,J}(G^{J\mu\nu}(W) - 2i\bar{\psi}_\lambda\gamma^{\mu\nu}\gamma^\lambda\Omega^J)\Big) \\
&\qquad - e^{-1}\tfrac{1}{4}f_{A,JK}\epsilon^{\lambda\mu\nu\rho\sigma}W^A_\lambda F^J_{\mu\nu}(W)F^K_{\rho\sigma}(W)\ .
\end{aligned}
\tag{4·8}
$$

The action is given by the sum of Eqs. (4·7) and (4·8), where the indices J and K run over the whole group G while the (external) index A of $f_A(M)$ is restricted to run only over the Abelian subset of G. However, interestingly, this action can be shown to give an invariant action even if we extend the the index A of $f_A(M)$ to I running over the whole group G. In that case, the function $f_I(M)$ for the indices I belonging to the non-Abelian factor groups G_i of G should of course be a function giving the adjoint representation of G_i to satisfy the G-invariance, and the Chern-Simons term should also be generalized to the corresponding one. (A similar situation is also observed to happen in 6D case.[8]) Then the product $M^I f_I(M)$ becomes a general G-invariant *homogeneous cubic* polynomial in M, which, with changing the sign, is called 'norm function' and denoted as $\mathcal{N}(M)$ following Günayden, Sierra and Townsend:[11]

$$\mathcal{N}(M) \equiv c_{IJK} M^I M^J M^K \ \ (= -M^I f_I(M)), \tag{4·9}$$

where the coefficient c_{IJK} is totally symmetric with respect to the indices. Now the resultant action is characterized solely by this cubic polynomial $\mathcal{N}(M)$ and we find vector multiplet action:

$$\begin{aligned}
&e^{-1}\mathcal{L}_{\mathrm{VL}} \\
&= +\tfrac{1}{2}\mathcal{N}\left(4C + 16t\cdot t + \tfrac{1}{2\alpha}F_{ab}(A)(4v^{ab} + i\bar{\psi}_c\gamma^{abcd}\psi_d) - 8i\bar{\psi}\cdot\gamma\chi - 4i\bar{\psi}_a\gamma^{ab}t\psi_b\right) \\
&\quad - \mathcal{N}_I\begin{pmatrix} 2t\cdot Y^I - 8i\bar{\Omega}^I\chi - 4i\bar{\psi}\cdot\gamma t\Omega^I + ig[\bar{\Omega},\, \Omega]^I \\ - G^I_{ab}(W)(v^{ab} + \tfrac{i}{4}\bar{\psi}_c\gamma^{abcd}\psi_d) \end{pmatrix} \\
&\quad - \tfrac{1}{2}\mathcal{N}_{IJ}\begin{pmatrix} -\tfrac{1}{4}G^I(W)\cdot G^J(W) + \tfrac{1}{2}\mathcal{D}_a M^I\mathcal{D}^a M^J - Y^I\cdot Y^J \\ +2i\bar{\Omega}^I(\not{\mathcal{D}} - \tfrac{1}{2}\gamma\cdot v + t)\Omega^J + i\bar{\psi}_a(\gamma\cdot G(W) - 2\not{\mathcal{D}}M)^I\gamma^a\Omega^J \\ -2(\bar{\Omega}^I\gamma^a\gamma^{bc}\psi_a)(\bar{\psi}_b\gamma_c\Omega^J) + 2(\bar{\Omega}^I\gamma^a\gamma^b\psi_a)(\bar{\psi}_b\Omega^J) \end{pmatrix} \\
&\quad - \mathcal{N}_{IJK}\begin{pmatrix} -i\bar{\Omega}^I(\tfrac{1}{4}\gamma\cdot G(W) + Y)^J\Omega^K \\ +\tfrac{2}{3}(\bar{\Omega}^I\gamma^{ab}\Omega^J)(\bar{\psi}_a\gamma_b\Omega^K) + \tfrac{2}{3}(\bar{\psi}^i\cdot\gamma\Omega^{Ij})(\bar{\Omega}^J_{(i}\Omega^K_{j)}) \end{pmatrix} \\
&\quad + e^{-1}\mathcal{L}_{\text{C-S}}\ ,
\end{aligned} \tag{4·10}$$

where $\mathcal{N}_I = \partial\mathcal{N}/\partial M^I$, $\mathcal{N}_{IJ} = \partial^2\mathcal{N}/\partial M^I\partial M^J$, etc, and $\mathcal{L}_{\text{C-S}}$ is the Chern-Simons term:

$$\begin{aligned}
\mathcal{L}_{\text{C-S}} = \frac{1}{8}c_{IJK}\epsilon^{\lambda\mu\nu\rho\sigma}W^I_\lambda\Big(&F^J_{\mu\nu}(W)F^K_{\rho\sigma}(W) + \tfrac{1}{2}g[W_\mu, W_\nu]^J F^K_{\rho\sigma}(W) \\
&+ \tfrac{1}{10}g^2[W_\mu, W_\nu]^J[W_\rho, W_\sigma]^K\Big)\,.
\end{aligned} \tag{4·11}$$

§5. Hypermultiplet Action

Now let $\boldsymbol{H}^\alpha = (\mathcal{A}^\alpha_i,\, \zeta^\alpha,\, \mathcal{F}^\alpha_i)$ $(\alpha = 1, 2, \cdots, 2r)$ be a set of hypermultiplets which belongs to a representation ρ of the gauge group G. Under the G transformation it transforms as $\delta_G(\theta)\boldsymbol{H}^\alpha = \sum_{I=1}^n g\theta^I\rho(t_I)^\alpha{}_\beta\boldsymbol{H}^\beta$. The ordinary matrix notation used for the vector multiplet in the preceding section was, for instance, $M = M^I t_I$ and the matrix t_I denoted an adjoint representation $\mathrm{ad}(t_I)$ of G. The representation ρ here can of course be different from the adjoint representation ad. However, to avoid cumbersome looking, we abuse the matrix notation and write, e.g., $M\mathcal{A}^\alpha_i = M^\alpha{}_\beta\mathcal{A}^\beta_i$ to denote actually $\rho(M)^\alpha{}_\beta\mathcal{A}^\beta_i = M^I\rho(t_I)^\alpha{}_\beta\mathcal{A}^\beta_i$. (Note $M\mathcal{A}_{\alpha i} = M_{\alpha\beta}\mathcal{A}^\beta_i$.)

The invariant action for the hypermultiplets can be obtained directly from the action in 6D. Again we rewrite the supercovariant derivative $\hat{\mathcal{D}}_\mu$ in terms of the usual covariant derivative $\mathcal{D}_\mu$ which is covariant only with respect $\boldsymbol{M}_{ab}$, $\boldsymbol{U}_{ij}$, $\boldsymbol{D}$ and $\boldsymbol{G}$. (Note that covariantization with respect to the central charge $\boldsymbol{Z}$ transformation is also taken out.) Then we obtain the following action for the kinetic term of the hypermultiplets:

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\rm kin} = {} & \mathcal{D}^a\mathcal{A}_i^{\bar\alpha}\mathcal{D}_a\mathcal{A}_\alpha^i - 2i\bar\zeta^{\bar\alpha}\not{\mathcal{D}}\zeta_\alpha + \tfrac{i}{2\alpha}\bar\zeta^{\bar\alpha}\gamma\cdot F(A)\zeta_\alpha - i\bar\zeta^{\bar\alpha}\gamma\cdot v\zeta_\alpha \\
& + 2ig\bar\zeta^{\bar\alpha}M_\alpha{}^\beta\zeta_\beta + \mathcal{A}_i^{\bar\alpha}(t+gM)^2\mathcal{A}_\alpha^i - 4i\bar\psi_a^i\zeta_\alpha\gamma^b\gamma^a\mathcal{D}^a\mathcal{A}_i^{\bar\alpha} \\
& + \begin{pmatrix} 2i\bar\zeta_\alpha\gamma^{ab}\mathcal{R}_{ab}{}^i(Q) - 8i\bar\zeta_\alpha\chi^i \\ +\frac{i}{\alpha}\bar\psi_a^i\gamma^{abc}\zeta_\alpha\hat{F}_{bc}(A) - 4i\bar\psi^{ia}\gamma^b\zeta_\alpha v_{ab} + 4i\bar\psi_{aj}\gamma^a\zeta_\alpha t^{ij} \\ -8ig\bar\Omega_\alpha^i{}^\beta\zeta_\beta + 4ig\bar\psi_a^i\gamma^a M_\alpha{}^\beta\zeta_\beta \end{pmatrix}\mathcal{A}_i^{\bar\alpha} \\
& - 2i\bar\psi_a^{(i}\gamma^{abc}\psi_c^{j)}\mathcal{A}_j^{\bar\alpha}\mathcal{D}_b\mathcal{A}_{\alpha i} \\
& + \begin{pmatrix} C + \frac{1}{4}\mathcal{R}(M) + \frac{i}{2}\bar\psi_a\gamma^{abc}\mathcal{R}_{bc}(Q) \\ -2i\bar\psi_a\gamma^a\chi + \frac{1}{8\alpha^2}\hat{F}(A)^2 - v^2 + 2t\cdot t \\ -\frac{i}{4\alpha}\bar\psi_a\gamma^{abcd}\psi_b\hat{F}_{cd}(A) + i\bar\psi_a\psi_b v^{ab} - i\bar\psi_a^i\gamma^{ab}\psi_b^j t_{ij} \end{pmatrix}\mathcal{A}^2 \\
& + 2gY_{\alpha\beta}^{ij}\mathcal{A}_i^{\bar\alpha}\mathcal{A}_j^\beta + 4ig\bar\psi_a^{(i}\gamma^a\Omega_{\alpha\beta}^{j)}\mathcal{A}_i^{\bar\alpha}\mathcal{A}_j^\beta \\
& + 2ig\bar\psi_a^{(i}\gamma^{ab}\psi_b^{j)}\mathcal{A}_i^{\bar\alpha}M_\alpha{}^\beta\mathcal{A}_{\beta j} + (1 - A^aA_a/\alpha^2)\mathcal{F}_i^{\bar\alpha}\mathcal{F}_\alpha^i \\
& + \bar\psi_a\gamma_b\psi_c\bar\zeta^{\bar\alpha}\gamma^{abc}\zeta_\alpha - \tfrac{1}{2}\bar\psi^a\gamma^{bc}\psi_a\bar\zeta^{\bar\alpha}\gamma_{bc}\zeta_\alpha\ ,
\end{aligned}
\tag{5·1}
$$

where the contractions between pairs of a barred index $\bar\alpha$ and α imply

$$
\mathcal{A}_i^{\bar\alpha}\mathcal{A}_{\alpha j} \equiv \mathcal{A}_i^\beta d_\beta{}^\alpha\mathcal{A}_{\alpha j}, \qquad \mathcal{A}^2 \equiv \mathcal{A}_i^{\bar\alpha}\mathcal{A}_\alpha^i, \qquad \bar\zeta^{\bar\alpha}\zeta_\alpha \equiv \bar\zeta^\beta d_\beta{}^\alpha\zeta_\alpha, \tag{5·2}
$$

by using a G-invariant metric $d_\alpha{}^\beta$ introduced in Eqs. (I B·22) and (I B·23). This metric $d_\alpha{}^\beta$ is, in its standard form, diagonal and takes the values ± 1. [12] The hypermultiplets can have masses and the invariant action for the mass term is given by Eq. (I 4·14) which reads

$$
e^{-1}\mathcal{L}_{\rm mass} = m\eta^{\alpha\beta}\begin{pmatrix} -A^a\mathcal{D}_a\mathcal{A}_{\alpha i}\mathcal{A}_\beta^i - (1 - A^aA_a/\alpha^2)\alpha\mathcal{F}_{\alpha i}\mathcal{A}_\beta^i \\ -2i\bar\psi_a^i\zeta_\alpha A^a\mathcal{A}_{\beta i} + \alpha\mathcal{A}_{i\alpha}(t+gM)\mathcal{A}_\beta^i \\ +i(-\alpha\bar\zeta_\alpha\zeta_\beta + A_a\bar\zeta_\alpha\gamma^a\zeta_\beta) \\ +2i\mathcal{A}_{\alpha i}(-\alpha\bar\psi_a^i\gamma^a\zeta_\beta + \bar\psi_a^i\gamma^{ab}\zeta_\beta A_b) \\ +i\mathcal{A}_{\alpha i}\mathcal{A}_{\beta j}(-\alpha\bar\psi_a^i\gamma^{ab}\psi_b^j + \bar\psi_a^i\gamma^{abc}\psi_c^j A_b) \end{pmatrix}. \tag{5·3}
$$

(Note that m is a dimensionless parameter and the actual mass is $\propto m\langle\alpha\rangle$.) Here $\eta^{\alpha\beta}$ is a symmetric G-invariant tensor. [12] Interestingly, this mass term turns out to be automatically included in the previous kinetic term action (5·1) and need not be considered separately, provided that we complete the square for the terms containing the auxiliary fields $\mathcal{F}_i^\alpha$ in $\mathcal{L}_{\rm kin}+\mathcal{L}_{\rm mass}$. (Essentially the same observation was already made in Ref. 12).) Doing so, the $\mathcal{F}_i^\alpha$ terms become

$$
(1 - A^aA_a/\alpha^2)\tilde{\mathcal{F}}_i^{\bar\alpha}\tilde{\mathcal{F}}_\alpha^i \qquad \text{with}\ \ \tilde{\mathcal{F}}_i^\alpha \equiv \mathcal{F}_i^\alpha + \tfrac{1}{2}m\alpha(d^{-1})_\gamma{}^\alpha\eta^{\gamma\beta}\mathcal{A}_{\beta i}\,, \tag{5·4}
$$

and then, all the other terms in $\mathcal{L}_{\rm mass}$ can be absorbed in the kinetic lagrangian $\mathcal{L}_{\rm kin}$ if we extend the gauge index I of the generators t_I acting on the hypermultiplets to run also from 0 and introduce

$$
(gt_{I=0})^{\alpha\beta} \equiv \tfrac{1}{2}m(d^{-1})_\gamma{}^\alpha\eta^{\gamma\beta}, \tag{5·5}
$$

so that gW_μ in $\mathcal{D}_\mu$ and M are now understood to be

$$gW_\mu = \sum_{I=1}^{n} W_\mu^I (gt_I) + A_\mu (gt_0)\,,$$
$$gM = \sum_{I=1}^{n} M^I (gt_I) + \alpha (gt_0)\,. \tag{5·6}$$

§6. Rewriting the action

Now the invariant action for our Yang-Mills-matter system coupled to supergravity is given by the sum $\mathcal{L} = \mathcal{L}_{\rm VL}[(4{\cdot}10)] + \mathcal{L}_{\rm kin}[(5{\cdot}1)]$, where in $\mathcal{L}_{\rm kin}$ the $\mathcal{F}^2$ term is replaced by (5·4) and Eq. (5·6) is understood.

In order to make the Einstein term canonical we take

$$\mathcal{N}(M) = 1. \tag{6·1}$$

as a gauge fixing condition for the $\boldsymbol{D}$ gauge.[13)]

For the purpose mainly to solve the mixing between the Rarita-Schwinger field ψ_μ^i and the gaugino field component $\Omega_i \equiv \mathcal{N}_I \Omega_i^I$, we define new Rarita-Schwinger, gaugino and hypermultiplet fermions by

$$\psi^{\rm N}_{\mu i} \equiv \psi_{\mu i} - \frac{1}{3\mathcal{N}}\gamma_\mu \Omega_i\,, \tag{6·2}$$
$$\lambda_i^I \equiv \Omega_i^I - \frac{M^I}{3\mathcal{N}}\Omega_i = \mathcal{P}^I_J \Omega^J_i\,, \tag{6·3}$$
$$\xi_\alpha \equiv \zeta_\alpha - \frac{\mathcal{A}^i_\alpha}{\mathcal{N}}\Omega_i\,. \tag{6·4}$$

$\mathcal{P}^I_J$ is the projection operator

$$\mathcal{P}^I_J \equiv \delta^I_J - \frac{M^I \mathcal{N}_J}{3\mathcal{N}} \quad \to \quad \mathcal{P}^I_J M^J = \mathcal{P}^I_J \mathcal{N}_I = 0\,, \tag{6·5}$$

so that this new gaugino fields λ_i^I satisfy

$$\mathcal{N}_I \lambda_i^I = 0\,. \tag{6·6}$$

The new hypermultiplet fermion ξ_α is orthogonal to $\mathcal{A}_i^{\bar\alpha}$ on-shell:

$$\mathcal{A}_i^{\bar\alpha}\xi_\alpha = \zeta_i - \frac{\mathcal{A}^2}{2\mathcal{N}}\Omega_i = (\zeta_i + \Omega_i) - \frac{1}{2\mathcal{N}}\Omega_i(\mathcal{A}^2 + 2\mathcal{N})\,. \tag{6·7}$$

We also redefine the auxiliary fields by

$$\begin{aligned}
\tilde V_a^{ij} &= V_a^{ij} + \frac{1}{2\mathcal{N}}\Big(4i\bar\Omega^{(i}\psi_a^{j)} + \frac{2i}{3\mathcal{N}}\bar\Omega^i\gamma_a\Omega^j\Big),\\
\tilde v_{ab} &= v_{ab} - \frac{1}{2\alpha}F_{ab}(A) + i\bar\psi_a\psi_b + i\frac{2}{3\mathcal{N}}\bar\psi_{[a}\gamma_{b]}\Omega + \frac{i}{9\mathcal{N}^2}\bar\Omega\gamma_{ab}\Omega,\\
\tilde Y^{Iij} &= \mathcal{P}^I_J Y^{Jij} - \frac{2i}{3\mathcal{N}}\bar\lambda^{I(i}\Omega^{j)}\,,\\
\tilde t^{ij} &= t^{ij} - \frac{\mathcal{N}_I Y^{Iij}}{3\mathcal{N}} + \frac{i}{9\mathcal{N}^2}\bar\Omega^{(i}\Omega^{j)},
\end{aligned} \tag{6·8}$$

We now rewrite the action by using the field redefinitions (6·2), (6·3), (6·4) and (6·8) everywhere. From here on, *the Rarita-Schwinger field is always the new variable* $\psi_\mu^{\rm N}$ *and we omit the cumbersome superscript* N.

This is actually a very tedious computations. Surprisingly, however, all the terms containing $\Omega_i \equiv \mathcal{N}^I \Omega_{Ii}$ completely cancel out in the action. After all, the action is finally found to take the form

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\rm hyper} + \mathcal{L}_{\rm vector} + \mathcal{L}_{\rm C\text{-}S} + \mathcal{L}_{\rm aux} \ , \\
e^{-1}\mathcal{L}_{\rm hyper} &= \nabla^a \mathcal{A}_i^{\bar\alpha} \nabla_a \mathcal{A}_\alpha^i - 2i\bar\xi^{\bar\alpha}(\nabla\!\!\!\!/ + gM)\xi_\alpha \\
&\quad + \mathcal{A}_i^{\bar\alpha}(gM)^2{}_\alpha{}^\beta \mathcal{A}_\beta^i - 4i\bar\psi_a^i\gamma^b\gamma^a\xi_\alpha\nabla_b\mathcal{A}_i^{\bar\alpha} - 2i\bar\psi_a^{(i}\gamma^{abc}\psi_c^{j)}\mathcal{A}_j^{\bar\alpha}\nabla_b\mathcal{A}_{\alpha i} \\
&\quad + \mathcal{A}_i^{\bar\alpha}\Big(8ig\bar\lambda^i_{\alpha\beta}\xi^\beta - 4ig\bar\psi_a^i\gamma^a M_{\alpha\beta}\xi^\beta \\
&\qquad\qquad + 4ig\bar\psi_a^{(i}\gamma^a\lambda^{j)}_{\alpha\beta}\mathcal{A}_j^\beta - 2ig\bar\psi_a^{(i}\gamma^{ab}\psi_b^{j)}M_{\alpha\beta}\mathcal{A}_j^\beta\Big) \\
&\quad + \bar\psi_a\gamma_b\psi_c\bar\xi^{\bar\alpha}\gamma^{abc}\xi_\alpha - \tfrac{1}{2}\bar\psi^a\gamma^{bc}\psi_a\bar\xi^{\bar\alpha}\gamma_{bc}\xi_\alpha \ , \\
e^{-1}\mathcal{L}_{\rm vector} &= -\tfrac{1}{2}R(\omega) - 2i\bar\psi_\mu\gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho + (\bar\psi_a\psi_b)(\bar\psi_c\gamma^{abcd}\psi_d + \bar\psi^a\psi^b) \\
&\quad - \mathcal{N}_I\left(ig[\bar\lambda,\lambda]^I - \tfrac{i}{4}\bar\psi_c\gamma^{abcd}\psi_d F_{ab}(W)^I\right) \\
&\quad + a_{IJ}\begin{pmatrix} -\tfrac{1}{4}F(W)^I\cdot F(W)^J + \tfrac{1}{2}\nabla_a M^I\nabla^a M^J \\ +2i\bar\lambda^I\nabla\!\!\!\!/\lambda^J + i\bar\psi_a(\gamma\cdot F(W) - 2\nabla\!\!\!\!/ M)^I\gamma^a\lambda^J \\ -2(\bar\lambda^I\gamma^a\gamma^{bc}\psi_a)(\bar\psi_b\gamma_c\lambda^J) + 2(\bar\lambda^I\gamma^a\gamma^b\psi_a)(\bar\psi_b\lambda^J) \end{pmatrix} \\
&\quad - \mathcal{N}_{IJK}\begin{pmatrix} -i\bar\lambda^I\tfrac{1}{4}\gamma\cdot F(W)^J\lambda^K \\ +\tfrac{2}{3}(\bar\lambda^I\gamma^{ab}\lambda^J)(\bar\psi_a\gamma_b\lambda^K) + \tfrac{2}{3}(\bar\psi^i\cdot\gamma\lambda^{Ij})(\bar\lambda^J_{(i}\lambda^K_{j)}) \end{pmatrix} \\
&\quad + \tfrac{1}{8}\left(2\bar\psi_a\psi_b + \bar\xi^{\bar\alpha}\gamma_{ab}\xi_\alpha + a_{IJ}\bar\lambda^I\gamma_{ab}\lambda^J\right)^2 \\
&\quad + i\tfrac{1}{4}\mathcal{N}_I F(W)^I\left(2\bar\psi_a\psi_b + \bar\xi^{\bar\alpha}\gamma_{ab}\xi_\alpha + a_{IJ}\bar\lambda^I\gamma_{ab}\lambda^J\right) \\
&\quad + \left(\mathcal{A}^{\bar\alpha i}\nabla_a\mathcal{A}^j_\alpha + ia_{IJ}\bar\lambda^{Ii}\gamma_a\lambda^{Jj}\right)^2 \\
&\quad - \tfrac{1}{4}(a^{IJ} - M^I M^J)\mathcal{Y}_I^{ij}\mathcal{Y}_{J\,ij} \ . \qquad\qquad (6\cdot9)
\end{aligned}
$$

$\mathcal{L}_{\rm aux}$ is the perfect square terms of the auxiliary fields, which vanish on shell:

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\rm aux} &= C'''(\mathcal{A}^2 + 2) - 8i\chi''^i\mathcal{A}_i^{\bar\alpha}\xi_\alpha \\
&\quad + 2(\tilde v - \tilde v_{\rm sol})^2 - (\tilde V - \tilde V_{\rm sol})^{ij}(\tilde V - \tilde V_{\rm sol})_{ij} \\
&\quad - 3(\tilde t - \tilde t_{\rm sol})^{ij}(\tilde t - \tilde t_{\rm sol})_{ij} + a_{IJ}(\tilde Y^I - \tilde Y^I_{\rm sol})^{ij}(\tilde Y^J - \tilde Y^J_{\rm sol})_{ij} \\
&\quad + \left(1 - A^2/\alpha^2\right)(\mathcal{F}_i^{\bar\alpha} - \mathcal{F}^{\bar\alpha}_{{\rm sol}\,i})(\mathcal{F}^i_\alpha - \mathcal{F}^i_{{\rm sol}\,\alpha}) \ , \qquad (6\cdot10)
\end{aligned}
$$

$X_{\rm sol}$ are the solutions for the auxiliary fields:

$$
\begin{aligned}
\tilde V^{ij}_{{\rm sol}\,a} &= -\frac{1}{2\mathcal{N}}\Big(2\mathcal{A}^{\bar\alpha(i}\nabla_a\mathcal{A}^{j)}_\alpha - i\mathcal{N}_{IJ}\bar\lambda^{Ii}\gamma_a\lambda^{Jj}\Big), \\
\tilde v_{{\rm sol}\,ab} &= -\frac{1}{4\mathcal{N}}\Big\{\mathcal{N}_I F_{ab}(W)^I - i\left(6\mathcal{N}\bar\psi_a\psi_b + \bar\xi^{\bar\alpha}\gamma_{ab}\xi_\alpha - \tfrac{1}{2}\mathcal{N}_{IJ}\bar\lambda^I\gamma_{ab}\lambda^J\right)\Big\},
\end{aligned}
$$

$$\tilde{Y}^{Iij}_{\rm sol} = -\tfrac{1}{2}a^{IJ}\mathcal{P}_J^K\mathcal{Y}^{ij}_K = -\tfrac{1}{2}\mathcal{P}^I_J a^{JK}\mathcal{Y}^{ij}_K = -\left(\tfrac{1}{2}a^{IJ} - \tfrac{1}{3}M^IM^J\right)\mathcal{Y}^{ij}_J$$
$$\text{with}\quad \mathcal{Y}^{ij}_I \equiv 2\mathcal{A}^{(i}_\alpha(gt_I)^{\bar\alpha\beta}\mathcal{A}^{j)}_\beta + i\mathcal{N}_{IJK}\bar\lambda^{Ji}\lambda^{Kj},$$
$$\tilde{t}^{ij}_{\rm sol} = -\frac{1}{6\mathcal{N}}M^I\mathcal{Y}^{ij}_I = -\frac{1}{6\mathcal{N}}\Big(2\mathcal{A}^{(i}_\alpha(gM)^{\bar\alpha\beta}\mathcal{A}^{j)}_\beta + i\mathcal{N}_{IJ}\bar\lambda^{Ii}\lambda^{Jj}\Big), \tag{6·11}$$

where a^{IJ} is the inverse of the metric a_{IJ} of the vector multiplet kinetic terms:

$$a_{IJ} \equiv -\tfrac{1}{2}\frac{\partial^2}{\partial M^I\partial M^J}\ln\mathcal{N} = -\frac{1}{2\mathcal{N}}\Big(\mathcal{N}_{IJ} - \frac{\mathcal{N}_I\mathcal{N}_J}{\mathcal{N}}\Big), \quad a^{IJ} \equiv (a^{-1})^{IJ}, \tag{6·12}$$

which enjoys the properties

$$a_{IJ}M^J = \mathcal{N}_I/2\mathcal{N} \ \rightarrow\ a^{IJ}\mathcal{N}_J/2\mathcal{N} = M^I, \qquad a^{IJ}\mathcal{P}_J^K = \mathcal{P}^I_J a^{JK}. \tag{6·13}$$

The Lagrange multiplier auxiliary fields C and χ have been redefined into

$$\begin{aligned}
C'' &= C + \tfrac{1}{4}\mathcal{R}(M) + \tfrac{i}{2}\bar\psi_a\gamma^{abc}\mathcal{R}_{bc}(Q) - 2i\bar\psi_a\gamma^a\chi + \tfrac{1}{8\alpha^2}\hat{F}(A)^2 \\
&\quad - v^2 - \tfrac{i}{4\alpha}\bar\psi_a\gamma^{abcd}\psi_b\hat{F}_{cd}(A) + i\bar\psi_a\psi_b v^{ab} - i\bar\psi^i_a\gamma^{ab}\psi^j_b t_{ij} \\
&\quad + \tfrac{9}{4}b^2 + \tfrac{5}{2}t{\cdot}t + \tfrac{3}{2}e^{-1}\nabla_\mu(eb^\mu) + \tfrac{1}{2}V^{ij}_aV^a_{ij} + i\bar\psi^i_b\gamma^{bac}\psi^j_c V_{a\,ij} \\
&\quad + \frac{1}{2\mathcal{N}^2}\Big\{-2i\bar\Omega\not{\nabla}\Omega + \Big(\bar\psi_a\gamma_b\psi_c(\bar\Omega\gamma^{abc}\Omega) - \tfrac{1}{2}\bar\psi^a\gamma_{bc}\psi_a(\bar\Omega\gamma^{bc}\Omega)\Big) \\
&\qquad\qquad - i\bar\Omega\gamma{\cdot}(v - \tfrac{1}{2\alpha}F(A))\Omega\Big\} + \frac{i}{\mathcal{N}}e^{-1}\nabla_\mu(e\bar\psi_a\gamma^\mu\gamma^a\Omega)\,, \\
\chi''_i &= \chi_i - \tfrac{1}{4}\gamma^{ab}\mathcal{R}_{abi}(Q) + \tfrac{1}{8\alpha}\gamma^{abc}\psi_{ai}\hat{F}_{bc}(A) \\
&\quad + \tfrac{1}{2}\gamma_b\psi_{ai}v^{ab} + \tfrac{1}{2}t\gamma{\cdot}\psi_i + \gamma^a\gamma^b(\tfrac{1}{2}V_b - \tfrac{3}{4}b_b)\psi_{ai} \\
&\quad + \frac{1}{4\mathcal{N}}\Big\{\Big(e^{-1}\nabla_\mu(ee^\mu_a)\,\gamma^a\Omega_i + 2\not{\nabla}\Omega_i\Big) \\
&\quad + i\,\Big(\gamma^{abc}\Omega_i(\bar\psi_a\gamma_b\psi_c) - \tfrac{1}{2}\gamma^{bc}\Omega_i(\bar\psi^a\gamma_{bc}\psi_a)\Big) + \gamma{\cdot}(v - \tfrac{1}{2\alpha}F(A))\Omega_i\Big\}.
\end{aligned} \tag{6·14}$$

and the multiplier term $C''(\mathcal{A}^2+2\mathcal{N}) - 8i\bar\chi''(\zeta+\Omega)$ has been rewritten into the form of the first line in $\mathcal{L}_{\rm aux}$ by using Eq. (6·7) and redefining further the C'' field as

$$C''' = C'' - i\frac{4}{\mathcal{N}}\bar\chi''\Omega\,. \tag{6·15}$$

With this rewriting, the explicit Ω_i has been completely dispelled from the action. Note that the final action (6·9) with (6·10) is everywhere written in terms of the new variables although the superscript N was omitted. In particular, the spin connection ω^{ab}_μ in the covariant derivative ∇_μ and $R(\omega)$ is the new one given by Eq. (3·2) with the new ψ_μ used and b_μ set equal to 0. By using this ω^{ab}_μ, $R(\omega)$ is given as usual:

$$R_{\mu\nu}{}^{ab}(\omega) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{[ac}\omega_{\nu]c}{}^{b]}, \quad R_{ab}(\omega) \equiv R_{ac}{}^c{}_b(\omega)\,, \quad R(\omega) \equiv R_a{}^a(\omega)\,. \tag{6·16}$$

§7. Final form of supersymmetry transformation

Now the supersymmetry ($\boldsymbol{Q}$) transformation $\delta_Q(\varepsilon)$ is modified since we have fixed the $\boldsymbol{D}$ gauge by (6·1) and made various field redefinitions (6·2), (6·3) and (6·4). The proper $\boldsymbol{Q}$-transformation is found to be the following linear combination of the original transformations of $\boldsymbol{Q}$, dilatation $\boldsymbol{D}$, local-Lorentz $\boldsymbol{M}$ and $SU(2)$ $\boldsymbol{U}$:

$$\begin{aligned}
&\delta_Q^{\rm N}(\varepsilon) = \delta_Q(\varepsilon) + \delta_D(\rho(\varepsilon)) + \delta_M(\lambda^{ab}(\varepsilon)) + \delta_U(\theta^{ij}(\varepsilon)),\\
&\quad\rho(\varepsilon) \equiv -\tfrac{2i}{3\mathcal{N}}\bar{\varepsilon}\varOmega, \quad \lambda^{ab}(\varepsilon) \equiv \tfrac{2i}{3\mathcal{N}}\bar{\varepsilon}\gamma^{ab}\varOmega, \quad \theta^{ij}(\varepsilon) \equiv -\tfrac{2i}{\mathcal{N}}\bar{\varepsilon}^{(i}\varOmega^{j)}.
\end{aligned}\tag{7·1}$$

The $\boldsymbol{Q}$-transformation laws of the Weyl multiplet are given by

$$\begin{aligned}
\delta e_\mu{}^a &= -2i\bar{\varepsilon}\gamma^a\psi_\mu\,,\\
\delta\psi_\mu^i &= \mathcal{D}_\mu\varepsilon^i + \gamma_\mu\tilde{t}^i{}_j\varepsilon^j + \tfrac{1}{2}\gamma_{\mu ab}\varepsilon^i\tilde{v}^{ab} + \tfrac{\mathcal{N}_I}{12\mathcal{N}}\gamma_\mu\gamma\cdot\hat{F}^I(W)\varepsilon^i + \tfrac{\mathcal{N}_{IJ}}{3\mathcal{N}}\gamma_\mu\lambda^{Ii}(2i\bar{\lambda}^J\varepsilon)\,,\\
\delta\tilde{V}_\mu^{ij} &= -4i\bar{\varepsilon}^{(i}\gamma_\mu\tilde{\chi}^{j)} - i\bar{\varepsilon}^{(i}\gamma_{\mu ab}\hat{\mathcal{R}}^{abj)}(Q) + 4i\bar{\varepsilon}^{(i}\gamma\cdot\left(\tilde{v} + \tfrac{\mathcal{N}_I}{4\mathcal{N}}\hat{F}^I(W)\right)\psi_\mu^{j)}\\
&\quad - 6i(\bar{\varepsilon}\psi_\mu)\tilde{t}^{ij} + \tfrac{4\mathcal{N}_{IJ}}{\mathcal{N}}\left((\bar{\psi}_\mu\lambda^I)\bar{\varepsilon}^{(i}\lambda^{j)J} - (\bar{\varepsilon}\lambda^I)\bar{\psi}_\mu^{(i}\lambda^{j)J}\right),\\
\delta\tilde{t}^{ij} &= 4i\bar{\varepsilon}^{(i}\tilde{\chi}^{j)} + i\bar{\varepsilon}^{(i}\gamma\cdot\hat{\mathcal{R}}^{j)}(Q) + \tfrac{2i\mathcal{N}_{IJ}}{3\mathcal{N}}\left(\bar{\varepsilon}^{(i}\hat{\not{\mathcal{D}}}M^I\lambda^{j)J} - (\bar{\varepsilon}\lambda^I)\tilde{Y}^{Jij}\right),\\
\delta\tilde{v}_{ab} &= -2i\bar{\varepsilon}\gamma_{ab}\tilde{\chi} - \tfrac{i}{2}\bar{\varepsilon}\gamma_{ab}\gamma\cdot\hat{\mathcal{R}}(Q) + \tfrac{i}{4}\bar{\varepsilon}\gamma_{abcd}\hat{\mathcal{R}}^{cd}(Q) - i\bar{\varepsilon}\hat{\mathcal{R}}_{ab}(Q)\,,\\
\delta\tilde{\chi}^i &= \tfrac{1}{2}\varepsilon^i\tilde{C} - \tfrac{1}{2}(\hat{\not{\mathcal{D}}}\boldsymbol{\Gamma}')\varepsilon^i - \tfrac{1}{4}\gamma\cdot\tilde{v}\,\boldsymbol{\Gamma}'\varepsilon^i + \tfrac{1}{4}\gamma\cdot\hat{\mathcal{R}}(U)\varepsilon^i\\
&\quad+\tfrac{1}{2}\gamma^a\boldsymbol{\Gamma}'\left(\gamma_a\tilde{t}\varepsilon^i + \tfrac{1}{2}\gamma_{abc}\varepsilon^i\tilde{v}^{bc} + \tfrac{\mathcal{N}_I}{12\mathcal{N}}\gamma_a\gamma\cdot\hat{F}^I(W)\varepsilon^i - \tfrac{\mathcal{N}_{IJ}}{3\mathcal{N}}\gamma_a\lambda^{Ii}(2i\bar{\varepsilon}\lambda^J)\right),\\
\delta\tilde{C} &= -2i\bar{\varepsilon}\hat{\not{\mathcal{D}}}\tilde{\chi} + \tfrac{1}{2}i\bar{\varepsilon}\{\gamma^{ab},\,\boldsymbol{\Gamma}'\}\hat{\mathcal{R}}_{ab}(Q) + \tfrac{2i}{3}\bar{\varepsilon}\boldsymbol{\Gamma}'\tilde{\chi} + \tfrac{i}{3}\bar{\varepsilon}\gamma\cdot\tilde{v}\tilde{\chi}\,,\\
\delta\omega_\mu{}^{ab} &= -2i\bar{\varepsilon}\gamma^{[a}\hat{\mathcal{R}}_\mu{}^{b]}(Q) - i\bar{\varepsilon}\gamma_\mu\hat{\mathcal{R}}^{ab}(Q)\\
&\quad - 2i\bar{\varepsilon}\gamma^{abcd}\psi_\mu\left(\tilde{v}_{cd} + \tfrac{\mathcal{N}_I}{6\mathcal{N}}\hat{F}_{ab}^I(W)\right) + 2i\bar{\varepsilon}\psi_\mu\tfrac{\mathcal{N}_I}{3\mathcal{N}}\hat{F}_{ab}^I(W)\\
&\quad - 4i\bar{\varepsilon}^i\gamma^{ab}\psi_\mu^j\tilde{t}_{ij} + \tfrac{4\mathcal{N}_{IJ}}{3\mathcal{N}}\left((\bar{\varepsilon}\lambda^I)\bar{\psi}_\mu\gamma^{ab}\lambda^J - (\bar{\varepsilon}\gamma^{ab}\lambda^I)\bar{\psi}_\mu\lambda^J\right),
\end{aligned}\tag{7·2}$$

where $\mathcal{D}_\mu$ is the covariant derivative which is covariant only with respect to homogeneous transformations $\boldsymbol{M}_{ab}, \boldsymbol{U}^{ij}$ and $\boldsymbol{G}$. $\boldsymbol{\Gamma}$ is a field-dependent matrix acting on a spinor with an $SU(2)$ index which is defined by

$$\boldsymbol{\Gamma}\varepsilon^i \equiv (-\gamma\cdot\tilde{V} + 3\tilde{t})^i{}_j\varepsilon^j + \gamma\cdot\tilde{v}\varepsilon^i + \tfrac{\mathcal{N}_I}{4\mathcal{N}}\gamma\cdot\hat{F}^I(W)\varepsilon^i + \tfrac{\mathcal{N}_{IJ}}{\mathcal{N}}\lambda^{iI}(2i\bar{\lambda}^J\varepsilon)\,,\tag{7·3}$$

and the prime on $\boldsymbol{\Gamma}$ implies that $\boldsymbol{U}$-gauge field in $\boldsymbol{\Gamma}$ is removed: $\boldsymbol{\Gamma}\varepsilon^i = \boldsymbol{\Gamma}'\varepsilon^i - \gamma\cdot\tilde{V}_j^i\varepsilon^j$. We have redefined the auxiliary fields χ'' and C''' so as to become covariant quantities:

$$\tilde{\chi}^i \equiv \chi''^i + \tfrac{1}{2}\gamma^a\boldsymbol{\Gamma}\psi_a^i\,, \qquad \tilde{C} \equiv C''' + 2i\bar{\psi}\cdot\gamma\tilde{\chi} - i\bar{\psi}_a\gamma^{ab}\boldsymbol{\Gamma}\psi_b\,.\tag{7·4}$$

The supersymmetry transformation laws of the vector multiplet are

$$\begin{aligned}
\delta W_\mu^I &= -2i\bar{\varepsilon}\gamma_\mu\lambda^I + 2i\bar{\varepsilon}\psi_\mu M^I,\\
\delta M^I &= 2i\bar{\varepsilon}\lambda^I,\\
\delta\lambda_i^I &= \mathcal{P}^I{}_J(-\tfrac{1}{4}\gamma\cdot\hat{F}^J(W)\varepsilon_i - \tfrac{1}{2}\hat{\not{\mathcal{D}}}M^J\varepsilon_i + \tilde{Y}_{ij}^J\varepsilon^j) - \frac{M^I\mathcal{N}_{JK}}{3\mathcal{N}}2i\bar{\varepsilon}\lambda^J\lambda_i^K,
\end{aligned}$$

$$
\begin{aligned}
\delta\tilde{Y}^{Iij} &= 2i\bar{\varepsilon}^{(i}\mathcal{P}^I{}_J\hat{\not{\mathcal{D}}}\lambda^{Jj)} - i\bar{\varepsilon}^{(i}\gamma\cdot\tilde{v}\lambda^{Ij)} - i\tfrac{\mathcal{N}_J}{6\mathcal{N}}\bar{\varepsilon}^{(i}\gamma\cdot\hat{F}^J(W)\lambda^{Ij)} \\
&\quad + 2i\bar{\varepsilon}^{(i}\tilde{t}^{j)}{}_k\lambda^{Ik} + 4i\bar{\varepsilon}\lambda^I\tilde{t}^{ij} - 2ig\bar{\varepsilon}^{(i}[M,\,\lambda]^{Ij)} \\
&\quad + \tfrac{4\mathcal{N}_{JK}}{3\mathcal{N}}\bar{\varepsilon}\lambda^J\bar{\lambda}^{K(i}\lambda^{Ij)} - \tfrac{M^I\mathcal{N}_{JK}}{3\mathcal{N}}2i\bar{\varepsilon}\lambda^J\tilde{Y}^{Kij}.
\end{aligned}
\tag{7·5}
$$

Finally, the hypermultiplet transformation laws are given by

$$
\begin{aligned}
\delta\mathcal{A}^i_\alpha &= 2i\bar{\varepsilon}^i\xi_\alpha\,, \\
\delta\xi_\alpha &= -\hat{\not{\mathcal{D}}}\mathcal{A}^i_\alpha\varepsilon_i + \varepsilon_i gM_{\alpha\beta}\mathcal{A}^{i\beta} + \boldsymbol{\Gamma}'\varepsilon_i\mathcal{A}^i_\alpha + \Big(1 + \frac{\not{A}}{\alpha}\Big)\varepsilon_i\tilde{\mathcal{F}}^i_\alpha\,, \\
\delta\tilde{\mathcal{F}}^i_\alpha &= -2i\bar{\varepsilon}^i\Big(1 - \frac{\not{A}}{\alpha}\Big)^{-1}\Big(\hat{\not{\mathcal{D}}}\xi_\alpha + 2\tilde{\chi}_j\mathcal{A}^j_\alpha + gM_{\alpha\beta}\xi^\beta + 2g\lambda_{j\alpha\beta}\mathcal{A}^{j\beta} \\
&\qquad\qquad + \tfrac{1}{2}\gamma\cdot\tilde{v}\,\xi_\alpha + \tfrac{2}{\alpha}\lambda^0_j\tilde{\mathcal{F}}^j_\alpha\Big) + \tfrac{2i}{\alpha}\bar{\varepsilon}\lambda^0\tilde{\mathcal{F}}^i_\alpha\,.
\end{aligned}
\tag{7·6}
$$

Here $gM = M^I gt_I$ and $g\lambda_i = \lambda^I_i gt_I$ includes the $I = 0$ part with $gt_{I=0}$ defined in Eq. (5·5), and the G covariantization for $I = 0$ in $\hat{\mathcal{D}}_\mu$ is understood to be $-A_\mu(gt_0)$ instead of the original central charge transformation $-\delta_Z(A_\mu)$.

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