Noncommutative Geometry and Vacuum String Field Theory

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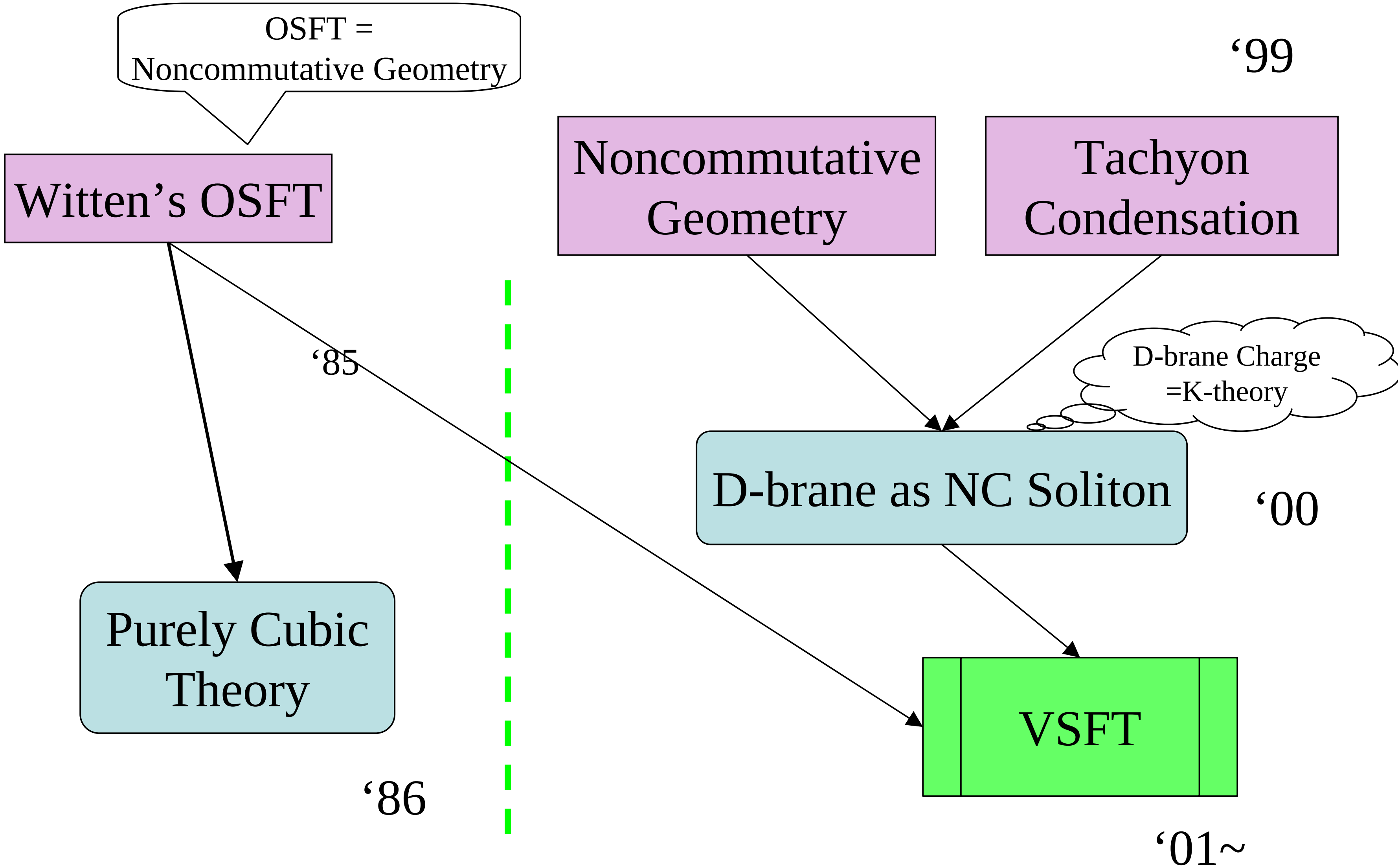
July 2002, YITP WS on QFT

1. Introduction and Motivation

A Brief History of VSFT



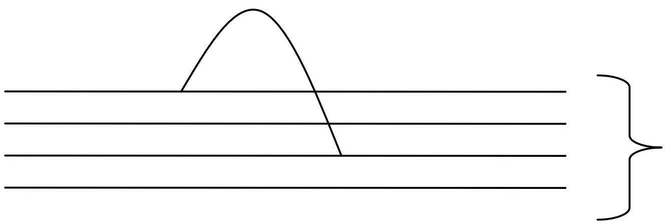
Motivation: *What is D-brane*?

- In effective field theory
  - D-brane = Soliton of closed string
    - Black hole like object
- In (full) string theory
  - D-brane = Boundary condition for open string
  - Described by (abstract) Boundary state

$$(L_n - \overline{L}_{-n}) | B \rangle = 0$$

They should be understood as "Solution"
To the second quantized field theory

Why Noncommutative Geometry is relevant to understand D-brane?

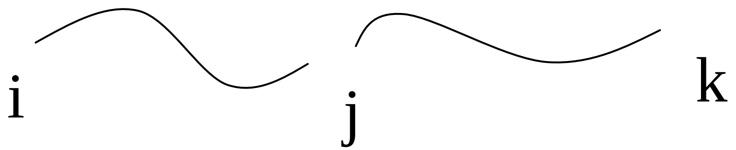


Open string has Chan-Paton index

$$\Phi_{ij}$$

: i,j : Chan-Paton Index

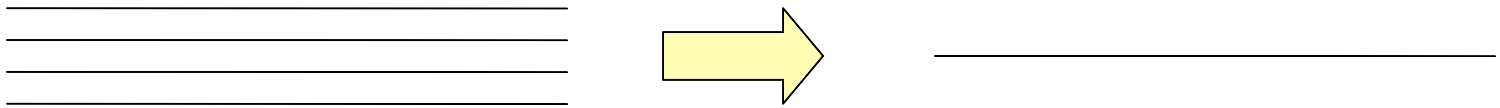
Composition of two open strings



$$\sum_k \Phi_{ij}\Phi_{jk} = \Phi_{ik}$$

= *Multiplication of matrices*

To pick up one D-brane, we use Projector to one specific Chan-Paton Index



Projector P

N-dim space

k dim linear subspace

$$P^2 = P$$

$$rank(P) = k$$

Matrix → Noncommutative Geometry

Projector → Noncommutative Soliton

D-(p-2) brane out of D-p brane

Idea: Use D-p brane world volume instead of Chan-Paton factor

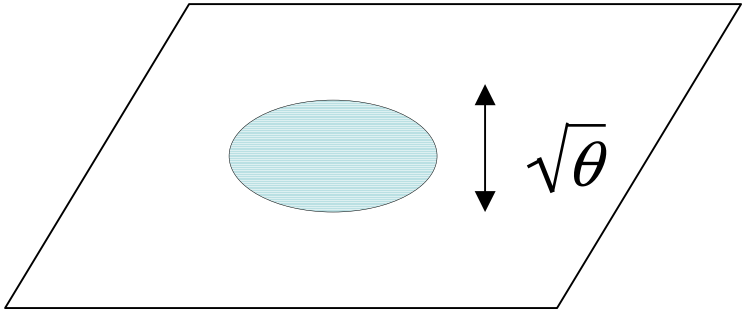
Start from D-p brane with *non-zero B-field*

Zero mode of open string becomes noncommutative

$$(f*g)(x) = e^{\frac{i}{2}\theta_{ij}(\partial_i\partial'_j - \partial_j\partial'_i)} f(x)g(x')\Big|_{x'=x}$$ *Moyal Product*

Moyal plane is the simplest example of NC geometry

Projector equation $$f*f=f \Rightarrow f=\exp\left(-\frac{1}{2\theta}(x_1^2+x_2^2)\right)$$

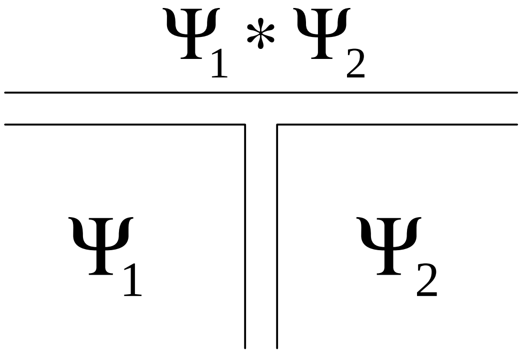


Blob with size θ is interpreted D-(p-2) brane

Open string as a whole as Matrix

Witten’s star product

$$\Psi_1 * \Psi_2$$



$$(\Psi_1 * \Psi_2)(X) = \int DYDZ \left(\prod_{\sigma=0}^{\pi/2} \delta(X(\sigma)-Y(\sigma))\delta(Y(\pi-\sigma)-Z(\sigma))\delta(Z(\pi-\sigma)-X(\pi-\sigma)) \right) \Psi_1(Y)\Psi_2(Z)$$

Path Integral for the overlap looks like matrix multiplication

Witten’s argument

1. *-product::noncommutative and Associative
2. Q: BRST operator : $Q^2=0$
3. Integration

Triplet $(*, Q, \int)$
defines
Noncommutative Geometry

Idea of VSFT

D-brane is NC soliton for Witten's star product

Matrix equation

$$\Psi * \Psi = \Psi$$

One to one correspondence between solutions?

Matrix=gl(∞)

↓ ?

Virasoro

Conformal Invariance

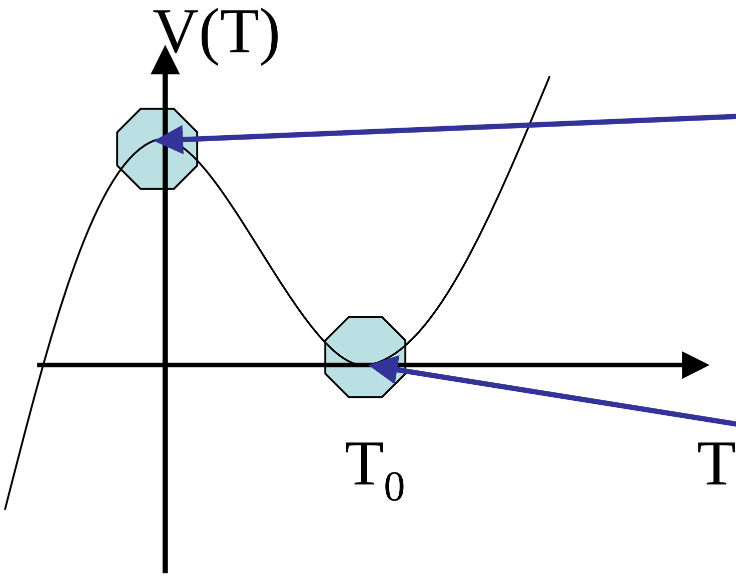
$$(L_n - \bar{L}_{-n})\,|B\rangle = 0$$

Open string

↓ ?

Closed String

Progress until 2002/7



A: Witten's OSFT

= D25 brane background

$$S = \int \left(\tfrac{1}{2}\Psi * Q\Psi + \tfrac{1}{3}\Psi^3 \right)$$

B: Tachyon Vacuum Ψ_0

solution by level truncation

$$\frac{\left| S[\Psi_0] - S[0] \right|}{\tau_{25}} = 0.9999....$$

Sen, RSZ, Berkovits, Taylor…

B should be universal for any D-brane

We want re-expand the theory from point B

No analytic solution known for Ψ_0

Ansatz of the theory at B = VSFT

RSZ

- Q → Q^{VSFT} : Pure ghost BRST operator
  - *NO COHOMOLOGY*
- Splitting of variable in wave function

$$\Psi = \Psi^{matter} \otimes \Psi^{ghost}$$

$$\Downarrow$$

$$Q^{VSFT}\Psi^{ghost} + \Psi^{ghost} * \Psi^{ghost} = 0$$

$$\Psi^{matter} * \Psi^{matter} = \Psi^{matter}$$

Exactly solvable !

Candidate of D-brane = Sliver state

Kostelecky-Potting solution

$$|\Xi\rangle \propto e^{\frac{1}{2}a^{+}CTa^{+}}|0\rangle$$

$$T = \frac{1}{2M_0}\left(1 + M_0 \pm \sqrt{(1+M_0)(1-3M_0)}\right)$$

Use of square root
And infinite product
Is the origin of
Trouble

Wedge state and Sliver state

$$|n\rangle = |0\rangle^{n} \qquad |\mathrm{n}\rangle * |m\rangle = |n+m\rangle$$

$$|\Xi\rangle = \lim_{n\to\infty}\left(|n\rangle\right)$$

2. Recent developments of VSFT

Topics

- Explicit correspondence with NC Geometry
  - Half string Formulation
  - Mapping Witten’s star product to Moyal product
- Appearance of Closed string
- Construction of Physical State
  - Can variation around sliver reproduces open string spectrum?
  - Hata-Kawano state, Okawa state, …

2.1 Explicit correspondence with NC Geometry

Witten's argument uses the path integral formally.

For explicit correspondence, we need to use mode expansion.

1. Split string formulation

Bordes et. al. ,

RSZ, Gross-Taylor

$X(\sigma)$

$l(\sigma)$ $r(\sigma)$

$$\begin{cases} l(\sigma) = X(\sigma) & 0 \le \sigma \le \frac{\pi}{2} \\ r(\sigma) = X(\pi - \sigma) & \frac{\pi}{2} \le \sigma \le \pi \end{cases}$$

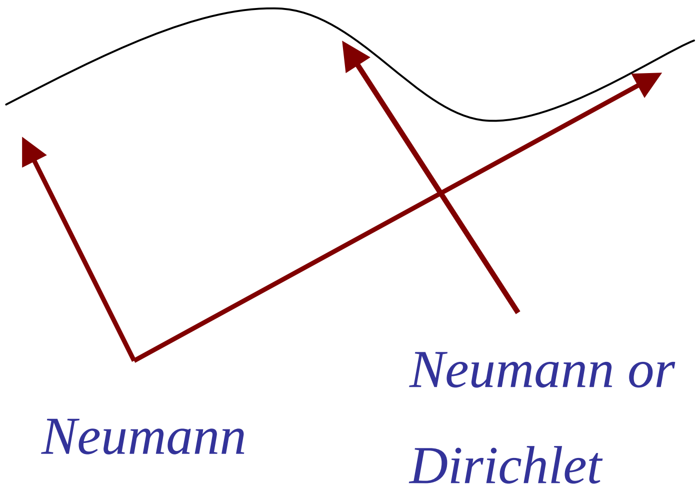
$$\Psi(X) \to \Psi(l, r)$$

$$(\Psi_1 * \Psi_2)(l, r) = \int dt \Psi_1(l, t) * \Psi_2(t, r)$$

*Except for the path integral, * product looks like matrix multiplication*

<u>*Subtlety in split string*</u>

Boundary condition at the midpoint ?



Neumann at M

$$l(\sigma) = l_0 + \sqrt{2}\Sigma_e l_e \cos(e\sigma) \quad (e \text{ even, positive})$$

$$r(\sigma) = r_0 + \sqrt{2}\Sigma_e r_e \cos(e\sigma)$$

Labeled by Even integers

Dirichlet at M

$$l(\sigma) = \sqrt{2}\Sigma_o l_o \cos(o\sigma) \quad (o \text{ even, positive})$$

$$r(\sigma) = \sqrt{2}\Sigma_o r_o \cos(o\sigma)$$

Labeled by Odd integers

Original Variable

$$X(\sigma) = x_0 + \sqrt{2}\Sigma_{n\geq 0} x_n \cos(n\sigma)$$

Labeled by Even and Odd integers

Translation between even and odd mode

$$T_{eo} = \frac{\pi}{4}\int_0^{\pi/2} d\sigma \cos(e\sigma)\cos(o\sigma) = \frac{2(-1)^{(e+o-1)/2}}{\pi}\left(\frac{1}{o+e}+\frac{1}{o-e}\right)$$

$$R_{oe} = \left(\bar{T}\right)_{oe} - (-1)^{e/2} T_{0e}$$

"X" in Gross-Jevicki, Gross-Taylor

$$H^{odd} \underset{R}{\overset{T}{\rightleftarrows}} H^{even} \qquad TR = RT = 1$$

Zero mode part

$$v_o = \frac{1}{\sqrt{2}} T_{0,o} \in H^{odd}, \quad w_e = \sqrt{2}(-1)^{e/2+1} \in H^{even}$$

with $Tv = 0,\ v = \bar{T}w,\ T\bar{T} = 1,\ \bar{T}T = 1 - v\bar{v}$

These relation breaks associativity…

$$(RT)v = v \quad \text{but} \quad R(Tv) = 0$$

$$(T\overline{T})w = w \quad \text{but} \quad T(\overline{T}w) = Tv = 0$$

- It is not very clear that this anomaly produces the associativity anomaly of * product itself.

- As we see later, any string amplitude can be written in terms of only one matrix written in terms of T and vector by w.

- In the following discussion, we will use the finite dimensional regularization and use ordinary multiplication rule of matrix everywhere.

Associativity anomaly in purely cubic theory

Purely cubic theory (Yoneya, Friedan, Witten)

$$S^{cubic} = \frac{1}{3}\int \Psi^3 \quad \Rightarrow \text{e.o.m } \Psi^2 = 0$$

Solution (Horowitz,Lykken, Rohm, Strominger)

$$\Psi_0 = Q_L I$$

I : Identity operator

Q_L : half BRST operator $\quad Q_L = \int_0^{\pi/2} j_{BRS}(\sigma) d\sigma$

Expansion around Ψ_0 Reproduces Witten's action

$$S^{cubic}[\Psi_0 + \Psi_1] = S^{Witten}[\Psi_1]$$

It reproduce correct
Open string spectrum!

Closed string sector in (old) VSFT

How to write space-time reparametrization by open string degree of freedom?

Space-Time translation *(Horowitz, Strominger)*

$$\Lambda = P_L \left| I \right\rangle , \qquad [\Lambda, \Psi[X]]_* = \frac{\partial}{\partial \varepsilon} \Psi[X+\varepsilon]$$

It breaks associativity explicitly.

$$(P_{1L} + P_{2L})\left| V_4 \right\rangle = 0, \quad (\bar{x}_1 - \bar{x}_3)\left| V_4 \right\rangle = 0$$

but $\left[P_{1L} + P_{2L}, \bar{x}_1 - \bar{x}_3 \right] = -i/2$

Closed string sector breaks associativity?

In terms of split string variables, $P_L = \Sigma_o v_o \partial_{l_o}, P_R = \Sigma_o v_o \partial_{r_o}$

Anomaly of T, R, v, w ⟺ Anomaly from closed string

Moyal Formulation *(Bars, Bars-Matsuo)*

Split string

$$(\Psi_1 * \Psi_2)(l,r) = \int_{-\infty}^{\infty} \Psi_1(l,t)\Psi_2(t,r)dt$$

Fourier Transformation

$$A(x,p) = \int_{-\infty}^{\infty} \Psi\left(\frac{x+y}{2}, \frac{x-y}{2}\right) e^{-ipy} dy \equiv F(\Psi)(x,p)$$

$$F(\Psi_1) * F(\Psi_2) = F(\Psi_1 * \Psi_2)$$

Moyal

$$(A_1 * A_2)(x,p) = e^{\frac{i}{2}(\partial_x \partial_p' - \partial_p \partial_x')} A_1(x,p) A_2(x',p') \Big|_{\substack{x=x' \\ p=p'}}$$

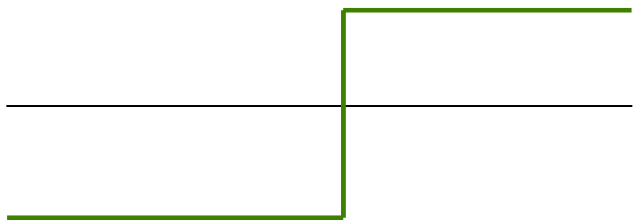
Extension to OSFT

$$A(x_{even}, x_{odd}) = \int \Pi_o dx_o e^{-2i\Sigma_{e,o} p_e T_{eo} x_o} \Psi(x_0, x_e, x_o)$$

1. Matrix T is needed to translate p_{odd} to p_{even}
2. On LHS, we do not need split string wave function but original wave function
3. Witten's star product is now realized infinite direct product of Moyal planes with same θ for all the planes…

Note

Associativity breaking mode *(Moore-Taylor, Bars-Matsuo)*

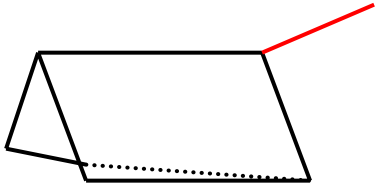


Kink at the midpoint

= zero mode of K_1 (RSZ)

= generator of space-time translation

Closed string vertex *(Hashimoto-Izhaki, GRSZ)*



$$\delta S = \int V\left(\pi/2\right)\Psi$$

V : closed string vertex

Gauge invariant form

Another formulation of MSFT

Liu, Douglas, Moore, Zwiebach

$$[x(\kappa), y(\kappa')]_* = i\theta(\kappa)\delta(\kappa-\kappa')$$

$$\theta(\kappa) = 2tanh\left(\pi\kappa/4\right) \;, \quad \kappa \geq 0 \;, \text{ Continuous parameter}$$

$$x(\kappa) = \sqrt{2}\Sigma_{e=2}^{\infty} v_e(\kappa)\sqrt{e}\,x_e, \;\; y(\kappa) = -\sqrt{2}\Sigma_{o>0}\frac{v_o(\kappa)}{\sqrt{o}}p_o$$

In terms of discrete variable *x*, *p*,

$$[x_e, p_o]_* = i\Theta^{e,o} \;, \qquad n, m \geq 1$$

$$\Theta^{e,o} = 2T_{e,o}$$

Comparison with Bars' : Fourier transformation without *T*

Explicit computation in MSFT

Bars, Matsuo

Any SFT computation is drastically simplified in MSFT

	Operator Formalism		*MSFT*
Identity:	$e^{\Sigma_n a_n^+ (-1)^n a_n^+} \lvert 0 \rangle$	$\Leftrightarrow$	1
Projector:	$\psi = e^{-a^+ CTa^+} \lvert 0 \rangle$		$A = e^{-\bar{\xi} M \xi}, \ \xi = \begin{pmatrix} x_e \\ p_e \end{pmatrix}$
	$MT^2 - (1+M)T + M = 0$		$m^2 = 1, \ (m = M\sigma)$
	$M = CV_3^{[rr]}$		$\sigma = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$
Nontrivial *Building block*	Neumann Coefficients		Perturbative vacuum

Wedge state and sliver in MSFT

Wedge state

$$|0\rangle \Leftrightarrow A_0 = N_0 \exp\left(-\bar{\xi} M_0 \xi\right) \;,\; M_0 = \begin{pmatrix} \kappa_e & 0 \\ 0 & Z \end{pmatrix},\; Z = T\kappa_o^{-1}\bar{T}$$

$$(A_0)_*^n = N_n \exp\left(-\bar{\xi} M_n \xi\right), \quad M_n\sigma = \frac{(1+m_0)^n - (1-m_0)^n}{(1+m_0)^n + (1-m_0)^n},\; m_0 = M_0\sigma$$

Sliver state

$$m_s = M_s\sigma = \lim_{n\to\infty} M_n\sigma = m_0 \Big/ \sqrt{m_0^{\,2}} \;,\quad m_s^{\,2} = 1$$

$$m_0 v^{(\kappa)} = tanh\left(\tfrac{\pi}{4}\kappa\right) v^{(\kappa)} \;\Rightarrow\; m_s v^{(\kappa)} = \varepsilon(\kappa) v^{(\kappa)}$$

$$-\infty < \kappa < \infty \,,\quad \text{at } \kappa = 0 \text{ indefinite}$$

Singularity at $\kappa=0$ *!*

Relation between OSFT and MSFT

Every Neumann coeffs are expressed in terms of M_0 and w

$$\langle V_n | \Psi_1 \rangle \otimes \cdots \otimes | \Psi_n \rangle = Tr(A_1 * \cdots * A_n)$$

$$A_i = F(|\Psi_i\rangle)$$

For example, 3-string vertices are expressed as

$$M_0 = \frac{\underline{m}_0^2 - 1}{\underline{m}_0^2 + 3}, \quad M_+ = 2\frac{\underline{m}_0 + 1}{\underline{m}_0^2 + 3}, \quad M_- = 2\frac{1 - \underline{m}_0}{\underline{m}_0^2 + 3}$$

$$V_0 = \frac{4\underline{m}_0^2}{3(\underline{m}_0^2 + 3)} W, \quad \mathrm{V}_+ =$$

$$V_{00} = \overline{W} \frac{4\underline{m}_0^2}{\underline{m}_0^2 + 3} W$$

Which satisfies all Gross-Jevicki's nonlinear identities.

Spectroscopy of Neumann coefficients RSZ

M_0, $M_{+/-}$ are simultaneously diagonalized

$$K_1 = L_1 + L_{-1}, \quad K_1 v^{(\kappa)} = \kappa v^{(\kappa)}, \quad \kappa \geq 0$$

$$\sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}} v_n^{(\kappa)} = \frac{1}{\kappa}\left(1 - \exp\left[-\kappa\, tan^{-1} z\right]\right)$$

$$M_0 v_n^{(\kappa)} = -\frac{1}{1+2\cosh(\pi\kappa/2)} v_n^{(\kappa)}, M_\pm v_n^{(\kappa)} = \frac{1+e^{\pm\pi\kappa/2}}{1+2\cosh(\pi\kappa/2)} v_n^{(\kappa)}$$

In Moyal language, this is automatic

Every Neumann coefficients are written by single matrix m_0

$$\underline{m}_0 = tanh\left(\tfrac{\pi}{4} K_1\right)$$

2.3 Physical States

Expansion around Sliver state should reproduce open string living on corresponding D-brane (up to gauge transformation)

Variation around Ψ_0 $(\Psi_0^2 = \Psi_0)$

$$\Psi' = \Psi_0 + \Psi_1, \quad \Psi'^2 = \Psi'$$

$$\Psi_1 = \Psi_0 * \Psi_1 + \Psi_1 * \Psi_0$$

← Very simple!

The Issue

- For finite dim noncommutative geometry (=finite matrix), any such variation becomes pure gauge
- Naively, there is no matter Virasoro in E.O.M. How can it reproduce every physical state correctly?

Possible Solutions

- Midpoint subtlety
- Infinite dimensionality
- Infinite conformal transformation associated with sliver state

① Hata-Kawano tachyon state

② Okawa state

Hata-Kawano state

Hata-Kawano

Ansatz $$|T\rangle = e^{\Sigma_n t_n a_n^+ a_0} e^{ipx_0} |\Xi\rangle$$

By tuning t_n, tachyon state satisfies e.o.m

If we expand, $x_0 = \bar{x} + \Sigma_{e>0} w_e x_e$, $\bar{x} = x(\pi/2)$

… Roughly speaking. We need delicate deviation from that to reproduce correct mass-shell condition.

Parameters *{t}* are tuned in such way to cancel x_e dependence of ipx_0

$$\Rightarrow \quad |T\rangle = e^{ip\bar{x}} |\Xi\rangle$$

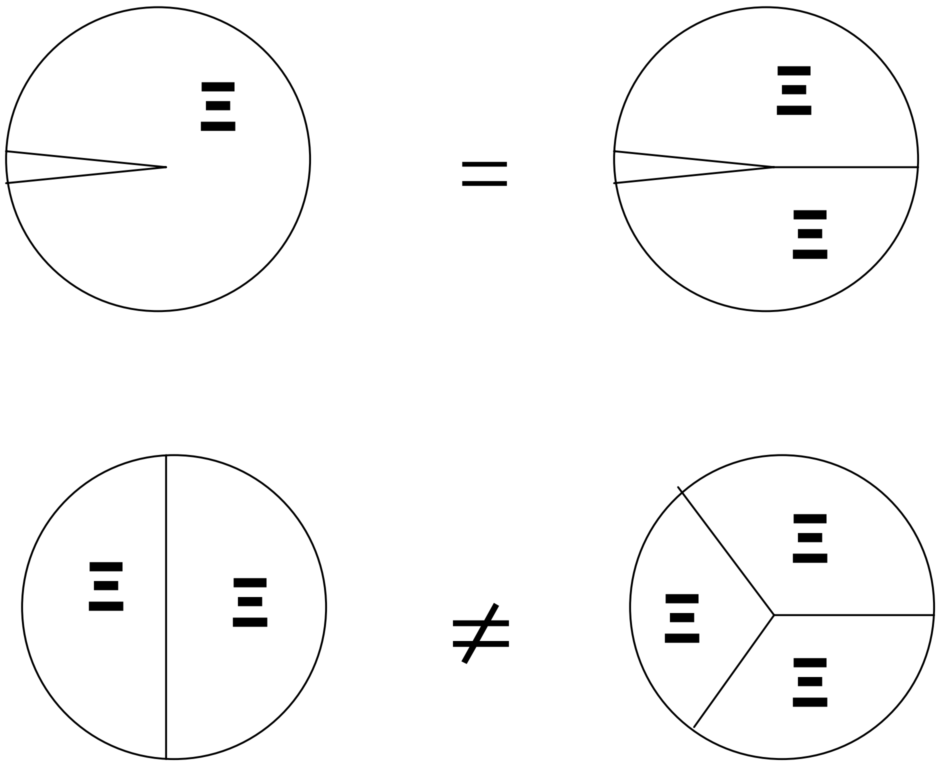
With this form, e.o.m follows directly.

Pathology from infinite product

$\langle \phi | e.o.m \rangle = 0$ for ϕ in Fock space

but

$\langle \phi | e.o.m \rangle = 0$ for ϕ in sliver state



We have to be very
Careful to define
The definition
Of Hilbert space
Where e.o.m. is
imposed

Okawa's state

BCFT consideration (Abstract argument)

D-brane ⟺ Boundary state $(L_n - \bar{L}_{-n})|B\rangle = 0$

Physical open string states on D-brane

$$\delta|B\rangle = \oint d\sigma\, j(\sigma)|B\rangle$$

$$(L_n - \bar{L}_{-n})\delta|B\rangle = 0$$

Solution in closed string sector

Mapping from boundary state to sliver

RSZ, YM

Closed string $|B\rangle$ $\longrightarrow$ $|B\rangle + \delta|B\rangle$

$\updownarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\downarrow$

Open string Hilbert space $|0\rangle_{BB} \in H_{BB}$

$\downarrow$

Sliver $|\Xi\rangle_{BB} = \left(|0\rangle_{BB}\right)_*^{\infty}$ $\longrightarrow$ *Okawa's state*

Some features of Okawa state

- It correctly reproduces *mass-shell condition* for any vertex operator
  - Conformal invariance requires the vertex operator to have dimension one
- *The brane tension* computed from three tachyon coupling gives correct value.

Remaining questions

- Both HK and Okawa states solve e.o.m. It seems that there are *too many solutions*. We need to re-examine the definition of Hilbert space more carefully.
- So far only (infinite) conformal transformation associated with sliver gives the right mass-shell conditions. Only conformal dimension gives on-shell condition. Does it also describe gauge degree of freedom correctly?

Conclusion

- Noncommutative geometry
  - MSFT gives handy description of OSFT
  - Now we do not need Neumann coefficients!
- Correct description of physical state on D-brane seems to be given.
- Many problems remain
  - Associativity anomaly
  - Extra (unphysical) solutions
  - Closed string sector
  - Supersymmetric extension