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Lie 3-algebra & Multiple M2-branes

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Based on

[1] P.-M. Ho, R.-C. Hou, YM, arXiv:0804.2110

[2] P.-M. Ho, YM, arXiv:0804.3629

[3] P.-M. Ho, Y. Imamura, YM, arXiv:0805.1202

[4] P.-M. Ho, Y. Imamura, YM, S. Shiba, arXiv:0805.2898

see also [0] P.-M. Ho, YM, hep-th/0701130

Before BLG

Nambu bracket

Y. Nambu (1973) Generalization of Poisson bracket

$$\{f_1, f_2, f_3\} = \sum_{ijk} \epsilon_{ijk} \partial_i f_1 \partial_j f_2 \partial_k f_3$$

Dynamical system with two Hamiltonians

$$H, K : \text{ two Hamiltonians} \qquad \frac{dO}{dt} = \{O, H, K\}$$

Application to the motion of tops

Quantization $[f_1, f_2, f_3] = \sum_{\sigma} (-1)^{\sigma} f_{\sigma(1)} f_{\sigma(2)} f_{\sigma(3)}$?

Long lasting difficult questions!

Nambu bracket and membrane

Membrane action can be written in terms of Nambu bracket

$$
\begin{aligned}
S &= \int d^3\sigma \sqrt{-\det G_{ij}} + C_{\mu\nu\lambda}\partial_0 X^\mu \partial_1 X^\nu \partial_2 X^\lambda \\
G_{ij} &= \frac{\partial X^\mu}{\partial \sigma_i} \cdot \frac{\partial X_\mu}{\partial \sigma_j} \\
\det G &= \{X^\mu, X^\nu, X^\lambda\}\{X_\mu, X_\nu, X_\lambda\} \\
& \quad \{\bullet, \bullet, \bullet\} : \text{ Nambu bracket}
\end{aligned}
$$

Multiplicities of M-branes

N D-branes : Gauge symmetry U(N)
DOF=N^2=Number of open strings
Expressed by Matrices

M-theory (Review : D. Berman arXiv:0710.1707)

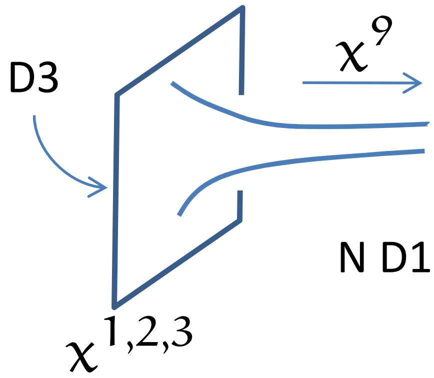
$$\mathrm{N\ \ M2\text{-}branes} \quad \mathrm{O}(\mathrm{N}^{3/2})$$
$$\mathrm{N\ \ M5\text{-}branes} \quad \mathrm{O}(\mathrm{N}^{3})$$

- AdS/CFT absorption cross section
- Brane thermodynamics
- Anomaly cancellation

Basu-Harvey

hep-th/0412310

D1-D3 system



N D1=BPS monopole

D3: 0 1 2 3
D1: 0 9

From D3: BPS Monopole solution

$$X^9 = \frac{N\pi\alpha'}{\sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2}}$$

(multiple) D1 viewpoint

Nahm equation: $$\frac{\partial X^i}{\partial X^9} \pm \frac{i}{2}\epsilon^{ijk}[X^j, X^k] = 0 \qquad X^i : \ N \times N \text{ matrices}$$

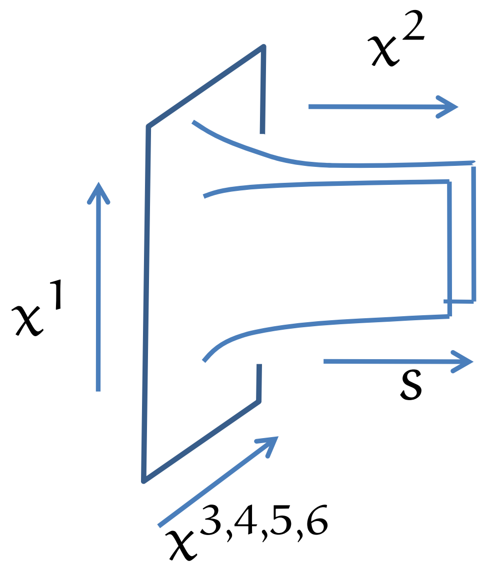
Solution $$X^i = \pm\frac{1}{2X^9}\alpha^i, \quad [\alpha^i, \alpha^j] = 2i\epsilon^{ijk}\alpha^k$$

$$\alpha^i \ : \ N \text{ dim representation of } SU(2)$$

$$R = \sqrt{\frac{(2\pi\alpha')^2}{N}\sum_i \mathrm{Tr}(X^i)^2} \sim \frac{\pi\alpha' N}{X^9}$$

Match!

M2-M5



M2 0 1 2
M5 0 1 3 4 5 6

Self-dual string on M5

$$s \sim \frac{N}{R^2}, \quad \text{“ridge” solution}$$

$$R^2 = (X^3)^2 + (X^4)^2 + (X^5)^2 + (X^6)^2$$

Eq. that corresponds to Nahm's equation

$$\frac{dX^i}{ds} = \frac{\lambda M_{11}^3}{4!8\pi} \epsilon_{ijkl}[G, X^j, X^k, X^l]$$

$$X^i: \text{ fuzzy } S^3, \quad G^2 = 1$$

How to derive Basu-Harvey equation?

→ Motivation for Bagger-Lambert-Gustavsson model

Bagger-Lambert-Gustavsson (BLG) model

J. Bagger, N. Lambert, arXiv:0611108, 0711.0955, 0712.3738
A. Gustavsson: 0709.1260

Lie 3-algebra

T^a $(a = 1 \sim \dim \mathcal{A} = D)$ basis

" Lie 3-algebra": $[T^a, T^b, T^c] = f^{abc}{}_d T^d$

"Metric": $\langle T^a, T^b \rangle = h^{ab}$

"Fundamental identity (FI)" (= generalized Jacobi law)

$$[T^a, T^b, [T^c, T^d, T^e]] = [[T^a, T^b, T^c], T^d, T^e] + [T^c, [T^a, T^b, T^d], T^e] + [T^c, T^d, [T^a, T^b, T^e]]$$

Invariance of metric

$$\langle [T^a, T^b, T^c], T^d \rangle + \langle T^c, [T^a, T^b, T^d] \rangle = 0$$

BL action

$$L = -\frac{1}{2}\langle D^{\mu}X^{I}, D_{\mu}X^{I}\rangle + \frac{i}{2}\langle\bar{\Psi}, \Gamma_{\mu}D^{\mu}\Psi\rangle$$
$$+\frac{i}{4}\langle\bar{\Psi}, \Gamma_{IJ}[X^{I}, X^{J}, \Psi]\rangle - V(X) + L_{CS}$$

$$V(X) = \frac{1}{12}\langle[X^{I}, X^{J}, X^{K}], [X^{I}, X^{J}, X^{K}]\rangle$$

$$L_{CS} = \frac{1}{2}\epsilon^{\mu\nu\lambda}(f^{abcd}A_{\mu ab}\partial_{\nu}A_{\lambda cd} + \frac{2}{3}f^{cda}{}_{g}f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef})$$

$$X^{I} = X^{I}_{a}T^{a}\ (I = 1 \sim 8), \quad \Psi = \Psi_{a}T^{a}$$
$$(D_{\mu}X^{I})_{a} = \partial_{\mu}X^{I}_{a} - f^{cdb}{}_{a}A_{\mu cd}X^{I}_{b}$$

Gauge symmetry

$$\delta_\Lambda X^I_a = f^{bcd}{}_a \Lambda_{bc} X^I_d$$
$$\delta_\Lambda \Psi_a = f^{bcd}{}_a \Lambda_{bc} \Psi_d$$
$$\delta_\Lambda A_{\mu ab} = \partial \Lambda_{ab} - f^{cde}{}_a A_{\mu cd} \Lambda_{eb} + f^{cde}{}_b A_{\mu cd} \Lambda_{ea}$$

Supersymmetry (N=8 maximal SUSY)

$$\delta X^I = i\bar{\epsilon}\Gamma^I \Psi$$
$$\delta \Psi = D_\mu X^I \Gamma^\mu \Gamma^I \epsilon - \frac{1}{6}[X^I, X^J, X^K]\Gamma_{IJK}\epsilon$$
$$\delta(\tilde{A}_\mu)^b_a = i\bar{\epsilon}\Gamma_\mu \Gamma_I X^I_a \Psi_d f^{cdb}{}_a, \quad (\tilde{A}_\mu)^b_a = f^{cdb}{}_a A_{\mu cd}$$

Properties of BLG model

- Gauge symmetry based on Lie 3-algebra
- Maximal SUSY (N=8)
- No arbitrary parameter except for structure constant
- Gauge field described by Chern-Simons Lagrangian (No propagating d.o.f)
- Lagrangian is defined only through structure constant (adjoint representation)
- BPS equation takes the form of Basu-Harvey

NO GO theorem

First example: A_4 -- SO(4) inv. algebra (BLG)

$$f^{abc}{}_{d} = \epsilon_{abcd}\,, \quad h^{ab} = \delta^{ab}$$

cf. Kawamura

Other algebra? many studies

- FI
- h > 0
- finite D

⇒ Only possible Lie 3-algebra is A_4 and its direct sum !
(NO GO THEOREM)

HHM(1) Conjecture
Papadopoulos 0804.2662
Gauntlett & Gutowski 0804.3078

Escape from NO-GO theorem

With milder condition, there exists other Lie 3-algebras

HHM (1)

- FI
- finite D
- negative/null norm generators

→

Lie algebra+ extra
Fuzzy S3
Examples from linear NP
Many examples with null states

- FI
- $h > 0$
- infinite D

→

Nambu-Poisson bracket on 3 dim manifold

Examples (w/neg, null norm gen.)

Lie algebra + 1 extra generator

$$\begin{cases} T^i & (i = 1, \cdots, \dim g) \\ T^0 & \end{cases}$$

cf. Awata, Li, Minic, Yoneya '99

$$[T^0, T^i, T^j] = f^{ij}{}_k T^k, \quad [T^i, T^j, T^k] = 0$$

Fuzzy S^3 (for N=1, it coincides with A_4)

$$\{f_1, f_2, f_3\} = x_i \epsilon_{ijkl} \partial_j f_1 \partial_k f_2 \partial_l f_3$$

$$X_{i_1 \cdots i_l} = x_{i_1} \cdots x_{i_l}, \quad l \leq N$$

$$[X_{i_1, \cdots, i_l}, X_{j_1, \cdots, j_m}, X_{k_1, \cdots, k_n}]$$

$$= \begin{cases} \{x_{i_1} \cdots x_{i_l}, \cdots, x_{k_1} \cdots x_{k_n}\} & l + m + n \leq N + 2 \\ 0 & \text{otherwise} \end{cases}$$

Nambu-Poisson bracket

$$\{f_1, f_2, f_3\} = \underline{P^{\mu_1, \mu_2, \mu_3}}(x)\partial_{\mu_1} f_1 \partial_{\mu_2} f_2 \partial_{\mu_3} f_3$$

anti-symmetric tensor

FI

$$\{f_1, f_2, \{f_3, f_4, f_5\}\} = \{\{f_1, f_2, f_3\}, f_4, f_5\} + \{f_3, \{f_1, f_2, f_4\}, f_5\} + \{f_3, f_4, \{f_1, f_2, f_5\}\}$$

It implies a very strong constraint on P !!

Decomposability

$$P_i := P_i^{\mu}\partial_\mu \qquad (i = 1, 2, 3)$$

$$P = P^{\mu_1 \mu_2 \mu_3}\partial_{\mu_1} \wedge \partial_{\mu_2} \wedge \partial_{\mu_3} = P_1 \wedge P_2 \wedge P_3$$

Nambu-Poisson bracket exists only in 3 dimensions (at least locally)

Direct product is not allowed

$$P = \partial_1 \wedge \partial_2 \wedge \partial_3 + \partial_4 \wedge \partial_5 \wedge \partial_6$$

$$\{y_1 y_4, y_2, \{y_3, y_5, y_6\}\} = 0 \quad \text{but}$$

$$\{\{y_1 y_4, y_2, y_3\}, y_5, y_6\} + \cdots = 1$$

FI is violated!!

N: 3 dim manifold with NP structure

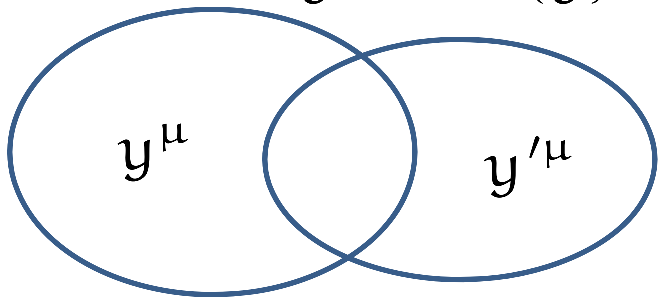
Takhtajan and others

$$\chi^a(y) \;:\; \text{basis of functions on N}$$

$$\{\chi^a, \chi^b, \chi^c\} = f^{abc}{}_d \chi^d \quad (a = 1, 2, \cdots, \infty)$$

$$f^{abc}{}_d \text{ satisfies FI}$$

$$y'^\mu = f^\mu(y)$$



Transition function f^μ must be volume preserving diffeo. to keep NP structure

Implication to BLG

finite D, Negative/Null norm generator

J.Gomis et.al 0805.1012
S.Benvenuti et.al. 0805.1087
HIM(3) 0805.1202

It turns out to be possible to escape from ghost by carefully choose algebra and new Higgs mechanism !!
→ *Multiple D2*

infinite D 3-algebra from Nambu-Poisson manifold *N*

$$3\mathrm{d} \left\{ \begin{array}{l} X^I \ (I = 1 \sim 8) \\ \Psi \\ A_{\mu ab} \end{array} \right. \Longrightarrow 6\mathrm{d} \left\{ \begin{array}{l} X^i \ (I = i \sim 5) \\ \Psi \\ B_{\mu\nu} \ (\text{self-dual}) \end{array} \right.$$

→ *BL Lagrangian describes M5 brane*

HM (2) HIMS (4)

In both cases, they describe various branes in M/String theories!

M5 from M2

HM, arXiv:0804.3629
HIMS, arXiv:0805.2898

$\mathcal{M}$:original membrane worldvolume x^{μ}

$\mathcal{N}$:3 dim mfd where NP structure is defined $y^{\dot{\mu}}$

M2 (3d) → M5 (6d)

$\mathcal{M}$ → $\mathcal{M} \times \mathcal{N}$

$$X^I_a T^a = X^I_a \chi^a(y) \quad \rightarrow \quad X^I(x,y)$$

$$\Psi_a T^a = \Psi_a \chi^a(y) \quad \rightarrow \quad \Psi(x,y)$$

$$A_{\mu ab}\chi^a(y)\chi^b(y') \quad \rightarrow \quad A_\mu(x,y,y')$$

Gauge field bi-local in $y^{\dot{\mu}}$ *?*

Gauge field bilocal in N?

→ No! They appear only in the combination

$$
\begin{aligned}
A_{\mu a b} f^{abc}{}_{d} &= A_{\mu a b} \langle \{ \chi^a, \chi^b, \chi^c \}, \chi_d \rangle \\
&= \int_{\mathcal{N}} \epsilon_{\dot{\mu}\dot{\nu}\dot{\lambda}} \frac{\partial}{\partial y^{\dot{\mu}}} \frac{\partial}{\partial y'^{\dot{\nu}}} A_\mu(x, y, y') \Big|_{y'=y} \frac{\partial \chi^c}{\partial y^{\dot{\lambda}}} \chi^d
\end{aligned}
$$

So what we need is,

$$
\frac{\partial}{\partial y'^{\dot{\nu}}} A_\mu(x, y, y') \Big|_{y'=y} = \underline{b_{\mu\dot{\nu}}(x, y)}
$$

local field in 6d

Division of transverse space

$$I = 1 \sim 8$$

$$\dot{\mu} = \dot{1} \sim \dot{3} \quad : \quad \text{identify with } \mathcal{N}$$

$$i = 1 \sim 5 \quad : \quad \text{transverse direction of M5}$$

We put,

$$X^{\dot{\mu}}(x, y) = y^{\dot{\mu}} + b^{\dot{\mu}}(x, y) = y^{\dot{\mu}} + \frac{1}{2}\epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}} b_{\dot{\nu}\dot{\lambda}}(x, y)$$

Fields on M5

$$X^i, \Psi, b_{\mu\dot{\nu}}, b_{\dot{\mu}\dot{\nu}}$$

Self-dual two form after field redefinition

Gauge symmetry on M5

$$
\begin{aligned}
\delta_\Lambda \Phi &= \Lambda_{ab}(x) f^{abc}{}_d \Phi_c \chi^d(y) \\
&= \Lambda_{ab}\{\chi^a, \chi^b, \Phi\} \\
&= \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}} \Lambda_{ab}(x) \partial_{\dot{\mu}} \chi^a \partial_{\dot{\nu}} \chi^b \partial_{\dot{\lambda}} \Phi \\
&= \delta_\Lambda y^{\dot{\lambda}} \partial_{\dot{\lambda}} \Phi
\end{aligned}
$$

$$
\begin{aligned}
\delta_\Lambda y^{\dot{\lambda}} &= \epsilon^{\dot{\lambda}\dot{\mu}\dot{\nu}} \partial_{\dot{\mu}} \Lambda_{\dot{\nu}}(x, y) \\
\Lambda_{\dot{\mu}}(x, y) &= \partial'_{\dot{\mu}} (\Lambda_{ab}(x) \chi^a(y) \chi^b(y'))|_{y'=y}
\end{aligned}
$$

$$
\partial_{\dot{\mu}} \delta_\Lambda y^{\dot{\mu}} = 0 \quad \leftrightarrow \quad \delta_\Lambda : \ \text{Volume preserving diffeo}
$$

BLG gauge symmetry reduces to volume preserving diffeo of N

Covariant derivative

Covariant derivative in x (M) direction

$$D_\mu\Phi = \partial_\mu\Phi - \epsilon^{\dot\mu\dot\nu\dot\rho}\partial_{\dot\mu} b_{\lambda\dot\nu}(x,y)\partial_{\dot\rho}\Phi$$

Covariant derivative in y (N) direction

$$\begin{aligned} D_{\dot\mu}\Phi &= \frac{1}{2}\epsilon_{\dot\mu\dot\nu\dot\lambda}\{X^{\dot\nu}, X^{\dot\lambda}, \Phi\} \\ &= \partial_{\dot\mu}\Phi + (\partial_{\dot\lambda} b^{\dot\lambda}\partial_{\dot\mu}\Phi - \partial_{\dot\mu} b^{\dot\lambda}\partial_{\dot\lambda}\Phi) + \cdots \end{aligned}$$

This is covariant because of fundamental identity

$$\delta_\Lambda\{\Phi_1,\Phi_2,\Phi_3\} = \{\delta_\Lambda\Phi_1,\Phi_2,\Phi_3\} + \{\Phi_1,\delta_\Lambda\Phi_2,\Phi_3\} + \{\Phi_1,\Phi_2,\delta_\Lambda\Phi_3\}$$

M5 Action

Quadratic part (HM2, complete formula HIMS4)

$$
\begin{aligned}
L^{quad} \;=\;& -\frac{1}{2}[(\partial_\mu X^i)^2 + (\partial_{\dot\mu} X^i)^2] + \frac{i}{2}\langle \bar\Psi, (\Gamma^\mu \partial_\mu + \Gamma^{\dot\mu}\partial_{\dot\mu})\Psi \\
& -\frac{1}{4}F^2_{\mu\dot\nu\dot\lambda} - \frac{1}{12}{F_{\dot\mu\dot\nu\dot\lambda}}^2 - \frac{1}{2}\epsilon^{\mu\nu\lambda}\epsilon^{\dot\mu\dot\nu\dot\lambda}\partial_\mu b_{\nu\dot\mu}\partial_{\dot\nu} b_{\lambda\dot\lambda}
\end{aligned}
$$

Lorentz covariant for X^i and Ψ
but non-covariant for b fields

However, the e.o.m becomes covariant even for b fields

→ self-dual two form fields on M5

Summary of M5 construction

- We apply inverse of KK reduction to BLG model which reproduces M5 action
- Gauge symmetry on M5 is VPD of fiber N. The linear part of transformation is usual gauge symmetry on M5
- There are `nonassociativity' from three form background which define Nambu-Poisson structure on N
- Action is not covariant but the equation of motion becomes covariant.

D2 from M2

HIM(3) arXiv:0805.1202
J.Gomis, G.Milousi, J.G.Russo 0805.1012
S.Benvenuti, D.Rodriguez-Gomez, E.Tonin, H.Verlinde
arXiv: 0805.1087

Lie 3 from Lie 2

Lie algebra g $[T^i, T^j] = f^{ij}{}_k T^k$

↓ extra generators $T^0 \quad T^{-1}$

Lie 3-algebra

$$[T^0, T^i, T^j] = f^{ij}{}_k T^k$$

$$[T^i, T^j, T^k] = f^{ijk} T^{-1}$$

$$[T^{-1}, T^a, T^b] = 0$$

T^0 appear only on LHS

T^{-1} center

Invariant metric

$$\langle T^{-1}, T^0 \rangle = -1 \,, \quad \langle T^i, T^j \rangle = h^{ij}$$ Killing form

Lorentzian signature

We expand

$$
\begin{aligned}
X^I &= X^I_0 T^0 + X^I_{-1} T^{-1} + \hat{X}, \quad \hat{X}^I = X^I_i T^i \\
A_\mu &= T^{-1} \otimes A_{\mu(-1)} - A_{\mu(-1)} \otimes T^{-1} \\
&\quad + T^0 \otimes \hat{A}_\mu - \hat{A}_\mu \otimes T^0 + A_{\mu ij} T^i \otimes T^j \\
\hat{A}_\nu &= A_{\mu 0 i} T^i, \quad A'_\mu := A_{\mu ij} f^{ij}{}_k T^k
\end{aligned}
$$

The bosonic part of the lagrangian becomes

$$
\begin{aligned}
L &= -\frac{1}{2}(\hat{D}_\mu \hat{X}^I - A'_\mu X^I_0)^2 + \frac{1}{4}(X_0)^2[\hat{X}^I, \hat{X}^J]^2 - \frac{1}{2}(X^I_0[\hat{X}^I, \hat{X}^J])^2 \\
&\quad + \frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu} A'_\lambda + (\text{fermion}) + L_{gh}
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{D}_\mu \hat{X}^I &= \partial_\mu \hat{X}^I - [\hat{A}_\mu, \hat{X}^I] \\
\hat{F}_{\mu\nu} &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - [\hat{A}_\mu, \hat{A}_\nu]
\end{aligned}
$$

Treatment of ghost lagrangian

$$L_{gh} = -\partial_\mu X_0^I A'_\mu \hat{X}^I + \partial_\mu X_0^I \partial_\mu X_{-1}^I + (\text{fermion})$$

Variation of X_{-1}

$$\Rightarrow \quad \partial^2 X_0^I = 0, \quad \Gamma^\mu \partial_\mu \Psi_0 = 0$$

X_0 and Ψ_0 are free field!

We can treat them as classical fields!

One can set $X_0^I = \text{const.}(= v\delta^I_{10}), \quad \Psi_0 = 0$
without losing consistency of e.o.m.

SUSY and Gauge symmetry can be kept

After this L_{gh}=0! No ghost in the theory

Fix to absorb ghost → New gauge symmetry

Bandres, Lipstein, Schwarz /Gomis, Rodriguez-Gomez, Raamsdonk, H. Verlinde

D2 action

One can integrate A'_μ in the Lagrangian

$$L_{eff} = -\frac{1}{2}(\hat{D}_\mu X^A)^2 + \frac{v^2}{4}[X^A, X^B]^2 + \frac{i}{4}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} - \frac{1}{4v^2}\hat{F}^2_{\mu\nu} \qquad (A, B = 3 \sim 9)$$

Higgs-like mechanism :

X^{10} is converted into d.o.f. of gauge fields

cf. Mukhi, Papageorgakis 0803.3218

We obtained multiple D2 action with arbitrary gauge group

Derivation from M5

One may derive multiple D2 action by another path

M2 → M5 → D4 → D2

- M2 → M5: Nambu-Poisson
- M5 → D4: Winding in 10th dim
- D4 → D2: Quantization of Nambu bracket

Interpretation of T_0 → Winding mode of M5 along X^{10}

$$\mathfrak{f}, \mathfrak{g}: \text{ functions of } y^{\dot{1}}, y^{\dot{2}}$$

$$\{y^{\dot{3}}, \mathfrak{f}, \mathfrak{g}\}_{\mathrm{NP}} = \{\mathfrak{f}, \mathfrak{g}\}_{\mathrm{Poisson}}$$

$$\{\mathfrak{f}, \mathfrak{g}, \mathfrak{h}\}_{\mathrm{NP}} = \text{center}$$

Conclusion

- BLG model is a real breakthrough to understand M-theory
- It is the first realistic model where 3-algebra becomes the gauge symmetry
- It can describe multiple M2, D2, D4, M5 in the same framework
- This is still the first stage of long works!

Future directions

- How to derive $N^{3/2}$ law?
  - Genuine (?) three algebra needed?
- Quantum Nambu bracket?
- Multiple M5, derivation of N^3 law
- Other proposals
  - M2 brane based on Lie algebra: *Gaiotto and Witten; Aharony, Bergman, Jafferis, Maldacena*