Associativity Anomaly

in

Open String Field Theory

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hep-th/0202030

0202XXX

(Moyal Formulation of
Open String Field Theory)

§1. Introduction

What is D-brane?

Soliton of Open (Closed) String

Classical Solution of OSFT?

Vacuum String Field Theory (RSZ)

$$S = \frac{1}{2}\int \Psi \star \mathcal{Q}\Psi + \frac{1}{3}\int \Psi \star \Psi \star \Psi$$

$\mathcal{Q}$: pure ghost BRST operator

- no cohomology
- background independent

$$\Psi = \Psi^{gh} \otimes \Psi^{matter}$$

$$\mathcal{Q}\Psi^{gh} + \Psi^{gh} \star \Psi^{gh} = 0$$

$$\boxed{\Psi^{Matter} \star \Psi^{Matter} = \Psi^{Matter}}$$

If one can replace

$$\star^{\text{Witten}} \Rightarrow \star^{\text{Moyal}}$$

$\boxed{\quad}$ is the equation of motion of

Noncommutative Soliton

The issue

To what extent we can regard
Open string Field Theory
as the noncommutative geometry
in infinite dimensional space ?

§2. Map from $\star^{\text{Witten}}$ to $\star^{\text{Moyal}}$

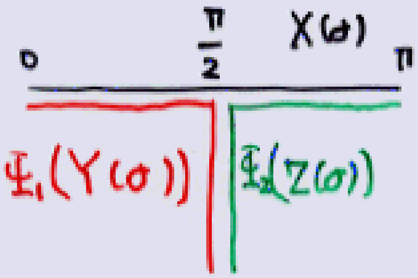
(after Bars' work hep-th/0106157)

Witten's star product

$$\Phi_1 \star \Phi_2 (X(\sigma)) = \int \prod_{\sigma=\frac{\pi}{2}}^{\pi} dY(\sigma) \prod_{\sigma'=0}^{\pi/2} dZ(\sigma')$$

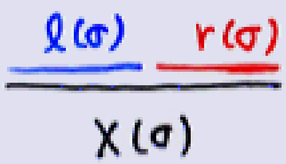
$$\prod_{\sigma=\frac{\pi}{2}}^{\pi} \delta\left(Y(\sigma) - Z(\pi-\sigma)\right) \prod_{\sigma=0}^{\frac{\pi}{2}} \delta\left(X(\sigma) - Y(\sigma)\right) \prod_{\sigma=\frac{\pi}{2}}^{\pi} \delta\left(X(\sigma) - Z(\sigma)\right)$$

$$\times\ \Phi_1(Y(\sigma))\ \Phi_2(Z(\sigma))$$



Split string formalism

$X(\sigma) \Longrightarrow l(\sigma),\ r(\sigma)$



$$(\Phi_1 \star \Phi_2)(l, r) = \int D(t)\ \Phi_1(l, t)\ \Phi_2(t, r)$$

looks like matrix multiplication

$$\sum_j A_{ij} B_{jk}$$

Split string $\longrightarrow$ Moyal product (one variable example)

Split string

$$\boxed{\psi(l,r)}$$

$$(\psi_1 \star \psi_2)(l,r) \equiv \int_{-\infty}^{\infty} dt \; \psi_1(l,t)\, \psi_2(t,r)$$

$$A(x,p) = \int dy\, e^{-iyp}\, \psi\left(\frac{x+y}{2}, \frac{x-y}{2}\right)$$

(Wigner function!)

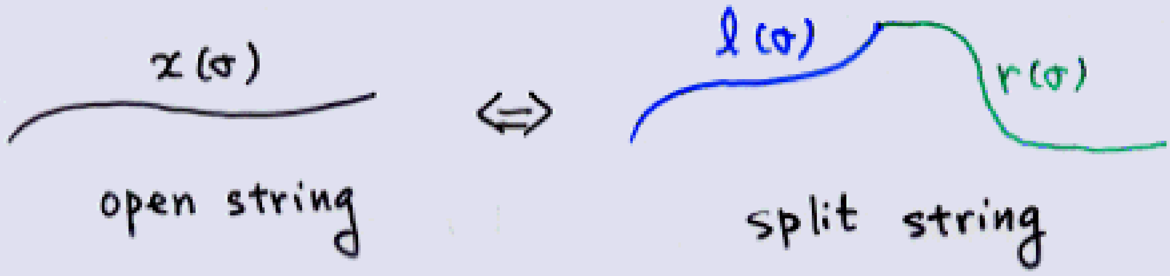
Moyal

$$\boxed{A(x,p)}$$

$$(A_1 \star A_2)(x,p) = e^{\frac{i}{2}(\partial_x \partial_{p'} - \partial_{x'} \partial_p)} \times A_1(x,p)\, A_2(x',p')\Big|_{(x',p')=(x,p)}$$

Split string $\approx$ Moyal formalism

Split string formalism



Subtle point

Boundary condition of l, r at mid point

(A) Dirichlet : Gross - Taylor

$$l\left(\tfrac{\pi}{2}\right) = r\left(\tfrac{\pi}{2}\right) = \bar{x}$$

$$l(\sigma) = \bar{x} + \sqrt{2} \sum_n l_{2n-1} \cos(2n-1)\sigma$$

$$r(\sigma) = \bar{x} + \sqrt{2} \sum_n r_{2n-1} \underline{\cos(2n-1)\sigma}$$

odd mode expansion

(B) Neumann : Bordes et. al.

$$l'\left(\tfrac{\pi}{2}\right) = r'\left(\tfrac{\pi}{2}\right) = 0$$

$$l(\sigma) = l_0 + \sqrt{2} \sum_{n=1}^{\infty} l_{2n} \cos(2n\sigma)$$

$$r(\sigma) = r_0 + \sqrt{2} \sum_{n=1}^{\infty} r_{2n} \underline{\underline{\cos(2n\sigma)}}$$

even mode expansion

Open string

$$x(\sigma) = x_0 + \sqrt{2} \sum_{n=1}^{\infty} x_n \underbrace{\cos(n\sigma)}_{\text{even \& odd}}$$

Open string $\qquad\qquad$ Split String

$$\begin{pmatrix} \text{odd} \\ \text{even} \end{pmatrix} \to \begin{pmatrix} \text{odd} \\ \text{odd} \end{pmatrix} \text{ or } \begin{pmatrix} \text{even} \\ \text{even} \end{pmatrix}$$

Translation between (odd) $\leftrightarrow$ (even) is mandatory

$$T_{2n,2m-1} = \frac{4}{\pi} \int_0^{\pi/2} d\sigma \, \cos(2n\sigma) \cos(2m-1)\sigma$$

$$= \frac{2(-1)^{m+n+1}}{\pi} \left(\frac{1}{2m-1+2n} + \frac{1}{2m-1-2n} \right)$$

$$R_{2m-1,2n} = \frac{4}{\pi} \int_0^{\pi/2} d\sigma \left(\cos(2n\sigma) - (-1)^n \right) \cos(2m-1)\sigma$$

$$= T_{2n,2m-1} - (-1)^n T_{0,2m-1}$$

$$\mathcal{H}^{odd} \underset{R}{\overset{T}{\rightleftarrows}} \mathcal{H}^{even}$$

$$\begin{cases} TR = \mathrm{Id}_{\mathcal{H}^{even}} \\ \underline{RT} = \mathrm{id}_{\mathcal{H}^{odd}} \end{cases}$$

infinite sum is absolutely convergent

With the help of this matrix, we can identify

$$\mathcal{H}^{odd} \underset{(?)}{\approx} \mathcal{H}^{even}$$

$$\psi(x) \underset{\text{Splitting}}{\longrightarrow} \tilde{\psi}(\ell, r) \rightarrow A(x_{2n}, p_{2n})$$

$$\propto \int dx^{odd}\, \psi(x^{even}; x^{odd}) \times e^{-\frac{i}{2} p_{2n} T_{2n,2\ell-1} x_{2\ell-1}}$$

$$\boxed{\text{String field algebra}} \underset{?}{\approx} \boxed{\text{Infinite direct product of Moyal planes}}$$

$$\{x_{2n}, p_{2m}\}_\star = i\, \delta_{2n,2m}$$

§3. Associativity anomaly

Basic example:

$$\begin{cases} v_{2n-1} = \frac{1}{\sqrt{2}} T_{0,2n-1} = \frac{2\sqrt{2}}{\pi} \frac{(-1)^{n+1}}{2n-1} & \in \mathcal{H}^{\text{odd}} \\ w_{2n} = \sqrt{2}\,(-1)^{n+1} & \in \mathcal{H}^{\text{even}} \end{cases}$$

① $\boxed{Tv = 0}$

$\because \sum_n T_{2m,2n-1} T_{0,2n-1}$

$$\propto \int_0^{\pi/2} \cos(2m\sigma)\, d\sigma = 0 \qquad (m > 0)$$

↑ completeness of $\{\cos(2n+1)\sigma\}$

However,

$$RT = 1$$

$$(RT)v = 1 \cdot v = v$$

$$\Updownarrow$$

$$R(Tv) = R \cdot 0 = 0$$

Associativity is broken!

② $v = \bar{T} w$

∵ $T_{0,2n-1} + 2\sum_{k=1}^{\infty} (-1)^k T_{2k,2n-1} = 0$

$$\Leftrightarrow \sum_{k=1}^{\infty} (-1)^k \int_0^{\pi/2} \cos(2n-1)\sigma \cos 2k\sigma \, d\sigma = 0$$

$$\Leftrightarrow \int_0^{\pi/2} \cos((2n-1)\sigma) \, \delta(\sigma - \tfrac{\pi}{2}) \, d\sigma = 0$$

On the other hand, we have $T\bar{T} = 1$

$$(T\bar{T})w = 1 \cdot w = w$$

$$\Updownarrow$$

$$T(\bar{T}w) = T \cdot v = 0$$

anomaly!

$$\boxed{\text{Associativity anomaly arises from midpoint subtlety of half-strings!}}$$

Associativity anomaly in half string formalism

Dirichlet at midpoint $\quad y^{\pm} = \dfrac{l^{(0)} \pm r^{(0)}}{2}$

$$\begin{pmatrix} x_0 \\ x^{(e)} \\ x^{(o)} \end{pmatrix} = \begin{pmatrix} 1 & \bar{v} & 0 \\ 0 & T & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ y^+ \\ y^- \end{pmatrix} \equiv \mathcal{J} \begin{pmatrix} \bar{x} \\ y^+ \\ y^- \end{pmatrix}$$

or

$$\begin{pmatrix} \bar{x} \\ y^+ \\ y^- \end{pmatrix} = \begin{pmatrix} 1 & -\bar{w} & 0 \\ 0 & R & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x^{(e)} \\ x^{(o)} \end{pmatrix} \equiv \mathcal{R} \begin{pmatrix} x_0 \\ x^{(e)} \\ x^{(o)} \end{pmatrix}$$

$$\mathcal{J}\mathcal{R} = \mathcal{R}\mathcal{J} = 1 \quad (\Leftrightarrow TR = RT = 1 ,\ \bar{v}R = \bar{w} ,\ \bar{v} = \bar{w}T)$$

but for $\mathcal{V} = \begin{pmatrix} -1 \\ v \\ 0 \end{pmatrix}$

$$\mathcal{J}\mathcal{V} = \begin{pmatrix} -1 + \bar{v}v \\ Tv \\ 0 \end{pmatrix} = 0$$

Anomaly

$$(\mathcal{R}\mathcal{J})\mathcal{V} = \mathcal{V} \qquad \text{versus} \qquad \mathcal{R}(\mathcal{J}\mathcal{V}) = 0$$

Interpretation

$\delta y^+ = \epsilon v$ & $\delta \bar{x} = \epsilon$ gives the same variation in open string mode

Redundancy in split string variable!

Relation with Horowitz-Strominger Anomaly

Horowitz-Strominger

Question : How to express space-time translation in the framework of open string field theory

Answer : Possible but Anomalous

$$\Lambda \equiv P_L |I\rangle$$

Adjoint action of Λ induces translation

$$\Lambda * \Phi(X) - \Phi * \Lambda = \frac{\partial}{\partial \epsilon} \Phi(X(\sigma) + \epsilon)$$

Anomalous

$$(P_{1R} + P_{2L}) |V_{1234}\rangle = 0$$

$$(\bar{x}_1 - \bar{x}_3) |V_{1234}\rangle = 0$$: midpoint is always fixed

but

$$[P_{1R} + P_{2L}, \bar{x}_1 - \bar{x}_3] = -\frac{i}{2}$$

In terms of split string variable

$$P_L^\mu = \sum \underline{v_{2n-1}} \, \partial^\mu_{\ell_{2n-1}}, \quad P_R^\mu = \sum_n \underline{v_{2n-1}} \, \partial_{r_{2n-1}}$$

exactly the element that induces anomaly!

§4. Control of Anomaly

(not yet complete!)

① Projecting out anomalous sector

$$T : \mathcal{H}_{odd} \to \mathcal{H}_{even} \qquad R : \mathcal{H}_{even} \to \mathcal{H}_{odd}$$

problem: $v \in \mathcal{H}_{odd}$ $\quad Tv = 0$

Project out subspace proportional to v

$$\mathcal{H}_{odd} \Rightarrow \mathcal{H}'_{odd} \equiv P\,\mathcal{H}_{odd} = (\bar{T}\,T)\,\mathcal{H}_{odd}$$

$$(\bar{T}\,T = 1 - v\bar{v})$$

Note: v translation of whole space-time

$\hat{}$ Elements of closed string sector

$$\mathcal{H} \nearrow \mathcal{H}^{open} \qquad \underbrace{\delta y^- = v}_{\text{"Unbalanced"}}, \quad \underbrace{\text{all others}}_{\text{Moyal pair}}$$

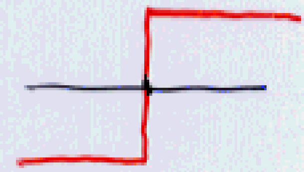
$$\mathcal{H} \searrow \mathcal{H}^{closed} \quad : \quad \delta y^+ = v$$

Note: Similar structure appeared in the paper by Douglas - Liu - Moore - Zwiebach

Subtlety of vertex operator

We have to project out

$\delta l, \delta r = \text{const}$ sector



except for $\delta l = -\delta r$

kink at the midpoint

Remark: vertex operator $\Leftrightarrow$ kink

$$\delta\phi \propto \theta(\sigma_0 - \sigma)$$

Except for $\sigma_0 = \frac{\pi}{2}$, it induces associativity anomaly!

We cannot express all the operators of free boson theory!

Relevant to this observation?

- Closed string vertex $\mathcal{O}_V = \int V(M)\, \Phi$

- GRSZ kinetic term

$$\mathcal{L} = \int \Phi * \left(c\left(\tfrac{\pi}{2}\right) \Phi\right)$$

(B) Finite dimensional cutoff

Basic relations among T & R

① $TR=1$, $RT=1$, $R=\bar{T}+v\bar{w}$

② $v=\bar{T}w$, $w=\bar{R}v$

③ $T\bar{T}=1$, $Tv=0$, $\bar{v}v=1$

Keep ①② and require associativity

③ → $T\bar{T}=1-\dfrac{w\bar{w}}{1+\bar{w}w}$, $Tv=\dfrac{w}{1+\bar{w}w}$

$$\bar{v}v=\frac{\bar{w}\hat{w}}{1+\bar{w}w}$$

(take limit $\bar{w}w\to\infty$ gives ③)

① Can we find a plausible candidate for T, R?

Spectrum condition

$$\bar{T}\,\kappa_e^2\,T=\kappa_o^2 \qquad \kappa_o=\mathrm{diag}\,(2m-1)$$

$$T\,\kappa_o^{-2}\,\bar{T}=\kappa_e^{-2} \qquad \kappa_e=\mathrm{diag}\,(2n)$$

Yes! but still remains some arbitrariness!

§5 Calculation of N-point vertex

Moyal product of Gaussian function

$$A_{N,M,\lambda} = N e^{-\bar{\xi} M \xi - \bar{\xi}\lambda}$$

$$\xi = \begin{pmatrix} x_{2n} \\ p_{2n} \end{pmatrix}$$

Composition rule $\left(\sigma = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)$

$$A_{N_1 M_1 \lambda_1} * A_{N_2 M_2 \lambda_2} = A_{N_{12} M_{12} \lambda_{12}}$$

$$M_{12} = (M_1 + M_2 \sigma M_1)(1 + \sigma M_2 \sigma M_1)^{-1} + (M_2 - M_1 \sigma M_2)(1 + \sigma M_1 \sigma M_2)^{-1}$$

$$\lambda_{12} = (1 - M_1 \sigma)(1 + M_2 \sigma M_1 \sigma)^{-1} \lambda_2 + (1 + M_2 \sigma)(1 + M_1 \sigma M_2 \sigma)^{-1} \lambda_1$$

$$N_{12} = N_1 N_2 \left(\det \left(\frac{M_{12}\sigma}{M_1 \sigma + M_2 \sigma} \right) \right)^{1/2} e^{\frac{1}{4} (\bar{\lambda}_1 + \bar{\lambda}_2)(M_1 + M_2)^{-1} (\lambda_1 + \lambda_2) - \bar{\lambda}_{12} M_{12}^{-1} \lambda_{12}}$$

Trace formula

$$\mathrm{Tr}\, A_{N,M,\lambda} = \frac{N}{\sqrt{\det(2M\sigma)}} e^{\frac{1}{4} \bar{\lambda} M^{-1} \lambda}$$

N-th product & its trace

$$\left(\begin{array}{l} M_i \sigma = m_i \\ [m_i, m_j] = 0 \end{array} \right)$$

$$A_{N_1 M_1 \lambda_1} \star \cdots \star A_{N_n M_n \lambda_n} = A_{N_{1\cdots n} M_{1\cdots n} \lambda_{1\cdots n}}$$

$$m_{12\cdots n} = M_{12\cdots n}\sigma = \frac{J^-_{12\cdots n}}{J^+_{12\cdots n}}$$

$$J^\pm_{12\cdots n} = \frac{1}{2}\left(\prod_{k=1}^{n} (1+m_k) \pm \prod_{k=1}^{n} (1-m_k) \right)$$

$$\lambda_{12\cdots n} = \sum_{i=1}^{n} \left(\frac{1}{J^+_{12\cdots n}} \prod_{k=1}^{i-1} (1-m_k) \prod_{l=k+1}^{n} (1+m_l) \right) \lambda_i$$

$$N_{12\cdots n} = \frac{N_1 \cdots N_n}{\det (J^+_{12\cdots n})^{1/2}} \exp\left(\frac{1}{4} K_{12\cdots n} \right)$$

$$K_{12\cdots n} = \sum_{i=1}^{n} \bar{\lambda}_i \frac{J^-_{12\cdots n}}{J^+_{12\cdots n}} \lambda_i - 2 \sum_{i<j} \bar{\lambda}_i \frac{\prod^{i-1}(1-m_k) \prod_{i+1}^{j-1} (1-m_k) \prod_{j+1}^{n} (1+m_k)}{J^+_{12\cdots n}} \lambda_j$$

$$\mathrm{Tr}\left(A_{N_1 M_1 \lambda_1} \star \cdots \star A_{N_n M_n \lambda_n} \right)$$

$$= \frac{N_1 \cdots N_n}{\det (2J^-_{12\cdots n})^{1/2}} \exp\left(\frac{1}{4} Q_{12\cdots n} \right)$$

$$Q_{12\cdots n} = \sum_{i} \bar{\lambda}_i \frac{J^+_{12\cdots n}}{J^-_{12\cdots n}} \lambda_i + \sum_{i\neq j} \bar{\lambda}_i \frac{\prod^{i-1}(1+m_k) \prod_{i+1}^{j-1} (1-m_k) \prod_{j+1}^{n} (1+m_k)}{J^-_{12\cdots n}} \lambda_j$$

Angle variable

$$\Theta_\ell \equiv \tan^{-1}(-i\, m_\ell) = \frac{1}{2i} \log\left(\frac{1+m_\ell}{1-m_\ell}\right)$$

$$\Theta_{12} = \tan^{-1}(-i\, m_{12}) = \Theta_1 + \Theta_2$$

$$\Downarrow$$

$$\Theta_{12\cdots n} = \Theta_1 + \cdots + \Theta_n$$

$$\tilde{\lambda}_\ell \equiv \frac{1}{\cos\Theta_\ell}\, \lambda_\ell$$

$$\tilde{\lambda}_{12} = e^{-i\Theta_1}\, \tilde{\lambda}_2 + e^{i\Theta_2}\, \tilde{\lambda}_1$$

$$\Downarrow$$

$$\tilde{\lambda}_{12\cdots n} = \sum_{\ell=1}^{n} e^{-i \sum_{k<\ell} \Theta_k + i \sum_{k>\ell} \Theta_k}\, \tilde{\lambda}_\ell$$

looks like Bethe ansatz ?

Application to SFT

$A_{N,M,\lambda} \Longleftrightarrow$ Generating function of perturbative Fock space

$$e^{\sum \mu_n a_n^\dagger} |0\rangle \equiv |\mu\rangle$$

$\mathrm{Tr}(A_{N_1 M_1 \lambda_1} \star \cdots \star A_{N_n M_n \lambda_n}) \Longleftrightarrow$ n-string vertex

$$\langle V_n | \, |\mu_1\rangle \otimes \cdots \otimes |\mu_n\rangle$$

Dictionary

Fock space $\longleftrightarrow$ x reps $\longleftrightarrow$ Moyal

$a_n^\dagger \cdots |0\rangle$

$$x_n = \frac{i}{2}\sqrt{\frac{2}{n}}\,(a_n - a_n^\dagger)$$

$$p_n = \sqrt{\frac{n}{2}}\,(a_n + a_n^\dagger)$$

$A = \int dx_{odd}\, e^{i p_e T x_o} \psi(x$

$$a_n|0\rangle = 0$$

$$\Longleftrightarrow |0\rangle = e^{-\sum x_n \frac{n}{2} x_n}$$

$|\mu\rangle \Longleftrightarrow A_{N, M_0, \lambda}$

$$M_0 = \begin{pmatrix} \kappa_e & 0 \\ 0 & \bar{T}\kappa_o^{-1}T \end{pmatrix} \qquad \lambda = \begin{pmatrix} i\kappa_e^{\frac{1}{2}}\mu_e \\ -T\kappa_e^{-1}\mu_o \end{pmatrix}$$

$$\kappa_e = \mathrm{diag}(2n), \quad \kappa_o = \mathrm{diag}(2n-1)$$

n-string vertex

overlap condition

$$X_R^i = X_L^{i+1}$$

$$Q_a(\sigma) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \omega^{a(i-1)} X^i(\sigma)$$

$$\boxed{Q_a(\sigma) = \omega^a Q_{n-a}(\pi - \sigma)}$$

overlap condition

(Gross - Jevicki)

$\Leftrightarrow$

$$(1 - Y_a)\,|Q_a)\,|V_n\rangle = (1 + Y_a)\,|P_a)\,|V_n\rangle = 0 \qquad (*)$$

$$Y_a = \mathrm{Re}(\omega^a) C + \mathrm{Im}(\omega^a) X$$

$$\omega = \exp(2\pi i/n)$$

$$C_{nm} = (-1)^n \delta_{n,m}$$

$$X_{n,m} = \frac{i}{\pi}(-1)^{(n-m-1)/2}(1 - (-1)^{n+m})\left(\frac{1}{n+m} + \frac{(-1)^n}{n-m}\right)$$

↑ essentially the same as T

- $\langle V_n | \mu_1\rangle \otimes \cdots \otimes |\mu_n\rangle = \exp\left(V_{nm}^{[rs]} \mu_n^{[r]} \mu_m^{[s]}\right) \equiv F_N(\mu)$

Overlap condition (*)

$\Leftrightarrow$ Diff. eq. for $F_N(\mu)$

$$\boxed{\text{Claim: } T_N \text{ satisfies the same PDE as } F_N(\mu)}$$

§6 Conclusion

- Moyal formulation gives a simple geometrical picture of OSFT

- N-th product gives off-shell string vertex correctly

- At the same time, OSFT is not a direct product of infinite Moyal planes because of associativity anomaly

- Anomaly seems to come from the existence of closed string hidden in OSFT

- So far, treatment of anomaly is still challenging, .. unfortunately!