Boundary states as exact solutions
of (vacuum) closed string field theory

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§1 Introduction

Analysis of nonperturbative phenomena

Soliton solutions are essential

Magnetic monopole | for Yang-Mills
Instanton
Blackhole | for Einstein gravity

They are solutions of nonlinear equation of motion \[ \Rightarrow \text{possible to carry nontrivial topological charge} \]

Soliton solutions in string theory

Magnetic 5 brane

D-brane \[\Leftarrow \text{completely stringy description} \]
is possible by Boundary state
Boundary state

Implementation of boundary conditions of open string in closed string Hilbert space

Example) D p-brane

\[ \partial_\sigma X^\mu |_{\sigma = 0, \pi} = 0 \quad \mu = 0, \ldots, p \]

\[ \partial_\sigma X^i |_{\sigma = 0, \pi} = 0 \quad i = p+1, \ldots, d-1 \]

(modal space)

Modular transformation (\( \sigma \leftrightarrow \pi \))

\[ \left\{ \begin{array}{l}
\partial_\tau X^\mu |_{\tau = 0, \pi} |B\rangle = 0 \\
\partial_\tau X^i |_{\tau = 0, \pi} |B\rangle = 0
\end{array} \right. \quad \text{relations between left/right movers} \]

\[ |B\rangle = \exp \left( \sum_{\nu=1}^{d} \frac{1}{m} S_{\nu \mu} \partial_\tau d^\mu - \partial_\mu d^\tau \right) |0; \vec{x}^0\rangle \]

\[ S_{\nu \mu} = \begin{pmatrix}
-1 & 1 & 0 & 0 \\
-1 & +1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \]

Note: Background dependent description
Background independent Characterization

\[(\ln - \tilde{\ln}) |B\rangle = 0 \quad \cdots \quad (**)

This linear condition is not enough!

Cardy condition

\[\langle B| q^{(k_0 + L_0)/2} |B\rangle = \chi(q)\]

\[= \sum_i N_{BB_i} \chi_i(q)\]

\[\chi_i(q) = \text{character of irreducible representation in open string channel}\]

\[N_{BB_i}: \text{must be positive integer}\]

\[\begin{array}{c}
\hat{q} \\
\uparrow \\
B' \\
\downarrow \\
B
\end{array}\]

\[\text{nonlinear constraint which must be imposed with (**)}\]

Can we translate it as the nonlinear equation in closed string field theory
Main claim

Many (all?) boundary states satisfy universal nonlinear equation (unipotency relation)

\[ |B\rangle \ast |B\rangle = (\text{constant}) \ |B\rangle \quad \cdots (\ast) \]

\( \ast \): star product of closed string field theory

HIKKO type

\[ \infty \]

Witten type

\[ \bigcirc \]

An analogue of VSFT eq. of motion for open string

\[ 2\mathcal{D} + \mathcal{D}^\ast \mathcal{D} = 0 \]

\( \mathcal{D} \): pure ghost BRST operator

Sliver butterfly \[ \leftrightarrow \] Boundary state
Variation of $|\phi\rangle$

$$\delta |B\rangle = \oint \frac{d\phi}{2\pi} V(\phi) |B\rangle$$

Insertion of operators at the boundary.

Note: Boundary state gives identification

$$X^L(\phi) \leftrightarrow X^R(\phi)$$

same as open string oscillator

Open string = Infinitesimal deformation of D-brane

We show that $\delta |B\rangle$ satisfies

$$|B\rangle \ast \delta |B\rangle + \delta |B\rangle \ast |B\rangle = (\text{const}) \delta |B\rangle$$

iff

$$\oint \frac{d\phi}{2\pi} V(\phi)$$

is marginal deformation

⇒ unipotency relation $|\phi\rangle$ knows

on-shell condition for open string
§2. Proof of unipotency relation

§2-1 Geometrical (Intuitive) proof

Boundary state

\[
\langle B | \Phi \rangle : \text{1 point function on Disk} \quad (z = e^{z+\phi})
\]

Role of boundary state

1. Cut the cylinder at \( \tau = 0 \)
2. Set the boundary condition

3-string vertex (Lightcone type)

Patching together 3 world sheets
\[ |B_{a}\rangle \times |B_{b}\rangle \]

When \( a = b \)

\[ |B_{a}\rangle \times |B_{a}\rangle \triangleq |B_{a}\rangle \]  \((*)\)

Boundary states are only states that have this property (stripping half cylinder)

\[ \uparrow \]

\((*)\) unipotency relation is satisfied only by Boundary state?
2. More explicit proof

1. Boundary state (D p-brane with flux $F_{\mu\nu}$)

$$|B(F)\rangle = e^{-\sum_{n} a_{n}^{\mu+} B_{\mu}^{n} a_{n}^{\mu+}} e^{\sum_{n} c_{n}^{\mu+} \bar{c}_{n}^{\mu+} c_{n}^{\mu+} \bar{c}_{n}^{\mu+}} |\rho_{p}, 0, x^{I}\rangle$$

- $O_{\mu}^{\nu} = \begin{cases} ( (1+F)^{-1} (1-F) )_{\mu}^{\nu} & \mu, \nu = 0, \ldots, p \\ -S_{\mu}^{\nu} & \mu, \nu = p+1, \ldots, d-1 \end{cases}$
- $F_{\mu\nu}$: constant magnetic flux on D-brane
- $|x^{I}\rangle = \frac{1}{(2\pi)^{d-1}} \int d^{d}x e^{i p^{\mu} x_{\mu}} |\Phi_{0}\rangle$

To take HIKKO type x product

- introduce a parameter

  (direct product with $|\Phi\rangle$)

- adjust ghost zero mode

$$|B(F)\rangle \Rightarrow \tilde{\Phi}_{F}(x; \alpha) = \mathcal{C}_{F} |B(F)\rangle \otimes |\Phi\rangle$$
HIKKO's star product

\[ \{ V(1,2,3) \} = \int \{ \Phi(1,2,3) \} \left[ \mu(1,2,3) \right]^2 \Phi^{(\alpha)}(x) \Phi^{(\beta)}(y) \times \prod_{n=1}^{3} \left( 1 + \frac{1}{\sqrt{2}} \omega^{(n)} \right) \xi^{(n)} \left( p_1, a_1 \right) \left( p_2, a_2 \right) \left( p_3, a_3 \right) \]

\[ F(1,2,3) = \sum_{n=1}^{3} \sum_{m=1}^{3} \sum_{a_1, a_2, a_3} \tilde{N}_n^{(m)} \left( 1 + a_m^{(1)} \alpha_1 + a_n^{(2)} \alpha_2 + \sqrt{n} \right) \left( a_m^{(1)} \alpha_1 + a_n^{(2)} \alpha_2 \right) + \frac{1}{2} \sum_{n=1}^{3} \sum_{m=1}^{3} \tilde{N}_n^{(m)} a_n^{(1)} a_m^{(2)} \right) \rho^2 - \frac{\tau_0}{\sqrt{d_1 d_2 d_3}} \rho^2 \]

\[ \rho = \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \]

\[ \omega^{(n)} = \ldots \]

\[ M(1,2,3) = \exp \left( -2 \Phi(1) \right) \quad \Phi(1) = (\psi + \beta + 1) \left( \frac{\log |\beta 1|}{\beta 1} - \frac{\log |\psi + 1|}{\beta} \right) \]

\[ \tau_0 = \sum_{r=1}^{3} \alpha_r \log |1 - d_r| \]

\[ \int \delta(1,2,3) = \int dp \, d\Phi \, (2\pi)^3 d^3 \left( p_1 + p_2 + p_3 \right) 2n \, \delta \left( d_1 + d_2 + d_3 \right) \]

\[ \Phi(x) = \int d\Phi \, e^{-\Phi} \left( N(x) - N(x) \right) \quad N(x) = \sum_{n} \left( a_m^{(1)} a_n^{(2)} + c \bar{c} c \bar{c} \right) \]

\[ \{ \Phi(1,2,3) \} = \int d\Phi \, d\Phi \, \langle \Phi(1,2,3) \rangle \]

\[ \langle \Phi(1,2) \rangle = \int d\Phi \, \langle \Phi(1,2) \rangle \]

\[ \langle \Phi(2,1) \rangle : \text{Reflector} \]
We prove (for \( x_1, x_2 > 0 \))

\[
\mathbf{E}_f(x_1, a_1) \times \mathbf{E}_f(x_2, a_2) = \delta^{d-1}(x_1 - x_2) \mathcal{G} \frac{\partial^2}{\partial x_1 \partial x_2} \mathbf{E}_f(x_1, a_1 + a_2)
\]

\[
\mathcal{G} = \mu(1, 2, 3)^k \left( \det (1 - r^2) \right)^{-12}
\]

\[
R_{mn} = \frac{(mn)^{1/2}}{m+n} \beta(-\sigma, \delta) \frac{f(m) f(n)}{f_m f_n}
\]

\[
\frac{f(m)}{f_m} = \frac{\Gamma(-m \beta) e^{m \left( \beta \log(\beta_1 - \log(1 + 1)) \right)}}{m! \Gamma(-m \beta + 1 - m)}
\]

So far, no closed form for determinant factor

After cutting-off \( r \) by size \( L \times L \)

\( \mathcal{G} \) behaves numerically

\[
\mathcal{G}(r) \sim L^3 e^{-0.707 + 0.866 \phi} \frac{1}{r}
\]

Mildly divergent due to singularity at intersection?
Some detail of computation (matter part)

boundary state

\[ |\tilde{3}, \gamma \rangle \otimes |\tilde{3}_s, \gamma \rangle = e^{\frac{i}{2} a^+ H a} \]

\[ \psi(\hat{\alpha}) \otimes |P_{\alpha, 1} d_s, \alpha \rangle \]

\[ \mu^t = \begin{pmatrix} a^{(\mu^t)} \\ a^{(\mu^t +)} \end{pmatrix} \]

\[ \alpha^{(\gamma^t)} = \begin{pmatrix} \alpha_n^{(\gamma^t)} \\ \alpha_n^{(\gamma^t +)} \end{pmatrix} \]

\[ M = \begin{pmatrix} 0 & -\delta_{MN} \delta_{\alpha \beta} \\ -\delta_{MN} \delta_{\alpha \beta} & 0 \end{pmatrix} \]

Exponential factor \( \propto V \)

\[ F(1, 2, 3) = \frac{1}{2} a^+ \eta a^+ + \alpha^\dagger \mu + a^{\mu^t} \eta^{\mu^t} a^+ - \frac{\eta}{4 \alpha^\dagger \mu} \]

\[ N = \eta_{MN} \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \]

\[ n = \begin{pmatrix} \tilde{N}^u & \tilde{N}^u \\ \tilde{N}^u & \tilde{N}^u \end{pmatrix} \]

\[ \mu = \begin{pmatrix} \mu^{(\gamma^t)} \\ \mu^{(\gamma^t +)} \end{pmatrix} \]

\[ \mu^{(\gamma^t)} = \begin{pmatrix} \tilde{N}^{u^t} a^{(\mu^t)} + \frac{1}{2} \tilde{N}^u \mu^t \\ \tilde{N}^{u^t} a^{(\mu^t +)} + \frac{1}{2} \tilde{N}^u \mu^t \end{pmatrix} \]

\[ \begin{array}{c}
|\tilde{3}, \gamma \rangle \otimes |\tilde{3}_s, \gamma \rangle = \det \eta \left(1 - \eta_{MN}\right) e^{H_1} |P_{\alpha, 1} d_s, \alpha \rangle \\
H = \frac{1}{2} a^{\dagger N} a^+ + \frac{1}{2} \tilde{N}^u \left(a^{(\mu^t)} + a^{(\mu^t +)}\right) - \frac{\eta}{4 \alpha^\dagger \mu} \mu \\
+ \frac{1}{2} \mu^T M \left(1 - \eta_{MN}\right)^T \mu
\end{array} \]

How to manage this factor?
Neumann matrix

\[ N_{mn} = \delta_{rs} \delta_{mr} - 2 \left( A^{(r)} A^{(s)} \right)_{mn} \]

\[ \tilde{N}_m = - \left( A^{(r)} \right)_{m} \]

\[ A^{(r)}_{mn} = - \frac{2 \overline{\phi}}{\pi} \int_{m \pi}^{m \pi} (-1)^{m+n} \frac{\beta \sin (m \pi \beta)}{n^2 - m^2 \beta^2} \, d \beta, \quad A^{(r)}_{mn} \text{ similar, } A^{(r)}_{mn} = \delta_{mn} \]

\[ B_m = - \frac{2 d \overline{\phi}}{\pi d_{1,2}} m^{-3/2} (-1)^m \sin (m \pi \beta) \]

\[ \tilde{\Gamma}_m = \delta_{mn} + \sum_{r=1,2} \left( A^{(r)} A^{(r)\tau} \right)_{mn} \]

\[ A^{(r)}, B : \text{ overlap of Fourier basis} \]

**Essential property of** \( A^{(r)} \)

**HA VE INVERSE** \( D^{(r)} \)

\[ \sum_{m \neq n} \left( A^{(r)} D^{(r)} \right)_{mn} = \delta_{mn} \]

\[ D^{(r)} = - \frac{\partial \overline{\phi}}{\partial r} C A^{(r)\tau} C^{-1} \quad C_m := m \delta_{mn} \]

In a sense, they are "rectangular" matrices

\[
\begin{pmatrix}
A^{(r)} & A^{(r)}
\end{pmatrix}
\begin{pmatrix}
D^{(r)}
\end{pmatrix}
= \begin{pmatrix} 1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
D^{(r)}
\end{pmatrix}
\begin{pmatrix}
A^{(r)} & A^{(r)}
\end{pmatrix}
= \begin{pmatrix} 1 & 0 \\
0 & 1
\end{pmatrix}
\]
One can compute the inverse now:

\[
(1 - NM)^{-1} = 
\begin{pmatrix}
(1 - n_1)^{-1} & -n_1 (1 - n_2)^{-1} \\
-n_2 (1 - n_1)^{-1} & (1 - n_2)^{-1}
\end{pmatrix}
\]

\[
(1 - n_1)^{-1} = A(n_1) P^{-2} A(n_1)
\]

\[
(1 - n_2)^{-1} = D(n_2) P^{-2} D(n_2)
\]

Some computations (after using this result) give:

\[
H_m = \frac{1}{2} a^+ M a^+ \quad (= \text{quadratic part of } \tilde{\Phi}_F)
\]

Determinant factor:

\[
\det (1 - MN)^{\frac{1}{2}} \rightarrow \mathcal{C}(\beta)
\]

We arrive at:

\[
\tilde{\Phi}_F(\mathbf{p}; \mathbf{d}_1) \times \tilde{\Phi}_F(\mathbf{q}; \mathbf{d}_2) = C(\beta) \frac{2}{\mathcal{C}_0} \tilde{\Phi}_F(\mathbf{p} + \mathbf{q}; \mathbf{d}_1 + \mathbf{d}_2)
\]

**Fourier transformation for \( p_1 \)**

(Ishibashi state \( \Rightarrow \) Cardy state)

\[
\tilde{\Phi}_F(\mathbf{x}_1; \mathbf{d}_1) \times \tilde{\Phi}_F(\mathbf{x}_2; \mathbf{d}_2) = C(\beta) \delta(\mathbf{x}_1 - \mathbf{x}_2) \frac{2}{\mathcal{C}_0} \tilde{\Phi}_F(\mathbf{x}_1 + \mathbf{x}_2; \mathbf{d}_1 + \mathbf{d}_2)
\]

\[
C(\beta) \frac{2}{\mathcal{C}_0} : \text{Universal factor for ANY Boundary state}
\]
33. Derivation of open string spectrum

Question: Does unipotency relation

$$\Xi_a \cdot \Xi_b = C(\beta) \delta_{ab} \frac{\delta}{\delta \phi_a} \Xi_a \quad \ldots \ldots \ (\nu)$$

remain true for any (curved) D-brane?

First nontrivial test

Check it for infinitesimal deformation around flat D-brane

$$S[B] = \int \frac{dt}{2\pi} V(\sigma) \, |B>$$

We examined two simplest examples

1. Tachyon
   $$V_T(\sigma) = e^{i k X(\sigma)}$$

2. Vector
   $$V_\nu(\sigma) = \left( \delta_{\mu\nu} \delta \phi \right) \cdot e^{i k X}$$

Variation of (\nu)

$$\delta \Xi_a \cdot \Xi_b + \Xi_a \cdot \delta \Xi_b = C(\beta) \frac{\delta}{\delta \phi_a} \delta \phi_b$$

implies

$$\begin{cases} k_{\mu} G^{\mu \nu} k_{\nu} = 2 & \text{for 1} \\ k_{\mu} G^{\mu \nu} k_{\nu} = 0 & \text{for 2} \end{cases}$$

\[ G^{\mu \nu} = ( (1+\Phi)^2 (1-\Phi)^2 )^{\mu \nu} \]

On-shell conditions of open string
Derivation of tachyon mass-shell condition

\[ \delta_\Sigma (d_i \times d_i) = - (\beta)^\frac{3}{2} k G k \langle \delta_\Sigma (d_i + d_i) \rangle \]

\[ \delta (d_i) \times \delta_\Sigma (d_i) = - (1 + \beta)^\frac{3}{2} k G k \langle \delta_\Sigma (d_i + d_i) \rangle \]

\[ \Rightarrow \text{on-shell condition } \Leftrightarrow (\beta)^\frac{3}{2} k G k + (1 + \beta)^\frac{3}{2} k G k = 0 \]

\[ \Leftrightarrow k_{\alpha_\beta} G_{\alpha_\beta} k_{\beta} = 0 \]

How we obtained \( \left( \frac{\log |B|}{\log (1+\beta)} \right) \) factor?

Coefficient of \( \frac{1}{k G k} \) has following form

\[
\int \frac{k^n}{A} \sum_{k=1}^{\infty} \frac{1}{k} - \sum_{p=1}^{\infty} \frac{1}{p^2} = \text{indeterminate}
\]

We need regularization (cut-off) of \( A \).

We note that

\[
\begin{bmatrix}
A_0 \\
A_0^T
\end{bmatrix}
\]

need to have inverse \( A \) is rectangular matrix.
We need to consider following cut-off

\[ A^u \]

\[ \to \]

\[ A^u \]

\[ -\beta L \quad (1+\beta) L \]

\[ \Omega = \frac{a_1}{a_1 + a_2}, \quad (1+\beta) = \frac{a_2}{a_1 + a_2} \]

It implies coeff of \( \frac{1}{k} k G_k \) is given by

\[
\lim_{L \to \infty} \left( \sum_{k=1}^{\frac{L}{\beta}} \frac{1}{k} - \sum_{p=1}^{\frac{L}{1+\beta}} \frac{1}{p} \right) = + \log |\beta| 
\]

for 1st term

2. Vector type fluctuation

\[
| \delta v_{u_1} (d_1) \times \overline{v}_{u_2} (d_2) \rangle = (-\beta)^{\frac{1}{2}} k G_{k+1} | \overline{v}_{u_1 + d_1} (d_2) \rangle 
\]

\[
| \overline{v}_{u_1} (d_1) + \delta v_{u_2} (d_2) \rangle = (1+\beta)^{\frac{1}{2}} k G_{k+1} | \overline{v}_{u_1 + d_1} (d_2) \rangle + \ldots
\]

On-shell condition

\[
(-\beta)^{\frac{1}{2}} k G_{k+1} + (1+\beta)^{\frac{1}{2}} k G_{k+1} = 1
\]

\[ \Leftrightarrow \quad k_k G^u \epsilon_k = 0 \]
There are MANY TERMS in ... but they are cancelled exactly...

A subtlety ... singularity at intersection point

... contains

\[ -i \beta \cdot g \cdot k \left( 2 \delta(\pi - \sigma - \theta) \sum_{\rho \equiv 1} \frac{\sin^2 \rho \theta}{\rho} \right) \]

\(( -\pi \leq \sigma + \theta \leq \pi )\)

... contains

\[ i \beta \cdot g \cdot k \left( 2 \delta(\sigma + \theta) \sum_{\rho \equiv 1} \frac{\sin^2 \rho \theta}{\rho} \right) \]

\(( 0 \leq \sigma + \theta \leq 2\pi )\)

\[ \sum_{\rho \equiv 1} \frac{\sin^2 \rho \theta}{\rho} \]

is divergent but A & B are cancelled.

Exactly after integration over \( \sigma \)

\[ \Leftarrow \text{No transversality condition} \]

\[ k_\mu g^{\nu \tau} \chi_\tau = 0 \]

but it is subtle

\( \Theta \) are subtleties where we need more explicit regularization scheme

CS. MSFT regularization
Note

There exists gauge symmetry for vector type deformation off-shell:

\[ \delta \sigma \rightarrow \delta \sigma + \epsilon \partial \mu \]

\[ \delta \psi |B\rangle \rightarrow \int \frac{d\sigma}{2\pi} \left( \delta \sigma + \epsilon \partial \mu \right) \delta_{\sigma} X^\mu \ e^{i\psi X(\sigma)} |B\rangle \]

\[ = \delta \psi |B\rangle + \int \frac{d\sigma}{2\pi} \epsilon \delta_{\sigma} \left( e^{i\psi X} \right) |B\rangle \]

Vector particle:

- Massless
- Gauge symmetry
- Transversality

\[ \begin{array}{c|c}
\text{Massless} & \text{Conjecture} \\
\text{Gauge symmetry} & \text{Variation of } \frac{1}{2} \mu^2 = C \mu \text{ reproduces complete open string spectrum} \\
\text{Transversality} & \text{Same as (conventional)}
\end{array} \]

- gauge particle
§4 Discussion

1 Unipotency relation for other closed string vertex?

Example:

\[ \begin{array}{c}
\text{Witten type} \\
\text{HIKKO (d_1, d_2 < 0)}
\end{array} \]

Yes! but VERY SINGULAR

\[ \Phi \ast \Phi = \bigcirc (\delta(0))^r \Phi \]

\( r \): number of overlap = \( \infty \)

Origin of singularity

\[ \begin{aligned}
\text{Boundary state} & \sim \prod \delta(X(\sigma) - \tilde{X}^{(r)}) \\
\text{Vertex} & \sim \prod_{r<s} \delta(X_r(\sigma_r) - X_s(\sigma_s)) \\
& \times \delta(\tilde{X}_r(\sigma_r) - \tilde{X}_s(\sigma_s)) \\
& = \delta(0) \delta(X(\sigma) - X(\sigma))
\end{aligned} \]
Is it a disaster to construct VSFT like scenario?

May not be so serious!

Example:

Inner product of Boundary States

\[ \langle B|B \rangle = (s(0))^n \quad \text{same type of infinity} \]

However, there exist a regularization scheme

\[ \langle j|q^{\frac{1}{2}}(P+\mathbf{E}_0)|k \rangle = \delta_{jk} \chi_j(E) \quad \text{\textless \textit{regulization}} \]

\[ = \delta_{jk} S_{jk} \chi_k(E) \]

\[ \lim_{q \to 1} \langle j|q^{\frac{1}{2}}(P+\mathbf{E}_0)|k \rangle \]

\[ = \lim_{q \to 0} \delta_{jk} S_{jk} \chi_k(E) \]

\[ = \delta_{jk} S_{jk} \quad \text{(This is finite!)} \]

infiniterly thin strip

\[ \quad \Leftarrow \text{only vacuum channel in open string} \]

\[ \quad \text{boson annihilation} \]
2. \( \Psi \ast \Psi = \Psi \) as background independent characterization of D-brane.

LPP formulation

\[
\langle V_\alpha | i \phi, i \phi', i \phi'' | i \phi, i \phi', i \phi'' \rangle \equiv \langle f, i \phi, f', i \phi', f'', i \phi'' \rangle
\]

\( f \) \( (i = 1, 2, 3) \) gluing conformal mapping

Only conformal transformation is involved

\[ \downarrow \]

* product is defined in background independent way

by LPP definition, we can prove

\[
(L^{\psi}_n - \tilde{L}^{\psi}_n) | \psi_i \rangle = 0
\]

\[
(L^{\psi}_n - \tilde{L}^{\psi}_n) | \psi_3 \rangle = 0
\]

\[
\downarrow
\]

\[
(L^{\psi}_n - \tilde{L}^{\psi}_n) (| \psi_i \rangle \chi | \psi_3 \rangle) = 0
\]

* product of boundary state is always boundary state

but without Cardy condition
We conjecture
\[ |B\rangle \times |B\rangle = |B\rangle \]
is satisfied only by Cardy state

In terms of Ishibashi states \( |i\rangle \rangle \)
this is equivalent to
\[ |i\rangle \rangle \times |j\rangle \rangle = \sum_k N_{ij}^k \ |k\rangle \rangle \]

\( N_{ij}^k \): Verlinde's fusion coefficient
\[ = \sum_k \frac{S_{ik} S_{kj} (S_{ik})^k}{S_{kk}} \]

Cardy state
\[ |a\rangle = \sum_k S_{ak} |k\rangle \]
\[ |a\rangle \times |b\rangle = \delta_{ab} \frac{1}{S_{aa}} |a\rangle \]

We check these relations for toroidal compactification
3. Similarity with VSFT

Open string $\implies$ Closed string

Witten's CSFT $\xrightarrow{\text{dual picture?}}$ HIKKO or Witten type

VSFT $\xrightarrow{\text{Tachyon vacuum?}}$ Closed SFT $\xrightarrow{\text{Hikko}}$

$\overline{\phi^*} \phi = \mathcal{C}_0 \overline{\phi^*} \phi$

D-brane $\implies$ Projector $\implies$ Boundary state = D-brane

- Basic picture is very similar
- Detail is quite different...
  - cf. massless excitation (vector particle)
Other Issues

- Supersymmetric extension
- Calculation of tension
- Higher excited mode
- Regularization
- Application to rolling tachyon (time dependent process)
- Closed string propagation