2018 S-semester

Quantum Field Theory

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Contents

§ 1 Introduction

about this lecture

(i) **Language**
(As a rule of the department, for the graduate course, if there are international students who prefer English, the lecture should be given in English. I'm happy to do so. On the other hand, for the undergraduate course, the lectures are usually given in Japanese. Now, this is a common lecture for both students, and there is no clear rule. . . .

Which one do you prefer?

		Speaking	
		E	J
Writing	E		(*)
	J		-

. . . (*) We choose this option.)

(If you find some wrong or unnatural English in the note, please tell me!)

(ii) **Web page**
Google: Koichi Hamaguchi → Lectures → Quantum Field Theory I

- All the announcements will also be given in this web page.
- The lecture note will also be uploaded and updated every week.

(iii) **Schedule**
April 9, 16, 23,
May 7, 14, 21, 28,
June 4, 11, 18, 25,
July 2, 9, 23.
(I don't check the attendance. You don't have to attend the classes if you can learn by yourself and submit the homework problems.)

(iv) **Grades**
based on the scores of homework problems. Details will be announced later.

(v) **Textbooks**
This course is not based on a specific textbook, but I often refer to the following textbooks during preparing the lecture note.

- M. Srednicki, Quantum Field Theory.
- M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory.
- S. Weinberg, The Quantum Theory of Fields volume I.
- M. D. Schwartz, Quantum Field Theory and the Standard Model.
- 「ゲージ場の量子論 I」九後汰一郎、培風館.
- 「場の量子論」坂井典佑、裳華房.

(vi) **Prerequisites for this lecture (前提知識)**
Basics of Quantum Mechanics and Special Relativity.

(You should be familiar with the following equations (taken from Srednicki [1]).

$$
\begin{aligned}
a^\dagger |n\rangle &= \sqrt{n+1}\,|n+1\rangle\,,\\
J_\pm |j,m\rangle &= \sqrt{j(j+1)-m(m\pm1)}\,|j,m\pm1\rangle\,,\\
A(t) &= e^{iHt}Ae^{-iHt},\\
H &= p\dot{q}-L,\\
E^2 &= \vec{p}^2c^2+m^2c^4.
\end{aligned}
$$

)

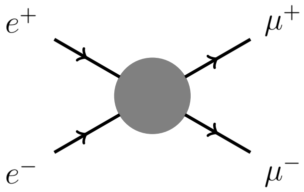
§ 1.1 Course objectives

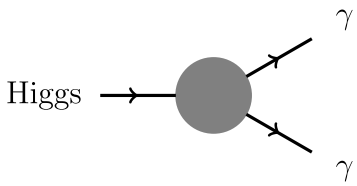
To learn the basics of Quantum Field Theory (QFT).

One of the goals is to understand how to calculate the transition probabilities (such as the cross section and the decay rate) in QFT. (→ § 5.)

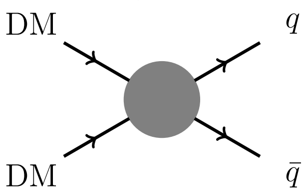
Examples

- at colliers





- in the early universe



(In this lecture: only the scalar interaction.)

§ 1.2 Quantum mechanics and quantum field theory

Quantum Field Theory (QFT) is just Quantum Mechanics (QM) applied to fields.

- QM: $q_i(t)$ $\quad i=1,2,\cdots$ discrete
- QFT: $\phi(\vec{x},t)$ $\quad \vec{x}\cdots$ continous (infinite number of degrees of freedom)
(Note: uncountably infinite, 非可算無限)

	QM	QFT
operators (Heisenberg picture)	$q_i(t), p_i(t)$ or $q_i(t), \dot{q}_i(t)$ $i = 1, 2, \cdots$ discrete $[q_i, p_j] = i\hbar\delta_{ij}$	$\phi(\vec{x}, t), \pi(\vec{x}, t)$ or $\phi(\vec{x}, t), \dot{\phi}(\vec{x}, t)$ $\vec{x} \cdots$ continous $[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y})$
states	(e.g., Harmonic Oscillator) $\|0\rangle$: ground state $a^\dagger \|0\rangle, a^\dagger a^\dagger \|0\rangle, \cdots$ $a^\dagger$ written in terms of q and p	 $\|0\rangle$: ground state $a^\dagger_{\vec{p}} \|0\rangle, a^\dagger_{\vec{p}} a^\dagger_{\vec{p}'} \|0\rangle, \cdots$ $a^\dagger_{\vec{p}}$ written in terms of ϕ and π
observables (expectation value)	$\langle \cdot \| \vec{p} \| \cdot \rangle, \langle \cdot \| H \| \cdot \rangle, \cdots$	$\langle \cdot \| \vec{p} \| \cdot \rangle, \langle \cdot \| H \| \cdot \rangle, \cdots$
transition probability	$P(i \to f) = \|\langle f \| i \rangle\|^2$	$P(i \to f) = \|\langle f \| i \rangle\|^2$

In this lecture, we focus on the relativistic QFT.
(QFT can also be applied to non-relativistic system: condensed matter, bound state,...)

Relativistic QFT is based on QM and SR (special relativity).

- QM: $\hbar \neq 0$ (important at small scale)
- SR: $c < \infty$ (important at large velocity)
- QFT: $\hbar \neq 0$ and $c < \infty$
  (physics at small scale & large velocity: Particle Physics, Early Universe,...)

§ 1.3 Notation and convention

- We will use the natural units

$$\hbar = c = 1\,,$$

where

$$\begin{aligned}\hbar &\simeq 1.055 \times 10^{-34}\ \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{sec}^{-1}\,,\\ c &= 2.998 \times 10^{8}\ \mathrm{m} \cdot \mathrm{sec}^{-1}\,.\end{aligned}$$

For instance, we write

  - $E^2 = p^2 + m^2$ instead of $E^2 = p^2c^2 + m^2c^4$, and
  - $[x, p] = i$ instead of $[x, p] = i\hbar$.

► We will use the following metric.

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

(The sign convention depends on the textbook. $g_{\mu\nu}(\text{here}) = g_{\mu\nu}^{\text{Peskin}} = -g_{\mu\nu}^{\text{Srednicki}}$)

$$\begin{aligned} x^\mu &= (x^0, x^1, x^2, x^3) = (t, \vec{x}) \\ x_\mu &= (x_0, x_1, x_2, x_3) = g_{\mu\nu} x^\nu = (t, -\vec{x}) \\ p^\mu &= (p^0, p^1, p^2, p^3) = (E, \vec{p}) \\ p_\mu &= g_{\mu\nu} p^\nu = (E, -\vec{p}) \\ p \cdot x &= p_\mu x^\nu = p^\mu x_\mu \\ &= p^0 x^0 - p^1 x^1 - p^2 x^2 - p^3 x^3 \\ &= p^0 x^0 - \vec{p} \cdot \vec{x} \\ &= Et - \vec{p} \cdot \vec{x} \end{aligned}$$

If p^μ is the 4-momentum of a particle with mass m,

$$\begin{aligned} p^2 &= p^\mu p_\mu = (p^0)^2 - |\vec{p}|^2 = E^2 - |\vec{p}|^2 \\ &= m^2 . \end{aligned}$$

§ 1.4 Various fields

		spin	equation of motion for free fields	
scalar field	$\phi(x)$	0	$(\Box + m^2)\phi = 0$	Klein-Gordon eq.
fermionic field	$\psi_\alpha(x)$	1/2	$(i\gamma^\mu \partial_\mu - m)\psi = 0$	Dirac eq.
gauge field (vector)	$A_\mu(x)$	1	$\partial^\mu F_{\mu\nu} = 0$ $(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$	(part of) Maxwell eq.

We start from the scalar field.

(The Standard Model of Particle Physics is also written in terms of QFT:

- quarks (u, d, s, c, b, t) and leptons (e, μ, τ, ν_i)... fermionic fields
- γ (photon), $W^\pm$, Z (weak bosons), g (gluon) ... gauge (vector) fields
- H (Higgs) ... scalar field)

§ 1.5 Plan

	spin	Free	Interaction	renormalization,...
scalar	0	§ 2 ①	③ § 5	
fermion	1/2	§ 4 ②	(A-semester ?)	
gauge	1			

(Last year: ① → ③ → ②)

§ 1 Introduction
§ 2 Free Scalar (spin 0) Field
§ 3 Lorentz transformation, Lorentz group and its representations
§ 4 Free Fermionic (spin 1/2) Field
§ 5 Interacting Scalar Field

§ 1.6 Hilbert space and Hamiltonian of (infinitely) many particles

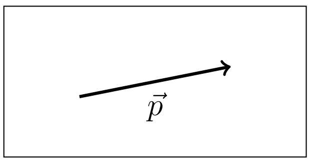
- Consider a scalar particle with mass m.

momentum $\vec{p}$

energy $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$

- Define a one-particle state with momentum $\vec{p}$.

$|\vec{p}\rangle$



which is an eigenstate of the Hamiltonian,

$$H\,|\vec{p}\rangle = E_{\vec{p}}\,|\vec{p}\rangle = \sqrt{|\vec{p}|^2 + m^2}\,|\vec{p}\rangle\,.$$

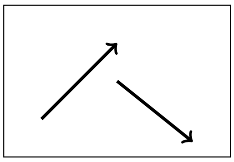
- We want a Hilbert space containing (infinitely) many one-particle states,

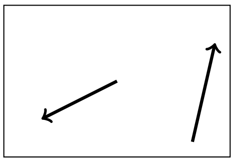
$$\{|\vec{p}_1\rangle\,,\ |\vec{p}_2\rangle\,,\ |\vec{p}_3\rangle\,,\ \cdots\}$$
$$H\,|\vec{p}_i\rangle = E_i\,|\vec{p}_i\rangle\,, \qquad E_i = \sqrt{|\vec{p}_i|^2 + m^2}.$$

$$\begin{pmatrix} & & \\ & H & \\ & & \end{pmatrix}\begin{pmatrix}|\vec{p}_1\rangle\\ |\vec{p}_2\rangle\\ \cdot\\ \cdot\\ \cdot\end{pmatrix} = \begin{pmatrix}E_1 & & & & \\ & E_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot\end{pmatrix}\begin{pmatrix}|\vec{p}_1\rangle\\ |\vec{p}_2\rangle\\ \cdot\\ \cdot\\ \cdot\end{pmatrix}$$

- We also want the Hilbert space to contain two particle states,

$$\{|\vec{p}_1, \vec{p}_2\rangle\ ,\ |\vec{p}_3, \vec{p}_4\rangle\ ,\ \cdots\}$$



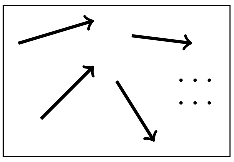


$\cdots\cdots$

$$\begin{aligned} H\,|\vec{p}_i, \vec{p}_j\rangle &= (E_i + E_j)\,|\vec{p}_i, \vec{p}_j\rangle \\ &= \left(\sqrt{|\vec{p}_i|^2 + m^2} + \sqrt{|\vec{p}_j|^2 + m^2}\right)|\vec{p}_i, \vec{p}_j\rangle\,. \end{aligned}$$

- and 3-particle, 4-particle,... n-particle states.

$$\{|\vec{p}_i, \vec{p}_j, \vec{p}_k\rangle\ ,\ \cdots|\vec{p}_1, \cdots\vec{p}_n\rangle\ ,\ \cdots\}$$
$$H\,|\vec{p}_1, \cdots\vec{p}_n\rangle = (E_1 + \cdots + E_n)\,|\vec{p}_1, \cdots\vec{p}_n\rangle$$



- We want <u>all these states</u> in the same Hilbert space,

$$\begin{pmatrix} & & \\ & H & \\ & & \end{pmatrix}\begin{pmatrix}|\vec{p}_1\rangle\\ \cdot\\ \cdot\\ |\vec{p}_1, \vec{p}_2\rangle\\ \cdot\\ \cdot\\ \cdot\\ |\vec{p}_1, \cdots\vec{p}_n\rangle\\ \cdot\end{pmatrix} = \begin{pmatrix}E_1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & E_1 + E_2 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & \sum_i^n E_n \\ & & & & & & & & \cdot\end{pmatrix}\begin{pmatrix}|\vec{p}_1\rangle\\ \cdot\\ \cdot\\ |\vec{p}_1, \vec{p}_2\rangle\\ \cdot\\ \cdot\\ \cdot\\ |\vec{p}_1, \cdots\vec{p}_n\rangle\\ \cdot\end{pmatrix} \quad (\star).$$

(With interactions, the off-diagonal elements become non-zero, and the transition between these states can occur, such as particle scattering and particle decay.)

- The Hilbert space and the Hamiltonian in eq.($\star$) can be expressed in a much simpler way, by creation and annihilation operators as

$$\begin{aligned}
&|\vec{p}\rangle \propto a^\dagger_{\vec{p}}\,|0\rangle\,, \\
&|\vec{p}_1,\vec{p}_2\rangle \propto a^\dagger_{\vec{p}_1}a^\dagger_{\vec{p}_2}\,|0\rangle\,, \\
&|\vec{p}_1,\cdots\vec{p}_n\rangle \propto a^\dagger_{\vec{p}_1}\cdots a^\dagger_{\vec{p}_n}\,|0\rangle\,, \\
&H = \int \frac{d^3q}{(2\pi)^3}E_q a^\dagger_{\vec{q}}a_{\vec{q}}
\end{aligned}$$

where $a^\dagger_{\vec{p}}$ and $a_{\vec{p}}$ are the annihilation and creation operators, respectively, and

$$\begin{aligned}
&[a_{\vec{p}}, a^\dagger_{\vec{q}}] = a_{\vec{p}}a^\dagger_{\vec{q}} - a^\dagger_{\vec{q}}a_{\vec{p}} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}), \\
&[a_{\vec{p}}, a_{\vec{q}}] = 0, \\
&[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0, \\
&\quad |0\rangle \text{ :vacuum state,} \quad a_{\vec{p}}\,|0\rangle = 0.
\end{aligned}$$

► Let's check it. (→ next week).

——————— on April 9, up to here. ———————

Questions after the lecture:

Q: What is the (3) of $\delta^{(3)}(\vec{p}-\vec{q})$?

A: It means $\delta^{(3)}(\vec{p}-\vec{q}) = \delta(p_1-q_1)\delta(p_2-q_2)\delta(p_3-q_3)$.

——————— on April 16, from here. ———————

(First of all, I am sorry that last week I misunderstood the time of the class. I thought it is from 10:30 to 12:15, but it should be 10:25 to 12:10.)

Where were we?

§ 1.6 Hilbert space and Hamiltonian of many particle states.

$$\begin{pmatrix} \\ & H & \\ \\ \end{pmatrix}\begin{pmatrix} |\vec{p}_1\rangle \\ \vdots \\ |\vec{p}_1,\vec{p}_2\rangle \\ \vdots \\ |\vec{p}_1,\cdots\vec{p}_n\rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} E_1 & & & & \\ & \ddots & & & \\ & & E_1+E_2 & & \\ & & & \ddots & \\ & & & & \sum_i^n E_n \\ & & & & & \ddots \end{pmatrix}\begin{pmatrix} |\vec{p}_1\rangle \\ \vdots \\ |\vec{p}_1,\vec{p}_2\rangle \\ \vdots \\ |\vec{p}_1,\cdots\vec{p}_n\rangle \\ \vdots \end{pmatrix}$$

——————————————($\star$).

can be expressed as

$$\begin{aligned}
|\vec{p}_1, \cdots \vec{p}_n\rangle &\propto a^\dagger_{\vec{p}_1} \cdots a^\dagger_{\vec{p}_n} |0\rangle\,, \\
H &= \int \frac{d^3q}{(2\pi)^3} E_q a^\dagger_{\vec{q}} a_{\vec{q}}
\end{aligned}$$

where

$$\begin{aligned}
[a_{\vec{p}}, a^\dagger_{\vec{q}}] &= (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}), \\
[a_{\vec{p}}, a_{\vec{q}}] &= 0, \\
[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] &= 0, \\
a_{\vec{p}} |0\rangle &= 0.
\end{aligned}$$

Let's check it.

$$\begin{aligned}
H|0\rangle &= \int \frac{d^3q}{(2\pi)^3} E_q a^\dagger_{\vec{q}} \underset{\sim\sim\sim}{a_{\vec{q}} |0\rangle} = 0 \\
\left[H, a^\dagger_{\vec{p}}\right] &= \int \frac{d^3q}{(2\pi)^3} E_q \left[a^\dagger_{\vec{q}} a_{\vec{q}}, a^\dagger_{\vec{p}}\right] \\
&= \int \frac{d^3q}{(2\pi)^3} E_q a^\dagger_{\vec{q}} \left[a_{\vec{q}}, a^\dagger_{\vec{p}}\right] \\
&= \int \frac{d^3q}{(2\pi)^3} E_q a^\dagger_{\vec{q}} (2\pi)^3 \delta^{(3)}(\vec{q}-\vec{p}) \\
&= E_p a^\dagger_{\vec{p}} \\
\therefore \quad H a^\dagger_{\vec{p}} &= a^\dagger_{\vec{p}} (E_p + H) \\
\therefore \quad H \left(a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} \cdots a^\dagger_{\vec{p}_n}\right) |0\rangle &= a^\dagger_{\vec{p}_1} (E_1 + H) a^\dagger_{\vec{p}_2} \cdots a^\dagger_{\vec{p}_n} |0\rangle \\
&= a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} (E_1 + E_2 + H) a^\dagger_{\vec{p}_3} \cdots a^\dagger_{\vec{p}_n} |0\rangle \\
&= \cdots \\
&= a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} \cdots a^\dagger_{\vec{p}_n} (\underbrace{H}_{\to 0} + E_1 + \cdots + E_n) |0\rangle \\
H |\vec{p}_1, \cdots \vec{p}_n\rangle &= (E_1 + \cdots + E_n) |\vec{p}_1, \cdots \vec{p}_n\rangle
\end{aligned}$$

- So, the states and the Hamiltonian in $(\star)$ are expressed in a simple way

$$\boxed{\begin{aligned}
|\vec{p}_1, \cdots \vec{p}_n\rangle &\propto a^\dagger_{\vec{p}_1} \cdots a^\dagger_{\vec{p}_n} |0\rangle\,, \\
H &= \int \frac{d^3q}{(2\pi)^3} E_q a^\dagger_{\vec{q}} a_{\vec{q}}
\end{aligned}}$$

We will see that they are nothing but the states and Hamiltonian of the free scalar QFT. $\to$§ 2

► By the way, we have seen a similar expression in QM !

$$H = \hbar\omega\left(\frac{1}{2} + a^\dagger a\right)$$

i.e., the harmonic oscillator (調和振動子) !

§ 1.6.1 Harmonic Oscillator and QFT

► Let's recall the QM of harmonic oscillator. We can start from a Lagrangian

$$L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\,,$$

where $\dot{q} = dq/dt$. The conjugate momentum and the Hamiltonian are given by

$$\begin{aligned} p &= \frac{dL}{d\dot{q}} = m\dot{q}, \\ H(q,p) &= p\dot{q} - L \\ &= \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2. \end{aligned}$$

Promoting q and p to operators $\hat{q}$ and $\hat{p}$, the canonical quantization is

$$[\hat{q},\hat{p}] = i\hbar.$$

Equivalently, we can express $\hat{q}$, $\hat{p}$ and $\hat{H}$ in terms of the creation and annihilation operators

$$\begin{aligned} \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + \frac{i}{m\omega}\hat{p}\right), \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} - \frac{i}{m\omega}\hat{p}\right) \end{aligned}$$

Then,

$$\begin{aligned} \hat{H} &= \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2 \\ &= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \end{aligned}$$

and

$$[\hat{q},\hat{p}] = i\hbar \qquad \Longleftrightarrow \qquad \left[\hat{a},\hat{a}^\dagger\right] = 1.$$

The ground state and the excited states are given by

$$\begin{aligned} \text{ground state}: &\quad |0\rangle\,, \quad \hat{a}|0\rangle = 0, \quad \hat{H}|0\rangle = \frac{1}{2}\hbar\omega|0\rangle\,, \\ \text{excited states}: &\quad |n\rangle \propto (\hat{a}^\dagger)^n|0\rangle\,, \quad \hat{H}|n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega|n\rangle\,. \end{aligned}$$

► If there are many harmonic oscillators, then

$$
\begin{aligned}
L &= \sum_i \left(\frac{1}{2} m_i \dot{q}_i^2 - \frac{1}{2} m_i \omega_i^2 q_i^2 \right) , \\
p_i &= \frac{dL}{d\dot{q}_i} = m\dot{q}_i , \\
[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij} &\Longleftrightarrow [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} , \\
\text{where } \ \hat{a}_i &= \sqrt{\frac{m_i\omega_i}{2\hbar}} \left(\hat{q}_i + \frac{i}{m_i\omega_i} \hat{p}_i \right) , \\
H &= \sum_i p_i \dot{q}_i - L , \\
\hat{H} &= \sum_i \left(\frac{1}{2m} \hat{p}_i^2 + \frac{1}{2} m_i \omega_i^2 \hat{q}_i^2 \right) \\
&= \hbar\omega \sum_i \left(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2} \right) .
\end{aligned}
$$

$$
\begin{aligned}
\text{ground state :} \quad & |0\rangle , \quad \hat{a}_i |0\rangle = 0 , \\
\text{excited states :} \quad & |n_1, n_2, \cdots\rangle \propto (\hat{a}_1^\dagger)^{n_1} (\hat{a}_2^\dagger)^{n_2} \cdots |0\rangle , \\
& \hat{H} |n_1, n_2, \cdots\rangle = \sum_i \left(n_i + \frac{1}{2} \right) \hbar\omega .
\end{aligned}
$$

► In § 2, we will see that free scalar QFT is essentially a QM of infinitely many harmonic oscillators. One important difference is that, in the QM of a harmonic oscillator,

$$(a^\dagger)^n |0\rangle$$

represents a <u>n-th excited state (of a single particle)</u>. In the QFT,

$$(a_{\vec{p}_1}^\dagger)^n |0\rangle$$

represents a <u>n-particle state</u>. In general, n-particle states are represented by

$$|\vec{p}_1, \cdots \vec{p}_n\rangle \propto a_{\vec{p}_1}^\dagger \cdots a_{\vec{p}_n}^\dagger |0\rangle .$$

§ 1.7 About homework problems (and the grade)

See the following link:
http://www-hep.phys.s.u-tokyo.ac.jp/~hama/lectures/lecture_files/QFT_2018_report1.pdf

§ 2 Free Scalar (spin 0) Field

We consider a real scalar field $\phi(x)$.

- real: $\phi(x)^\dagger = \phi(x)$ (Hermitian operator).
- scalar: Lorentz transformation of the field is given by (→ see § 3)

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)\,.$$

§ 2.1 Lagrangian and Canonical Quantization of Real Scalar Field

In quantum mechanics, we consider a Lagrangian

$$L = L(\dot{q}, q) \qquad \dot{} = \frac{\partial}{\partial t}\,.$$

In QFT, we also start from a Lagrangian

$$L = \int d^3x \; \underbrace{\mathcal{L}[\dot{\phi}(\vec{x},t), \phi(\vec{x},t)]}_{\text{Lagrangian density}}$$

In the case of free scalar theory, it is given by

$$\begin{aligned} L = L_{\text{free}} &= \int d^3x \; \mathcal{L}[\dot{\phi}(\vec{x},t), \phi(\vec{x},t)] \\ &= \int d^3x \; \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \right) \\ &= \int d^3x \; \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi - \frac{1}{2}m^2\phi^2 \right) \end{aligned}$$

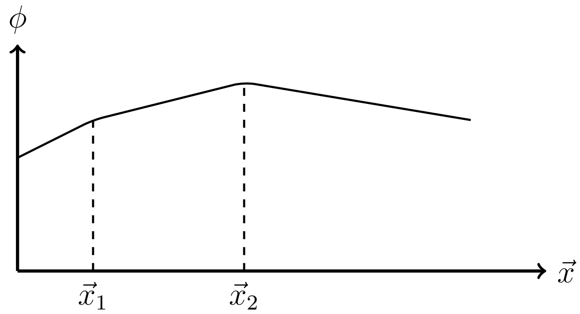
where

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

$$\partial_\mu\phi\partial^\mu\phi = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \left(\frac{\partial}{\partial x^0}\phi\right)^2 - \sum_{i=1}^{3}\left(\frac{\partial}{\partial x^i}\phi\right)^2 = \dot{\phi}^2 - \vec{\nabla}\phi\cdot\vec{\nabla}\phi$$

If we regard $\vec{x}$ as just a label,

$$\phi(\vec{x},t) = \{\phi_{\vec{x}_1}(t), \phi_{\vec{x}_2}(t), \cdots\}$$



QM of infinite number of degrees of freedom

$$L = \sum_{\vec{x}} \left(\frac{1}{2}\dot{\phi}_{\vec{x}}(t)^2 + \cdots \right)$$
$$\sim \sum_i \frac{1}{2}\dot{q}_i(t)^2 + \cdots$$

	QM	QFT
conjugate momentum	$p_i = \dfrac{\partial L}{\partial \dot{q}_i}$	$\pi(\vec{x},t) = \dfrac{\delta L}{\delta \dot{\phi}(\vec{x},t)} = \dot{\phi}(\vec{x},t)$ (functional derivative)
Hamiltonian	$H = \sum_i p_i \dot{q}_i - L$	$\begin{aligned} H &= \int d^3x \left(\pi(\vec{x},t)\dot{\phi}(\vec{x},t) - \mathcal{L} \right) \\ &= \int d^3x \left(\pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right) \\ &= \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right) \end{aligned}$
Canonical Quantization: Equal Time Commutation Relation		
	$[q_i(t), p_j(t)] = i\delta_{ij}$ $[q_i(t), q_j(t)] = 0$ $[p_i(t), p_j(t)] = 0$	$\boxed{\begin{array}{l} [\phi(\vec{x},t), \pi(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y}) \\ [\phi(\vec{x},t), \phi(\vec{y},t)] = 0 \\ [\pi(\vec{x},t), \pi(\vec{y},t)] = 0 \end{array}}$ equal time

Comments

(i) The action

$$S = \int dt L = \int dt d^3x \mathcal{L} = \int d^4x \mathcal{L}$$

is Lorentz invariant. $\rightarrow$ We will see it in § 3.

(ii) Schrödinger representation and Heisenberg representation:

In QFT, usually the Heisenberg representation is used.

	state	operator
S-rep.	$\lvert\Psi(t)\rangle_S$ time-dependent	$\mathcal{O}_S$ time-independent
H-rep.	$\lvert\Psi\rangle_H$ time-independent	$\mathcal{O}_H(t)$ time-dependent

S-rep.

$$i\frac{d}{dt}|\Psi(t)\rangle_S = H(p,q)|\Psi(t)\rangle_S$$

$$|\Psi(t)\rangle_S = e^{-iH(t-t_0)}|\Psi(t_0)\rangle_S$$

Expectation value of an operator: ${}_S\langle\Psi(t)|\mathcal{O}_S|\Psi(t)\rangle_S$

H-rep.

$$\begin{cases} |\Psi\rangle_H & \equiv |\Psi(t_0)\rangle_S = e^{iH(t-t_0)}\,|\Psi(t)\rangle_S \\ \mathcal{O}_H(t) & \equiv e^{iH(t-t_0)}\mathcal{O}_S e^{-iH(t-t_0)} \end{cases}$$

Expectation value: $\quad {}_H\langle\Psi|\mathcal{O}_H(t)|\Psi\rangle_H = \cdots = {}_S\langle\Psi(t)|\mathcal{O}_S|\Psi(t)\rangle_S$

$$\begin{aligned} i\frac{d}{dt}\mathcal{O}_H(t) &= -He^{iH(t-t_0)}\mathcal{O}_S e^{-iH(t-t_0)} + e^{iH(t-t_0)}\mathcal{O}_S H e^{-iH(t-t_0)} \\ &= -H\mathcal{O}_H(t) + \mathcal{O}_H(t)H \\ &= [\mathcal{O}_H(t), H]. \quad \text{Heisenberg eq.} \end{aligned}$$

——————— on April 16, up to here. ———————

Questions after the lecture:

Q: What is ϕ? Does it represent a real particle (in nature)?

A: Well, the scalar field theory here is a kind of a "toy model", so it does not necessarily represent a real particle in nature. As mentioned in § 1.4, the (only) elementary scalar particle known in the Standard Model of particle physics is the Higgs boson. So if you want, the ϕ field in this section can be regarded as a (free, non-interacting) Higgs boson. In addition, as composite particles, there are many scalar boson in nature, such as the pion.
The reason we start from a scalar field is just because it is simple. I guess you are more familiar with other particles such as the electron and the photon (than the Higgs boson which only particle physicists might care.) However, the electron is a fermion and the photon is a gauge boson, and in QFT, they are more difficult than the scalar. Thus, we start from an easier scalar QFT.

Q: In § 2.1, you wrote
$$L = \sum_{\vec{x}}\left(\frac{1}{2}\dot{\phi}_{\vec{x}}(t)^2 + \cdots\right) \sim \sum_i \frac{1}{2}\dot{q}_i(t)^2 + \cdots.$$
What do you mean by that?

A: It's just an analogy, or a correspondence between QFT and QM. $\phi \leftrightarrow q$, $\vec{x} \leftrightarrow i$, ...

Q: OK. In QM, $\dot{q}_i(t)$ corresponds to a velocity. Does $\dot{\phi}$ represent a velocity of a particle then?

A: No, no, it is just an analogy or correspondence. $\dot{\phi}$ has nothing to do with a velocity of the particle. In that sense, the symbol "$\sim$" is misleading. You can just forget about this equation if it is confusing.

——————— on April 23, from here. ———————

Where were we?

§ 2.1 Lagrangian and Canonical Quantization of Real Scalar Field

Comments (ii): S-rep. and H-rep.

QFT → usually H-rep.

H-rep.

$$\begin{cases} |\Psi\rangle_H & \equiv |\Psi(0)\rangle_S = e^{iHt}|\Psi(t)\rangle_S \\ \mathcal{O}_H(t) & \equiv e^{iHt}\mathcal{O}_S e^{-iHt} \end{cases}$$

$${}_H\langle\Psi|\mathcal{O}_H(t)|\Psi\rangle_H = \cdots = {}_S\langle\Psi(t)|\mathcal{O}_S|\Psi(t)\rangle_S$$

$$i\frac{d}{dt}\mathcal{O}_H(t) = -He^{iH(t-t_0)}\mathcal{O}_S e^{-iH(t-t_0)} + e^{iH(t-t_0)}\mathcal{O}_S He^{-iH(t-t_0)}$$

$\leftarrow$ [Last Week]
[today] $\rightarrow$

$$\begin{aligned} &= -H\mathcal{O}_H(t) + \mathcal{O}_H(t)H \\ &= [\mathcal{O}_H(t), H]. \quad \text{Heisenberg eq.} \end{aligned}$$

§ 2.2 Equation of Motion (EOM)

- There are two ways to derive the EOM.

  (i) From the action principle $\delta S = \delta \int dtL = 0$,

$$\partial_\mu \frac{\delta L}{\delta(\partial_\mu\phi)} - \frac{\delta L}{\delta\phi} = 0 \quad \text{Euler-Langrange eq.}$$

  (ii) From Heisenberg eqs.,

$$\begin{cases} i\dot{\phi} = [\phi, H] \\ i\dot{\pi} = [\pi, H] \end{cases}$$

- From (i), for the Lagrangian $L = \displaystyle\int d^3x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right)$, we obtain

$$\partial_\mu \frac{\delta L}{\delta(\partial_\mu\phi)} - \frac{\delta L}{\delta\phi} = \partial_\mu(\partial^\mu\phi) + m^2\phi = 0$$

or

$$\text{EOM} \quad \boxed{\left(\Box + m^2\right)\phi(x) = 0}$$

where $\Box = \partial_\mu\partial^\mu = \dfrac{\partial^2}{\partial t^2} - \vec{\nabla}^2$.

Problem

(b-1) Derive the EOM $\left(\Box + m^2\right)\phi(x) = 0$ from Heisenberg eqs.,
by using $H = \displaystyle\int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right)$
and the equal-time commutation relations of ϕ and π.
(Note: $\phi(t, \vec{x})$ and $\pi(t, \vec{x})$ depend on t, but H does not.)

§ 2.3 Solution of the EOM

- Starting from $\boxed{(\Box + m^2)\phi(x) = 0 \text{ (Klein-Gordon eq.)}}$, one can show that $\phi(x)$ can be expressed as

$$\boxed{\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x}\right)} \qquad \text{———}(1)$$

where $\phi(x)$, $a(\vec{p})$, $a^\dagger(\vec{p})$ are operators, $p\cdot x = p_\mu x^\mu = p^0 t - \vec{p}\cdot\vec{x}$, and $p^0 = E_p = \sqrt{\vec{p}^2 + m^2}$.

- Eq. (1) is a solution of the EOM, because

$$\begin{aligned}(\Box + m^2)e^{\pm ip\cdot x} &= (\partial_\mu\partial^\mu + m^2)e^{\pm ip\cdot x} \\ &= \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)e^{\pm ip\cdot x} \\ &= (-E_p^2 + \underbrace{\vec{p}^2 + m^2}_{E_p^2})e^{\pm ip\cdot x} = 0.\end{aligned}$$

- Now let's show that Eq.(1) is the general solution of the EOM.

(i) Fourier transform (FT) $\phi(x)$ with respect to $\vec{x}$:

$$\phi(x) = \underbrace{\phi(\vec{x}, t)}_{\text{operator}} = \int d^3p\, \underbrace{C(\vec{p}, t)}_{\text{operator}}\, e^{i\vec{p}\cdot\vec{x}} \qquad \text{———}(2)$$

(ii) From the condition $\phi = \phi^\dagger$ (real field),

$$\begin{aligned}\int d^3p C(\vec{p}, t)e^{i\vec{p}\cdot\vec{x}} &= \int d^3p C^\dagger(\vec{p}, t)e^{-i\vec{p}\cdot\vec{x}} \\ &= \int d^3p' C^\dagger(-\vec{p}', t)e^{i\vec{p}'\cdot\vec{x}} \quad (\vec{p}' = -\vec{p}) \\ &= \int d^3p C^\dagger(-\vec{p}, t)e^{i\vec{p}\cdot\vec{x}}\end{aligned}$$

Using inverse FT,

$$C(\vec{p}, t) = C^\dagger(-\vec{p}, t) \qquad \text{———}(3)$$

(iii) From $\left(\Box + m^2\right)\phi = \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi = 0$ and (2),

$$\int d^3p \left(\ddot{C}(\vec{p}, t) + C(\vec{p}, t)\underbrace{(\vec{p}^2 + m^2)}_{E_p^2}\right) e^{i\vec{p}\cdot\vec{x}} = 0$$

Using inverse FT,

$$\ddot{C}(\vec{p},t) + E_p^2 C(\vec{p},t) = 0$$
$$\therefore \quad C(\vec{p},t) = C(\vec{p})e^{-iE_pt} + C'(\vec{p})e^{+iE_pt}.$$

From (3), $C'(\vec{p}) = C^\dagger(-\vec{p})$, and hence

$$\underline{C(\vec{p},t) = C(\vec{p})e^{-iE_pt} + C^\dagger(-\vec{p})e^{+iE_pt}}.$$

(iv) Substituting it to (2) (and changing $\vec{p} \to -\vec{p}$ in the 2nd term),

$$\begin{aligned}\phi(\vec{x},t) &= \int d^3p \left(C(\vec{p})e^{-iE_pt}e^{i\vec{p}\cdot\vec{x}} + C^\dagger(\vec{p})e^{+iE_pt}e^{-i\vec{p}\cdot\vec{x}}\right)\\ &= \int d^3p \left(C(\vec{p})e^{-ip\cdot x} + C^\dagger(\vec{p})e^{+ip\cdot x}\right)\end{aligned}$$

Finally by normalizing as $a(\vec{p}) \equiv (2\pi)^3\sqrt{2E_p}\cdot C(\vec{p})$, we obtain (1). ■

► Note that the normalization depends on the convention (textbook).

$$a(\text{here}) = a(\text{Peskin}) = \frac{1}{\sqrt{2E_p}}a(\text{Srednicki}) = (2\pi)^{3/2}a(\text{Weinberg}).$$

► From (1), we can express the operators $a(\vec{p})$ and $a^\dagger(\vec{p})$ in terms of $\phi(x)$:

$$\boxed{\begin{cases} a(\vec{p}) = \dfrac{1}{\sqrt{2E_p}}\displaystyle\int d^3x e^{+ip\cdot x}\left[i\dot{\phi}(x) + E_p\phi(x)\right] \\ a^\dagger(\vec{p}) = \dfrac{1}{\sqrt{2E_p}}\displaystyle\int d^3x e^{-ip\cdot x}\left[-i\dot{\phi}(x) + E_p\phi(x)\right]\end{cases}} \qquad (4)$$

Problem

(b-2) Substitute (1) to the right-hand side (RHS) of (4) and show that it gives $a(\vec{p})$ & $a(\vec{p})^\dagger$.

(b-3) The RHS of (4) seems to depend on $x^0 = t$, but the LHS does not. Show that $\frac{\partial}{\partial t}$[RHS of (4)]$= 0$, using the EOM. (Hint: integration by parts (部分積分))

(b-4) Substitute (4) to the RHS of (1) and show that it gives LHS.

Pay attention to which variables are just the integration variable. For instance, let's solve (b-2):

$$\text{from (1)}, \quad \phi(x) = \int \underbrace{\frac{d^3q}{(2\pi)^3\sqrt{2E_q}}}_{=[dq]}\left(a(\vec{q})e^{-iq\cdot x} + a^\dagger(\vec{q})e^{+iq\cdot x}\right)$$

$$i\dot{\phi}(x) = \int [dq]\left(E_q a(\vec{q})e^{-iq\cdot x} - E_q a^\dagger(\vec{q})e^{+iq\cdot x}\right)$$

$$i\dot{\phi}(x) + E_p\phi(x) = \int [dq]\left((E_q + E_p)a(\vec{q})e^{-iq\cdot x} + (-E_q + E_p)a^\dagger(\vec{q})e^{-ip\cdot x}\right)$$

Thus,

$$\text{RHS of (4)} = \frac{1}{\sqrt{2E_p}}\int d^3x e^{+ip\cdot x}\int [dq]\left((E_q+E_p)a(\vec{q})e^{-iq\cdot x} + (-E_q+E_p)a^\dagger(\vec{q})e^{+iq\cdot x}\right)$$

$$\int d^3x e^{iE_p x^0} e^{-i\vec{p}\cdot\vec{x}} \cdot e^{-iE_q x^0} e^{i\vec{q}\cdot\vec{x}}$$

$$= (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\cdot e^{i(E_p-E_q)x^0} \qquad (2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\cdot e^{i(E_p+E_q)x^0}$$

$$= \frac{1}{\sqrt{2E_p}}\int \frac{d^3q}{\sqrt{2E_q}}\left((E_q+E_p)a(\vec{q})\delta^{(3)}(\vec{p}-\vec{q}) + \underbrace{(-E_q+E_p)}_{\to 0}a^\dagger(\vec{q})\delta^{(3)}(\vec{p}-\vec{q})e^{i(E_p+E_q)x^0}\right)$$

$$= a(\vec{p}) = \text{LHS of (4)} \quad \blacksquare$$

§ 2.4 Commutation relations of a and $a^\dagger$

- From the commutation relation in § 2.1, we have the following commutation relations (recall $\pi(x) = \dot{\phi}$):

$$\boxed{\begin{aligned}&[\phi(\vec{x},t),\dot{\phi}(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y})\\ &[\phi(\vec{x},t),\phi(\vec{y},t)] = 0\\ &[\dot{\phi}(\vec{x},t),\dot{\phi}(\vec{y},t)] = 0\end{aligned}} \Longleftrightarrow \boxed{\begin{aligned}&[a(\vec{p}),a^\dagger(\vec{q})] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\\ &[a(\vec{p}),a(\vec{q})] = 0\\ &[a^\dagger(\vec{p}),a^\dagger(\vec{q})] = 0\end{aligned}} \quad \text{——— (5)}$$

Problem

(b-5) Show that RHS of (5) $\Longrightarrow$ LHS of (5), using (1).

(b-6) Show that LHS of (5) $\Longrightarrow$ RHS of (5), using (4).

§ 2.5 $a^\dagger$ and a are the creation and annihilation operators.

- In this section we will see that

$a(\vec{p})$ ——— annihilate a particle with energy E_p, momentum $\vec{p}$.

$a^\dagger(\vec{p})$ ——— create a particle with energy E_p, momentum $\vec{p}$.

- First, we can show that $\boxed{\begin{aligned}&[H,a^\dagger(\vec{p})] = E_p a^\dagger(\vec{p})\\ &[H,a(\vec{p})] = -E_p a(\vec{p})\end{aligned}}$ ——— (6) (We will show it later.)

  We can also show $\boxed{\begin{aligned}&[\hat{\vec{P}},a^\dagger(\vec{p})] = \vec{p}a^\dagger(\vec{p})\\ &[\hat{\vec{P}},a(\vec{p})] = -\vec{p}a(\vec{p})\end{aligned}}$

  where $\hat{\vec{P}}$ is the "momentum" operator. ($\hat{\vec{P}} = -\int d^3x \pi\vec{\nabla}\phi = \int \frac{d^3q}{(2\pi)^3}\vec{q}a^\dagger(\vec{q})a(\vec{q})$. We skip the details here. (It can be obtained from Noether's current for translation.))

- Consider a state with energy E_X and momentum $\vec{p}_X$;

$$|X\rangle : \quad \begin{cases} H\,|X\rangle = E_X\,|X\rangle \\ \hat{\vec{P}}\,|X\rangle = \vec{p}_X\,|X\rangle \end{cases}$$

Then, for the state $a^\dagger(\vec{p})\,|X\rangle$,

$$\begin{aligned} H\left(a^\dagger(\vec{p})\,|X\rangle\right) &= \left([H, a^\dagger(\vec{p})] + a^\dagger(\vec{p})H\right)|X\rangle \\ &= \left(E_p a^\dagger(\vec{p}) + a^\dagger(\vec{p})E_X\right)|X\rangle \\ &= (E_p + E_X)\left(a^\dagger(\vec{p})\,|X\rangle\right), \\ \hat{\vec{P}}\left(a^\dagger(\vec{p})\,|X\rangle\right) &= \left([\hat{\vec{P}}, a^\dagger(\vec{p})] + a^\dagger(\vec{p})\hat{\vec{P}}\right)|X\rangle \\ &= \left(\vec{p}a^\dagger(\vec{p}) + a^\dagger(\vec{p})\vec{p}_X\right)|X\rangle \\ &= (\vec{p} + \vec{p}_X)\left(a^\dagger(\vec{p})\,|X\rangle\right). \end{aligned}$$

Thus, the state $a^\dagger(\vec{p})\,|X\rangle$ has energy $E_p + E_X$ and momentum $\vec{p} + \vec{p}_X$, namely, $a^\dagger(\vec{p})$ adds energy E_p and momentum $\vec{p}$. (creation operator)

- Similarly, we can show

$$\begin{aligned} H\left(a(\vec{p})\,|X\rangle\right) &= (E_X - E_p)\left(a(\vec{p})\,|X\rangle\right), \\ \hat{\vec{P}}\left(a(\vec{p})\,|X\rangle\right) &= (\vec{p}_X - \vec{p})\left(a(\vec{p})\,|X\rangle\right). \end{aligned}$$

and therefore $a(\vec{p})$ is an annihilation operator.

- Now let's show (6). There are two ways.

  (i) Express H in terms of a and $a^\dagger$.

  (ii) Use (4) and Heisenberg eqs.

Problem

(b-7) Do (ii): Show (6) by using (4) and Heisenberg eqs.

——— on April 23, up to here. ———

Questions after the lecture:

Q: In § 2.3, $C(\vec{p})$ and $C'(\vec{p})$ are independent at the beginning. Where does $C'(\vec{p})$ go?

A: It is (implicitly) shown in the sentence "From (3), $C'(\vec{p}) = C^\dagger(-\vec{p})$."
If we substitute $C(\vec{p}, t) = C(\vec{p})e^{-iE_p t} + C'(\vec{p})e^{+iE_p t}$ to eq.(3), then there are terms proportional to $e^{+iE_p t}$ and $e^{-iE_p t}$ in the both right- and left-hand sides. The equation should hold for any t. By comparing the coefficients of $e^{+iE_p t}$ and $e^{-iE_p t}$, we obtain $C'(\vec{p}) = C^\dagger(-\vec{p})$.

Q: In § 2.3, when solving (b-2), you used $-E_q + E_p = 0$ under $\delta^{(3)}(\vec{p} + \vec{q})$. Is this because $E_q = \sqrt{\vec{q}^2 + m^2}$ and $E_p = \sqrt{\vec{p}^2 + m^2}$, and therefore $E_q = E_p$ for $\vec{q} = -\vec{p}$?

A: Yes, you are right.

Q: Discussions around here seem to apply for Analytical Mechanics (解析力学), if we replace the commutation relations with the Poission brackets. For instance, the relation between ϕ and a, $a^\dagger$ seem to hold for a classical field as well. From which point does the *quantum* mechanics start?

A: Good question, and you just almost answered the question by yourself. The quantization started when we imposed the commutation relations, $[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$ etc, instead of the Poission brackets.

——— on May 7, from here. ———

Where were we?

§ 2 Free Scalar

§ 2.5 a = annihilation, $a^\dagger$ = creation

We can show $\boxed{\begin{array}{l} [H, a^\dagger(\vec{p})] = E_p a^\dagger(\vec{p}) \\ [H, a(\vec{p})] = -E_p a(\vec{p}) \end{array}}$ ——— (6) by

(i) Expressing H in terms of a and $a^\dagger$, and
(ii) Using (4) and Heisenberg eqs. ($\to$ (b-7).)

► Here we do (i). First,

$$\begin{aligned}\phi(x) &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a(\vec{p})e^{-iE_pt+i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{iE_pt-i\vec{p}\cdot\vec{x}}\right) \\ &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a(\vec{p})e^{-iE_pt} + a^\dagger(-\vec{p})e^{iE_pt}\right) e^{i\vec{p}\cdot\vec{x}} \quad (\vec{p}\to-\vec{p} \text{ in the 2nd term})\end{aligned}$$

Let's define,

$$A(\vec{p},t) \equiv \frac{1}{(2\pi)^3\sqrt{2E_p}} a(\vec{p})e^{-iE_pt}$$

and omit t for simplicity: $A(\vec{p}) = A(\vec{p},t)$. Then

$$\begin{aligned}\phi(x) &= \int d^3p \left(A(\vec{p}) + A^\dagger(-\vec{p})\right) e^{i\vec{p}\cdot\vec{x}} \\ \vec{\nabla}\phi(x) &= \int d^3p \left(A(\vec{p}) + A^\dagger(-\vec{p})\right) (i\vec{p})e^{i\vec{p}\cdot\vec{x}} \\ \dot{\phi}(x) &= \int d^3p(-iE_p) \left(A(\vec{p}) - A^\dagger(-\vec{p})\right) e^{i\vec{p}\cdot\vec{x}} \quad (\because \dot{A}(\vec{p},t) = (-iE_p)A(\vec{p},t))\end{aligned}$$

Therefore,

$$
\begin{aligned}
H &= \int d^3x \left(\frac{1}{2}\underbrace{\pi^2}_{=\dot{\phi}^2} + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right) \\
&= \underbrace{\int d^3x} \int d^3p \int d^3q \ \underbrace{e^{i\vec{p}\cdot\vec{x}}e^{i\vec{q}\cdot\vec{x}}} \quad \to (2\pi)^3\delta^{(3)}(\vec{p}+\vec{q}) \\
&\times \left[\frac{1}{2}(-iE_p)(-iE_q)\left(A(\vec{p}) - A^\dagger(-\vec{p})\right)\left(A(\vec{q}) - A^\dagger(-\vec{q})\right) \right. \\
&\qquad + \frac{1}{2}(i\vec{p})(i\vec{q})\left(A(\vec{p}) + A^\dagger(-\vec{p})\right)\left(A(\vec{q}) + A^\dagger(-\vec{q})\right) \\
&\qquad \left. + \frac{1}{2}m^2\left(A(\vec{p}) + A^\dagger(-\vec{p})\right)\left(A(\vec{q}) + A^\dagger(-\vec{q})\right) \right] \\
&= \cdots \\
&= \int d^3q (2\pi)^3 E_q^2 \left[A(-\vec{q})A^\dagger(-\vec{q}) + A^\dagger(\vec{q})A(\vec{q}) \right] \\
&= \int d^3q (2\pi)^3 E_q^2 \frac{1}{(2\pi)^6 2E_q} \left[a(\vec{q})a^\dagger(\vec{q}) + a^\dagger(\vec{q})a(\vec{q}) \right] \quad (\vec{q} \to -\vec{q} \ \text{ in the 1st term})
\end{aligned}
$$

Here, note that the t-dependence of $A(\vec{q}, t)$ cancels in H, and hence H is time independent. By using

$$
a(\vec{q})a^\dagger(\vec{q}) = a^\dagger(\vec{q})a(\vec{q}) + (2\pi)^3\delta^{(3)}(0),
$$

we obtain

$$
H = \int \frac{d^3q}{(2\pi)^3} E_q \left(a^\dagger(\vec{q})a(\vec{q}) + \underbrace{\frac{1}{2}(2\pi)^3\delta^{(3)}(0)} \right)
$$

The constant term,

$$
\int d^3q E_q \frac{1}{2}\delta^{(3)}(0)
$$

is the zero-point energy. (This corresponds to the $\frac{1}{2}\hbar\omega$ term in the energy spectrum of the harmonic oscillator, $E = \hbar\omega(a^\dagger a + \frac{1}{2})$.)

The zero-point energy cannot be observed (except through the gravitational force), so we neglect it in the following.

> In fact, there is an ordering ambiguity to quantize the theory from a classical level. If we define the Hamiltonian by
>
> $$ H =: \int d^3x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right] : \qquad : aa^\dagger :=: a^\dagger a : \quad \text{normal ordering} $$
>
> then there is no zero-point energy.

In any case, we have $\boxed{H = \int \frac{d^3q}{(2\pi)^3} E_q a^\dagger(\vec{q}) a(\vec{q})}$ (+ const.) $\quad$ ($\leftarrow$ § 1.6)

Therefore,

$$
\begin{aligned}
[H, a^\dagger(\vec{p})] &= \int \frac{d^3q}{(2\pi)^3} E_q a^\dagger(\vec{q}) \underbrace{[a(\vec{q}), a^\dagger(\vec{p})]}_{(2\pi)^3\delta^{(3)}(\vec{q}-\vec{p})} \\
&= E_p a^\dagger(\vec{p})
\end{aligned}
$$

similarly $\quad [H, a(\vec{p})] = -E_p a(\vec{p}) \quad \blacksquare$

Problem

Now that $\phi(x)$ and H are expressed in terms of a and $a^\dagger$, we can do some consistency check, based on (1), (6), and the commutation relations of a and $a^\dagger$.

- Heisenberg eq. $\quad i\dot{\phi}(x) = [\phi(x), H]$ ———— (i).
- $\phi(x)$ is a Heisenberg operator: $\quad \phi(x) = \phi(t, \vec{x}) = e^{iH(t-t_0)}\phi(t_0, \vec{x})e^{-iH(t-t_0)}$ — (ii).

(b-8) Show (i) by using (1) and (6).

(b-9) Show that $e^{iHt}a(\vec{p})^\dagger e^{-iHt} = a(\vec{p})^\dagger e^{iE_pt}$ and $e^{iHt}a(\vec{p})e^{-iHt} = a(\vec{p})e^{-iE_pt}$ by using (6).

(b-10) Show (ii) by using the result of (b-9) and eq.(1).

§ 2.6 vacuum state

The operator $a(\vec{p})$ decreases the energy:

$$
\begin{array}{lcccccc}
 & |X\rangle & \to & a(\vec{p})\,|X\rangle & \to & a(\vec{q})a(\vec{p})\,|X\rangle & \cdots \\
\text{energy} & E_X & & E_X - E_p & & E_X - E_p - E_q &
\end{array}
$$

The ground state (lowest energy state) $|0\rangle$ is a state which satisfies

$$a(\vec{p})\,|0\rangle = 0$$

and Lorentz invariant ($\to$ § 3)

$$U(\Lambda)\,|0\rangle = |0\rangle\,.$$

§ 2.7 One-particle state and n-particle state

- The one-particle state is given by (for free theory)

$$|\vec{p}\rangle = \sqrt{2E_p}a^\dagger(\vec{p})\,|0\rangle\ .$$

normalization

$$
\begin{aligned}
\langle \vec{q} | \vec{p} \rangle &= \sqrt{2E_q}\sqrt{2E_p} \langle 0 | a(\vec{q}) a^\dagger(\vec{p}) | 0 \rangle \\
&= \sqrt{2E_q}\sqrt{2E_p} \langle 0 | \Big(\underbrace{[a(\vec{q}), a^\dagger(\vec{p})]}_{(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})} + a^\dagger(\vec{p}) \underbrace{a(\vec{q})}_{\to 0} \Big) | 0 \rangle \\
&= (2\pi)^3 2E_p \delta^{(3)}(\vec{p}-\vec{q}) \,,
\end{aligned}
$$

Why $|\vec{p}\rangle \propto \sqrt{E_p} a^\dagger(\vec{p}) |0\rangle$, not just $a^\dagger(\vec{p}) |0\rangle$?
$\rightarrow$ because $E_p \delta^{(3)}(\vec{p}-\vec{q})$ is Lorentz covariant. ($\rightarrow$ § 3)

► n-particle state is given by

$$
|\vec{p}_1, \cdots \vec{p}_n\rangle = \sqrt{2E_1} \cdots \sqrt{2E_n} a^\dagger(\vec{p}_1) \cdots a^\dagger(\vec{p}_n) |0\rangle
$$

and satisfies

$$
\begin{aligned}
H |\vec{p}_1, \cdots \vec{p}_n\rangle &= (E_1 + \cdots E_n) |\vec{p}_1, \cdots \vec{p}_n\rangle \\
\hat{\vec{P}} |\vec{p}_1, \cdots \vec{p}_n\rangle &= (\vec{p}_1 + \cdots \vec{p}_n) |\vec{p}_1, \cdots \vec{p}_n\rangle
\end{aligned}
$$

As we promised in § 1.6.

§ 2.8 $[\phi(x), \phi(y)]$ for $x^0 \neq y^0$

For $x^0 = y^0 = t$, we have $[\phi(x), \phi(y)] = 0$. What if $x^0 \neq y^0$?
From Eq.(1) and the commutation relations of a and $a^\dagger$ in § 2.4, we have

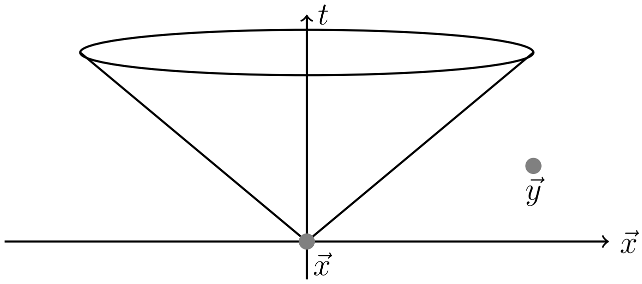
$$
\boxed{
\begin{aligned}
&[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3 2E_p} \left(e^{-ip\cdot(x-y)} - e^{+ip\cdot(x-y)} \right) \qquad \equiv i\Delta(x-y) \\
&\Delta(x) = (-i) \int \frac{d^3p}{(2\pi)^3 2E_p} \left(e^{-ip\cdot x} - e^{+ip\cdot x} \right)
\end{aligned}
}
$$

Properties of $\Delta(x)$:

(a) $(\Box + m^2)\Delta(x) = 0$.

(b) Lorentz invariant: $\Delta(\Lambda x) = \Delta(x)$.

(c) Local causality: $\Delta(x) = 0$ for $x^2 = (x^0)^2 - \vec{x}^2 < 0$ (space-like).



$[\phi(x), \phi(y)] = 0$

(a) is clear from the definition of $\Delta(x)$.
(b), (c) $\rightarrow$ § 3.

Problem

(b-11) Compute $[\phi(x), \dot{\phi}(y)]$, for *not* equal time ($x^0 \neq y^0$). (Use an integral, if necessary.)

(b-12) Take $x^0 = y^0$ in the $[\phi(x), \dot{\phi}(y)]$ obtained in (b-11), and show that it reproduces the equal-time commutation relation of ϕ and $\dot{\phi}$.

§ 3 Lorentz transformation, Lorentz group and its representations

§ 3.1 Lorentz transformation of coordinates

— is a linear, homogeneous change of coordinates from x^μ to x'^μ,

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \,,$$

where Λ is a 4×4 matrix satisfying

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma}, \qquad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\Longleftrightarrow \quad \Lambda^\mu{}_\rho g_{\mu\nu}\Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

$$\Longleftrightarrow \quad \Lambda^T g\Lambda = g \quad \text{(in matrix notation)}\,.$$

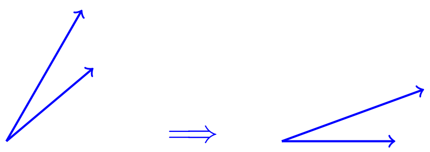
__Comments__

(i) It preserves inner products of four vectors:

$$\begin{aligned} & x\cdot y = g_{\mu\nu}x^\mu y^\nu \\ \rightarrow\ & x'\cdot y' = g_{\mu\nu}x'^\mu y'^\nu = g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma x^\rho y^\sigma = g_{\rho\sigma}x^\rho y^\sigma = x\cdot y\,. \end{aligned}$$

This is similar to orthogonal transformation $\vec{v}\to\vec{v'} = R\vec{v}$ where R is an orthogonal matrix satisfying $R^TR = \mathbf{1}$. (e.g., $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ in 2-dim.)

Inner products are preserved: $\vec{u}\cdot\vec{v}\to\vec{u'}\cdot\vec{v'} = (R\vec{u})\cdot(R\vec{v}) = \vec{u}^TR^TR\vec{v} = \vec{u}\cdot\vec{v}$.



(ii) The set of all Lorentz transformations (LTs) forms a group (Lorentz group).

- Product of two LTs Λ_1 and Λ_2 is defines as $(\Lambda_2\Lambda_1)^\mu{}_\nu = (\Lambda_2)^\mu{}_\rho(\Lambda_1)^\rho{}_\nu$.
- closure: if $\Lambda_1^Tg\Lambda_1 = g$ and $\Lambda_2^Tg\Lambda_2 = g$, then $(\Lambda_2\Lambda_1)^Tg(\Lambda_2\Lambda_1) = g$.
- associativity: $(\Lambda_1\Lambda_2)\Lambda_3 = \Lambda_1(\Lambda_2\Lambda_3)$.
- identity: $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$.
- inverse: $(\Lambda^{-1})^\mu{}_\nu = g^{\mu\rho}g_{\nu\sigma}\Lambda^\sigma{}_\rho = \Lambda_\nu{}^\mu$.
  (This satisfies $\Lambda^{-1}\Lambda = \mathbf{1}$, i.e., $(\Lambda^{-1})^\mu{}_\nu\Lambda^\nu{}_\rho = \delta^\mu{}_\rho$.)

Examples

$\boxed{\text{rotations}}$ around x, y, z axes

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_1 & \sin\theta_1 \\ & & -\sin\theta_1 & \cos\theta_1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & \cos\theta_2 & & -\sin\theta_2 \\ & & 1 & \\ & \sin\theta_2 & & \cos\theta_2 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & \cos\theta_3 & \sin\theta_3 & \\ & -\sin\theta_3 & \cos\theta_3 & \\ & & & 1 \end{pmatrix}.$$

$\boxed{\text{boosts}}$ in the x, y, z directions

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\eta_1 & \sinh\eta_1 & & \\ \sinh\eta_1 & \cosh\eta_1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \cosh\eta_2 & & \sinh\eta_2 & \\ & 1 & & \\ \sinh\eta_2 & & \cosh\eta_2 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \cosh\eta_3 & & & \sinh\eta_3 \\ & 1 & & \\ & & 1 & \\ \sinh\eta_3 & & & \cosh\eta_3 \end{pmatrix}.$$

$$\cosh\eta = \frac{e^\eta + e^{-\eta}}{2} = \gamma$$
$$\sinh\eta = \frac{e^\eta - e^{-\eta}}{2} = \beta\gamma = \sqrt{\gamma^2 - 1}$$

§ 3.2 infinitesimal Lorentz Transformation and generators of Lorentz group (in the 4-vector basis)

- Consider an infinitesimal Lorentz Transformation:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \qquad (\omega^\mu{}_\nu \ll 1),$$
$$\text{or} \qquad \Lambda = I + \omega.$$

where I is the identity matrix ($I^\mu{}_\nu = \delta^\mu{}_\nu$). Then, from $\Lambda^T g \Lambda = g$,

$$(I+\omega)^T g (I+\omega) = g$$
$$\therefore \omega^T g + g\omega = 0 \qquad (\text{up to } \mathcal{O}(\omega^2))$$
$$\therefore \underbrace{\omega^\rho{}_\mu \, g_{\rho\nu}}_{\substack{\| \\ g_{\nu\rho}\omega^\rho{}_\mu \\ \| \\ \omega_{\nu\mu}}} + \underbrace{g_{\mu\rho}\,\omega^\rho{}_\nu}_{\equiv\, \omega_{\mu\nu}} = 0$$
$$\therefore \underline{\omega_{\nu\mu} = -\omega_{\mu\nu} \qquad \text{anti-symmetric}}$$

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} \qquad \begin{array}{c} \text{6 independent degrees} \\ \Updownarrow \\ \text{3 rotations and 3 boosts} \end{array}$$

► In fact, the matrix ${\omega^\mu}_\nu = g^{\mu\rho}\omega_{\rho\nu}$ can be written as

$$
{\omega^\mu}_\nu = \begin{pmatrix} 0 & \eta_1 & \eta_2 & \eta_3 \\ \eta_1 & 0 & \theta_3 & -\theta_2 \\ \eta_2 & -\theta_3 & 0 & \theta_1 \\ \eta_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix}
$$

Note that
${\omega^0}_\nu = g^{00}\omega_{0\nu} = \omega_{0\nu}$, $\quad \omega_{0i} = \eta_i = -\omega_{i0}$.
${\omega^i}_\nu = g^{ij}\omega_{j\nu} = -\omega_{i\nu}$, $\quad -\omega_{ij} = \epsilon_{ijk}\theta_k = \omega_{ji}$.

—————— on May 7, up to here. ——————

—————— May 14, from here. ——————

Where were we?

§ 3 Lorentz…

§ 3.1 $x \to x' = \Lambda x$.

§ 3.2 $\quad \Lambda = I + \omega$

$\to \quad \omega_{\mu\nu} = -\omega_{\nu\mu}$.

the matrix ${\omega^\mu}_\nu = g^{\mu\rho}\omega_{\rho\nu}$ can be written as

$$
{\omega^\mu}_\nu = \begin{pmatrix} 0 & \eta_1 & \eta_2 & \eta_3 \\ \eta_1 & 0 & \theta_3 & -\theta_2 \\ \eta_2 & -\theta_3 & 0 & \theta_1 \\ \eta_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix}
$$

Note that
${\omega^0}_\nu = g^{00}\omega_{0\nu} = \omega_{0\nu}$, $\quad \omega_{0i} = \eta_i = -\omega_{i0}$.
${\omega^i}_\nu = g^{ij}\omega_{j\nu} = -\omega_{i\nu}$, $\quad -\omega_{ij} = \epsilon_{ijk}\theta_k = \omega_{ji}$.

(today $\to$)

and the rotations and boosts in § 3.1 can be expanded as

rotation around x axis

$$
\Lambda = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_1 & \sin\theta_1 \\ & & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \theta_1 \\ & & -\theta_1 & 0 \end{pmatrix} + \mathcal{O}(\theta_1^2)
$$

boost in the x direction

$$
\Lambda = \begin{pmatrix} \cosh\eta_1 & \sinh\eta_1 & & \\ \sinh\eta_1 & \cosh\eta_1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & \eta_1 & & \\ \eta_1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \mathcal{O}(\eta_1^2)
$$

► The matrix ${\omega^\mu}_\nu$ above can also be written as

$$
{\omega^\mu}_\nu = i\left[\theta_i {(J_i)^\mu}_\nu + \eta_i {(K_i)^\mu}_\nu\right]
$$

where

$$(J_1)^\mu{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix}, \quad (J_2)^\mu{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{pmatrix}, \quad (J_3)^\mu{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{pmatrix},$$

$$(K_1)^\mu{}_\nu = \begin{pmatrix} 0 & -i & & \\ -i & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad (K_2)^\mu{}_\nu = \begin{pmatrix} 0 & & -i & \\ & 0 & & \\ -i & & 0 & \\ & & & 0 \end{pmatrix}, \quad (K_3)^\mu{}_\nu = \begin{pmatrix} 0 & & & -i \\ & 0 & & \\ & & 0 & \\ -i & & & 0 \end{pmatrix}.$$

These 6 matrices are the <u>generators</u> of the Lorentz group in the 4-vector basis.

- Any group element can be uniquely written as

$$\Lambda = \exp\left(i\theta_i J_i + i\eta_i K_i\right)$$

up to some discrete transformations. (cf. § 3.A)

(We omit the proof.)
For example, for $\theta_1 \neq 0, \theta_2 = \theta_3 = \eta_i = 0$,

$$\begin{aligned} \Lambda &= \exp\left(i\theta_1 J_1\right) \\ &= \exp\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \theta_1 \\ & & -\theta_1 & 0 \end{pmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \theta_1 \\ & & -\theta_1 & 0 \end{pmatrix}^n = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_1 & \sin\theta_1 \\ & & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \end{aligned}$$

For $\theta_i, \eta_i \ll 1$,

$$\Lambda = I_{4\times 4} + \underbrace{i\left(\theta_i J_i + \eta_i K_i\right)}_{\omega} + \mathcal{O}(\theta_i, \eta_i)^2$$

- The generators J_i and K_i satisfy the following commutation relations

$$\boxed{\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k \\ [J_i, K_j] &= i\epsilon_{ijk} K_k \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k \end{aligned}} \qquad \left(\begin{array}{l} \text{Lie algebra} \\ \text{of Lorentz group } SO(1,3) \end{array}\right)$$

In § 3.4, we will see <u>the same commutation relations hold</u> for generators of <u>general representations</u> of Lorentz group.

§ 3.A Other (disconnected) Lorentz transformations

- The above LTs (rotations and boosts) are continuously connected to the identity element I by infinitesimal Lorentz transformations (LTs).

$$I \bullet \longrightarrow \underset{I+\omega_1}{\bullet} \longrightarrow \underset{I+\omega_2}{\bullet} \longrightarrow \cdots\cdots \longrightarrow \bullet\, \Lambda = (1+\omega_n)\cdots(1+\omega_2)(1+\omega_1)I$$

But there are also LTs which cannot be connected to I by infinitesimal LTs.

- From $g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\nu\sigma}$,

$$\text{(i)}\ \det g \cdot (\det \Lambda)^2 = \det g \qquad \therefore \underline{\det \Lambda = \pm 1}.$$

$$\begin{aligned} \text{(ii)}\ g_{00} &= g_{00}\Lambda^0{}_0\Lambda^0{}_0 + g_{ij}\Lambda^i{}_0\Lambda^j{}_0 \\ 1 &= (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 \qquad \therefore \underline{(\Lambda^0{}_0)^2 = 1 + (\Lambda^i{}_0)^2 \geq 1}. \end{aligned}$$

- Thus, LTs can be classified into 4 disconnected sets;

	$\det\Lambda = +1$ ("proper ")	$\det\Lambda = -1$ ("improper")
$\Lambda^0{}_0 \geq 1$ ("orthochronous")	connected to $I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	connected to $P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$
$\Lambda^0{}_0 \leq -1$ ("anti-orthochronous")	connected to $PT = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$	connected to $I = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

- In the following, we consider only the proper-orthochronous ($\det\Lambda = +1$ and $\Lambda^0{}_0 \geq 1$) LTs, which are connected to I.

Problem

(b-13) Show that the proper-orthochronous LTs form a subgroup.

§ 3.3 Lorentz transformation of scalar field

- LT of a field is represented by unitary operators acting on it:[1]

$$\begin{aligned} \Phi(x) \to \Phi'(x) & \cancel{= U(\Lambda)\Phi(x)U(\Lambda)^{-1}} \\ &= U(\Lambda)^{-1}\Phi(x)U(\Lambda) \qquad \Phi(x):\ \text{generic field}, \quad U(\Lambda)^{-1} = U(\Lambda)^{\dagger} \end{aligned}$$

[1] [Note added on May 22.] The definition in the original version was wrong. I'm sorry! See the "Comments after the lecture" on May 21 for details. To clarify the corrections, the wrong expressions are struck out in red.

► Scalar fields are the fields which transform as

$$\phi(x) \to \phi'(x) \cancel{= U(\Lambda)\phi(x)U(\Lambda)^{-1}}$$
$$= U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$$

Substituting $x = y' = \Lambda y$, it means $\underline{\phi'(y') = \phi(y)}$ (for all y).

§ 3.3.1 Lorentz transformation of a, $a^\dagger$, and one-particle state

$$\begin{aligned} U(\Lambda)a(\vec{p})U(\Lambda)^{-1} &= ??\\ U(\Lambda)a(\vec{p})^\dagger U(\Lambda)^{-1} &= ??\\ U(\Lambda)\left|\vec{p}\right\rangle &= ?? \end{aligned}$$

(i) First of all, for any $f(\vec{p})$,

$$\boxed{\int d^3p \frac{1}{2E_p} f(\vec{p}) = \int d^4p\,\delta(p^2 - m^2)\theta(p^0) f(\vec{p})}$$

where $\theta(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x \le 0) \end{cases}$ is the step function.

Problem

(b-14) Show it.

(ii) Therefore

$$\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x} \right) \\ &= \int d^4p\,\delta(p^2 - m^2)\theta(p^0)\frac{\sqrt{2E_p}}{(2\pi)^3} \left(a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x} \right) \end{aligned}$$

and its 4-momentum FT is given by

$$\begin{aligned} \widetilde{\phi}(k) &\equiv \int d^4x e^{ik\cdot x}\phi(x) \\ &= \int d^4p\,\delta(p^2 - m^2)\theta(p^0)\frac{\sqrt{2E_p}}{(2\pi)^3} \left(a(\vec{p})(2\pi)^4\delta^{(4)}(p-k) + a^\dagger(\vec{p})(2\pi)^4\delta^{(4)}(p+k) \right) \\ &= (2\pi)\delta(k^2 - m^2)\sqrt{2E_k}\left(\theta(k^0)a(\vec{k}) + \theta(-k^0)a^\dagger(-\vec{k}) \right) \text{———}(\star)\,. \end{aligned}$$

(iii) and its LT is

$$
\begin{aligned}
U(\Lambda)[\text{RHS of } (\star)]U(\Lambda)^{-1} &= (2\pi)\delta(k^2-m^2)\sqrt{2E_k} \\
&\quad\times \Big(\theta(k^0)U(\Lambda)a(\vec{k})U(\Lambda)^{-1} + \theta(-k^0)U(\Lambda)a^\dagger(-\vec{k})U(\Lambda)^{-1}\Big)\,. \\
U(\Lambda)[\text{LHS of } (\star)]U(\Lambda)^{-1} &= \int d^4x e^{ik\cdot x} U(\Lambda)\phi(x)U(\Lambda)^{-1} \\
&= \int d^4x e^{ik\cdot x} \cancel{\phi(\Lambda^{-1}x)}\ \phi(\Lambda x) \\
&= \cdots \big(\cancel{x=\Lambda y,\ \ k=\Lambda k'}\ \ x=\Lambda^{-1}y,\ \ k=\Lambda^{-1}k',\ \ k\cdot x = k'\cdot y\big) \\
&\qquad \left(d^4x = \left|\det\frac{\partial x}{\partial y}\right| d^4y = |\det \cancel{\Lambda}\,\Lambda^{-1}| d^4y = d^4y\right) \\
&= \int d^4y e^{ik'\cdot y}\phi(y) \\
&= \widetilde{\phi}(k') \\
&= (2\pi)\delta(k'^2-m^2)\sqrt{2E_{k'}}\,\Big(\theta(k'^0)a(\vec{k'}) + \theta(-k'^0)a^\dagger(-\vec{k'})\Big)\,.
\end{aligned}
$$

Comparing them, and using $k'^2 = k^2$ and $\theta(k'^0)=\theta(k^0)$, we obtain

$$
\begin{aligned}
&U(\Lambda)a(\vec{k})U(\Lambda)^{-1} = \sqrt{\frac{E_{k'}}{E_k}}a(\vec{k'}) \quad (\cancel{k'=\Lambda^{-1}k}\ k'=\Lambda k) \\
\text{and}\qquad &U(\Lambda)a(\vec{k})^\dagger U(\Lambda)^{-1} = \sqrt{\frac{E_{k'}}{E_k}}a(\vec{k'})^\dagger \qquad \blacksquare
\end{aligned}
$$

► LT of a one-particle state is then given by

$$
\begin{aligned}
U(\Lambda)\left|\vec{p}\right\rangle &= U(\Lambda)\sqrt{2E_p}a^\dagger(\vec{p})\left|0\right\rangle \\
&= \sqrt{2E_p}U(\Lambda)a^\dagger(\vec{p})U(\Lambda)^{-1}\ \underbrace{U(\Lambda)\left|0\right\rangle}_{=\left|0\right\rangle\quad(\S\,2.6)} \\
&= \sqrt{2E_p}\sqrt{\frac{E_{p'}}{E_p}}a(\vec{p'})^\dagger\left|0\right\rangle \qquad (\cancel{p'=\Lambda^{-1}p}\ p'=\Lambda p) \\
&= \sqrt{2E_{p'}}a(\vec{p'})^\dagger\left|0\right\rangle \\
&= \left|\vec{p'}\right\rangle \qquad \blacksquare
\end{aligned}
$$

§ 3.3.2 Let's check what we have written in § 2.

(§ 2.1) Let's show the invariance of the action,

$$
S = \int d^4x\mathcal{L} = \int d^4x\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right)
$$

under the Lorentz transformation

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) = \phi(y), \qquad y^\nu = (\Lambda^{-1})^\nu{}_\mu x^\mu .$$

The transformation of $\partial_\mu \phi$ is given by

$$\partial_\mu \phi(x) \to \frac{\partial}{\partial x^\mu}\phi(\Lambda^{-1}x) = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\phi(y) = (\Lambda^{-1})^\nu{}_\mu [\partial_\nu \phi](y)$$

and hence

$$\begin{aligned}\partial_\mu\phi\partial^\mu\phi(x) &= g^{\mu\rho}\partial_\mu\phi\partial_\rho\phi(x) \\ &\to \underbrace{g^{\mu\rho}(\Lambda^{-1})^\nu{}_\mu(\Lambda^{-1})^\sigma{}_\rho}_{=g^{\nu\sigma}}[\partial_\nu\phi](y)[\partial_\rho\phi](y) = [\partial_\nu\phi\partial^\nu\phi](y)\end{aligned}$$

Thus,

$$\begin{aligned}S &= \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi(x) - \frac{1}{2}m^2\phi(x)^2\right) \\ &\to \int \underbrace{d^4x}_{=d^4y}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi(y) - \frac{1}{2}m^2\phi(y)^2\right) \\ &= S\,. \quad \blacksquare\end{aligned}$$

(§ 2.7) $E_p\delta^{(3)}(\vec{p}-\vec{q})$

From § 2.7,

$$\begin{aligned}E_p\delta^{(3)}(\vec{p}-\vec{q}) &= \langle\vec{q}|\vec{p}\rangle \\ &= \langle\vec{q}|U(\Lambda)^{-1}U(\Lambda)|\vec{p}\rangle \\ &= \langle\vec{q'}|\vec{p'}\rangle \qquad (p' = \cancel{\Lambda^{-1}p}\ \Lambda p) \\ &= E_{p'}\delta^{(3)}(\vec{p'}-\vec{q'}) \qquad \blacksquare\end{aligned}$$

(§ 2.8) $\Delta(x)$

(b): $\Delta(\Lambda x) = \Delta(x)$ can be shown by using the equation in (i) of § 3.3.1:

Problem

(b-15) Show it.

(c): $\Delta(x) = 0$ for $x^2 < 0$ can be shown as follows. First,

$$\Delta(x^0 = 0, \vec{x}) = (-i)\int \frac{d^3p}{(2\pi)^3 2E_p}\left(e^{i\vec{p}\cdot\vec{x}} - e^{-i\vec{p}\cdot\vec{x}}\right) = 0\,.$$

On the other hand, for space-like x ($x^2 = (x^0)^2 - \vec{x}^2 < 0$), one can always Lorentz transform it to $\Lambda x = x' = (0, \vec{x}')$.

(For a Lorentz boost in the opposite direction to $\vec{x}$, x^0 is transformed as $x'^0 = \frac{1}{\sqrt{1-\beta^2}}(x^0 - \vec{\beta}\cdot\vec{x})$. Taking $\vec{\beta} = \frac{x^0}{\vec{x}^2}\vec{x}$, we have $x'^0 = 0$. Note that this is impossible for a time-like x, where $x^2 = (x^0)^2 - \vec{x}^2 > 0$, because $\left|\frac{x^0}{\vec{x}^2}\vec{x}\right| > 1$ in that case.)

Therefore, by using (b), we have $\Delta(x) = \Delta(\Lambda x) = \Delta(0, \vec{x'}) = 0$ for $x^2 < 0$.

——— on May 14, up to here. ———

Questions after the lecture:

Q: Concerning the LT of fields, shouldn't it be

$$\Phi'(x) = U(\Lambda)^{-1}\Phi(x)U(\Lambda),$$

not

$$\Phi'(x) = U(\Lambda)\Phi(x)U(\Lambda)^{-1} \quad ?$$

For instance, in the relation between the Shrödinger-rep. and Heisenberg-rep. ($\mathcal{O}_H = e^{iHt}\mathcal{O}_S e^{-iHt}$), it seems the latter convention is taken. Furthermore, the latter leads to $U(\Lambda)|p\rangle = |\Lambda p\rangle$, which seems more natural than $U(\Lambda)|p\rangle = |\Lambda^{-1}p\rangle$ given by the former one.

A: ~~Well,... I know there are different definitions, but I just chose one of them, and it is at least self-consistent within my lecture note, I hope.~~ → [Note added on May 22]. No, it is not self-consistent! It is not a matter of definition, as far as I want to keep the relation $U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2)$. I was wrong. You are right! I have corrected the corresponding parts. See also the "Comments after the lecture" on May 21 for details.

——— May 21, from here. ———

Where were we?

§ 3 Lorentz...
§ 3.1 $x \to x' = \Lambda x$.
§ 3.2 $\Lambda = I + \omega$.
§ 3.A...
§ 3.3 $\phi'(x') = \phi(x)$.
(today →)

§ 3.4 Lorentz transformations of other fields, and representations of Lorentz group.

§ 3.4.1 Lorentz transformations of general fields

► In § 3.3.2, we have shown that the scalar action

$$S = \int dtL = \int d^4x\mathcal{L}[\phi(x), \partial_\mu\phi(x)]$$

is invariant under the LTs of the scalar field,

$$
\begin{aligned}
&\underline{\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)} \\
\text{or} \quad &\underline{\phi'(x') = \phi(x)}.
\end{aligned}
$$

We'd like to generalize it as

$$
\begin{aligned}
&\boxed{\Phi_a(x) \to \Phi'_a(x) = D_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x)} \quad a, b = 1, \cdots N \\
\text{or} \quad &\boxed{\Phi'(x') = D(\Lambda)\Phi(x)}, \qquad x' = \Lambda x
\end{aligned}
$$

- The matrix $D(\Lambda)$ ($N \times N$ matrix) must be a representation of the Lorentz group.

$$\boxed{D(\Lambda_2\Lambda_1) = D(\Lambda_2)D(\Lambda_1)}$$

(Proof:) For two successive LTs,

$$
\begin{aligned}
&x \underset{\Lambda_1}{\longrightarrow} x' \underset{\Lambda_2}{\longrightarrow} x'' \\
&\Phi'(x') = D(\Lambda_1)\Phi(x) \\
&\Phi''(x'') = D(\Lambda_2)\Phi'(x') = D(\Lambda_2)D(\Lambda_1)\Phi(x)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&x'' = \Lambda_2 x' = \Lambda_2(\Lambda_1 x) = (\Lambda_2\Lambda_1)x \\
&\therefore \Phi''(x'') = D(\Lambda_2\Lambda_1)\Phi(x)
\end{aligned}
$$

Thus

$$D(\Lambda_2\Lambda_1) = D(\Lambda_2)D(\Lambda_1) \qquad \blacksquare$$

- What kind of representations does the Lorentz group have?
  ($\Longleftrightarrow$ What kind of fields (particles) are allowed in relativistic QFT?)

$$
\begin{cases}
\text{scalar field} : D(\Lambda) = 1 \quad (1 \times 1 \text{ matrix}) & \Phi = \phi \text{ has one component.} \\
\text{spinor field} : D(\Lambda) = ??(\to \S 3.5) \quad (2 \times 2 \text{ or } 4 \times 4 \text{ matrix}) & \Phi = \psi \text{ has 2 or 4 components.} \\
\text{vector field} : D(\Lambda) = \Lambda^\mu{}_\nu \quad (4 \times 4 \text{ matrix}) & \Phi = A_\mu \text{ has 4 components.}
\end{cases}
$$

§ 3.4.2 Infinitesimal Lorentz transformation and the generators

- Consider an infinitesimal LT

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu,$$

parametrized by 6 small parameters $\omega_{\mu\nu} \ll 1$ (or $\theta_i, \eta_i \ll 1$ see § 3.2), acting on a generic field Φ_a ($a = 1, \ldots N$),

$$\Phi'(x) = D(\Lambda)\Phi(\Lambda^{-1}x).$$

For $\omega_{\mu\nu} = 0$, $\Lambda = I_{4\times4}$ (no transformation), and

$$D_{ab}(\Lambda) = D_{ab}(I) = \delta_{ab} \qquad (N \times N \text{ matrix})$$

Thus we can expand $D(\Lambda)$ as

$$D(I + \omega) = I_{N\times N} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}, \qquad M^{\mu\nu} = -M^{\nu\mu}$$

$$\text{or} \qquad D_{ab}(I_{4\times4} + \omega) = \delta_{ab} + \frac{i}{2}\omega_{\mu\nu}\left[M^{\mu\nu}\right]_{ab}$$

where the $N \times N$ matrices $\left[M^{\mu\nu}\right]_{ab}$ are the 6 generators of Lorentz group in this representation.

- The 6 generators $M^{\mu\nu}$ satisfy the commutation relations of Lorentz algebra. Let's show it. Consider three successive LTs,

$$D(\Lambda_3\Lambda_2\Lambda_1) = D(\Lambda_3)D(\Lambda_2)D(\Lambda_1) \qquad \text{———————}(1)$$

with

$$\begin{cases} \Lambda_1 = I + \omega \\ \Lambda_2 = I + \bar{\omega} \\ \Lambda_3 = I - \omega \end{cases} \qquad \bar{\omega}_{\mu\nu}, \omega_{\mu\nu} \ll 1$$

Then,

$$\begin{aligned} \text{LHS of (1)} &= D\Big((I-\omega)(I+\bar{\omega})(I+\omega)\Big) \\ &= D(I + \bar{\omega} - \omega^2 - \omega\bar{\omega} + \bar{\omega}\omega - \omega\bar{\omega}\omega) \\ &= I + \frac{i}{2}(\bar{\omega} - \omega^2 - \omega\bar{\omega} + \bar{\omega}\omega)_{\alpha\beta}M^{\alpha\beta} + \mathcal{O}(\bar{\omega}, \omega^2, \omega\bar{\omega})^2 \\ \text{RHS of (1)} &= D(I-\omega)D(I+\bar{\omega})D(I+\omega) \\ &= \left(I - \frac{i}{2}\omega M + \mathcal{O}(\omega)^2\right)\left(I + \frac{i}{2}\bar{\omega} M + \mathcal{O}(\bar{\omega})^2\right)\left(I + \frac{i}{2}\omega M + \mathcal{O}(\omega)^2\right) \end{aligned}$$

Comparing both sides

$$\mathcal{O}(1), \mathcal{O}(\omega), \mathcal{O}(\bar{\omega}) \text{ eqs.} \to \text{trivial } (I = I, O = O, M = M)$$

$$\mathcal{O}(\omega^2), \mathcal{O}(\bar{\omega}^2) \text{ eqs.} \to \text{no closed relations.}$$

$$\mathcal{O}(\omega\bar{\omega}) \text{ eq.} \to \frac{i}{2}(-\omega\bar{\omega} + \bar{\omega}\omega)_{\alpha\beta}M^{\alpha\beta} = \frac{1}{4}\left(\omega M \cdot \bar{\omega} M - \bar{\omega} M \cdot \omega M\right) \qquad \text{———————}(2)$$

Now, comparing the $\omega_{\mu\nu}\bar{\omega}_{\rho\sigma}$ components in the both sides of (2), we obtain

$$\boxed{[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right)} \qquad \text{———————}(3)$$

Lie algebra of Lorentz group

Problem

(b-16) Show it. (Note: $(\bar{\omega}\omega - \omega\bar{\omega})_{\alpha\beta} = \bar{\omega}_{\alpha\gamma}\omega^{\gamma}{}_{\beta} - \omega_{\alpha\gamma}\bar{\omega}^{\gamma}{}_{\beta}$)

► Defining

$$\begin{cases} D(J_i) \equiv -\dfrac{1}{2}\epsilon_{ijk}M^{jk} \iff M^{ij} = -\epsilon_{ijk}D(J_k) \\ D(K_i) \equiv M^{0i} \end{cases} \qquad \text{——(4)},$$

$$(3) \iff \boxed{\begin{array}{l} [D(J_i), D(J_j)] = i\epsilon_{ijk}D(J_k) \\ [D(J_i), D(K_j)] = i\epsilon_{ijk}D(K_k) \\ [D(K_i), D(K_j)] = -i\epsilon_{ijk}D(J_k) \end{array}} \text{——(5)}$$

The same as those in § 3.2 ! (as we promised.)

► Recall that $M^{\mu\nu} \iff D(J_i), D(K_i)$ are the generators defined by

$$D(I_{4\times4} + \omega) = I_{N\times N} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu},$$

which represents infinitesimal rotations and boosts. Using the notation $\omega_{0i} = \eta_i$ and $\omega_{ij} = -\epsilon_{ijk}\theta_k$ for the 6 small parameters (see § 3.2), the above eq. becomes

$$D(I_{4\times4} + \omega) = I_{N\times N} + i\left[\theta_i D(J_i) + \eta_i D(K_i)\right]$$

which is the same form as the infinitesimal LT of coordinates in § 3.2.

► Now, define

$$\boxed{\begin{array}{l} D(A_i) = \dfrac{1}{2}\left(D(J_i) + iD(K_i)\right) \\ \\ D(B_i) = \dfrac{1}{2}\left(D(J_i) - iD(K_i)\right) \end{array}}$$

(Note that $D(B_i) \neq D(A_i)^\dagger$, because $D(K_i)^\dagger \neq D(K_i)$ (see discussion later).)

Then

$$(5) \iff \boxed{\begin{array}{l} [D(A_i), D(A_j)] = i\epsilon_{ijk}D(A_k) \\ [D(B_i), D(B_j)] = i\epsilon_{ijk}D(B_k) \\ [D(A_i), D(B_j)] = 0 \end{array}} \text{——(6)}$$

This is the algebra of SU(2)×SU(2), and therefore we can classify the representations of Lorentz group by using representations of SU(2).

► Before going ahead, let's summarize the discussion so far:

- LTs of fields: $\Phi'(x') = D(\Lambda)\Phi(x)$ with $x' = \Lambda x$.
- For infinitesimal LTs, $D(\Lambda) = D(I_{4\times4} + \omega) = I_{N\times N} + \dfrac{i}{2}\omega_{\mu\nu}M^{\mu\nu}$

- Three equivalent ways of representing the 6 generators:

$$\begin{array}{rcccccc} & M^{\mu\nu} & \Longleftrightarrow & D(J_i), D(K_i) & \Longleftrightarrow & D(A_i), D(B_i) \\ \text{satisfying} & (3) & \Longleftrightarrow & (5) & \Longleftrightarrow & (6) \end{array}$$

- So far, $D(\Lambda)$, $M^{\mu\nu}$, $D(J_i)$, $D(K_i)$, $D(A_i)$, $D(B_i)$, are all generic $N \times N$ matrices.

§ 3.4.3 Representation of "A-spin"

► What's the generic representation which satisfy (6) ? Let's concentrate on $D(A_i)$.

$$[D(A_i), D(A_j)] = i\epsilon_{ijk} D(A_k)$$

We know this from QM !!

$$\left[\hat{j}_i, \hat{j}_j\right] = i\epsilon_{ijk}\hat{j}_k.$$

Starting from this, we could show that generic representation is

$$\begin{aligned} &\underline{\text{spin-}j\text{ state}}: \quad |j, m\rangle \\ &j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots \\ &m = \underbrace{-j, -j+1, \cdots j-1, j}_{2j+1} \end{aligned}$$

$$\text{where} \quad \begin{cases} \hat{j}^2 |j, m\rangle = j(j+1) |j, m\rangle \\ \hat{j}_3 |j, m\rangle = m |j, m\rangle \end{cases} .$$

► $\boxed{\text{NOTE}}$ In QM, we have used the fact that $\hat{j}_i$ are Hermitian, $\hat{j}_i^\dagger = \hat{j}_i$. Here, $D(A_i)$ and $D(B_i)$ are not Hermitian, but we can derive a similar result assuming a finite dimensional representation. Let's see this.

► For simplicity, we denote

$$A_i = D(A_i) \qquad (N \times N \text{ matrix}).$$

Define

$$\begin{cases} A^2 = A_1^2 + A_2^2 + A_3^2, \\ A_\pm = A_1 \pm iA_2. \end{cases}$$

From (6), we can show

$$\begin{aligned} [A^2, A_3] &= 0 \quad \text{———} \ \text{(i)}, \\ [A^2, A_\pm] &= 0 \quad \text{———} \ \text{(ii)}, \\ [A_3, A_\pm] &= \pm A_\pm \quad \text{———} \ \text{(iii)}, \\ A^2 &= A_3(A_3+1) + A_-A_+ \\ &= A_3(A_3-1) + A_+A_- \quad \text{———} \ \text{(iv)}. \end{aligned}$$

From (i), there exists a simultaneous eigenvector of A^2 and A_3; $\Phi_{\lambda,\mu}$,[2]

$$\left.\begin{array}{l}\underbrace{\begin{pmatrix} & & \\ & A^2 & \\ & & \end{pmatrix}}\begin{pmatrix}\Phi_{\lambda,\mu}\end{pmatrix} = \lambda\begin{pmatrix}\Phi_{\lambda,\mu}\end{pmatrix} \\ \underbrace{\begin{pmatrix} & & \\ & A_3 & \\ & & \end{pmatrix}}_{N\times N}\begin{pmatrix}\Phi_{\lambda,\mu}\end{pmatrix} = \mu\begin{pmatrix}\Phi_{\lambda,\mu}\end{pmatrix}\end{array}\right\} N$$

Then, from (ii) and (iii), the vector

$$\Phi_{\lambda,\mu\pm1} \equiv A_\pm \Phi_{\lambda,\mu}$$

satisfy

$$\begin{cases} A^2\Phi_{\lambda,\mu\pm1} = \lambda\Phi_{\lambda,\mu\pm1} \\ A_3\Phi_{\lambda,\mu\pm1} = (\mu\pm1)\Phi_{\lambda,\mu\pm1}. \end{cases}$$

Continuing further, the vector $\Phi_{\lambda,\mu\pm n} = (A_\pm)^n\Phi_{\lambda,\mu}$ satisfy

$$\begin{cases} A^2\Phi_{\lambda,\mu\pm n} = \lambda\Phi_{\lambda,\mu\pm n} \\ A_3\Phi_{\lambda,\mu\pm n} = (\mu\pm n)\Phi_{\lambda,\mu\pm n}. \end{cases}$$

Now, assuming <u>finite dimensional representation</u>, there must be upper and lower bounds on A_3's eigenvalue

$$\begin{cases} \mu_{\max} = \mu + n_+, \\ \mu_{\min} = \mu + n_-, \end{cases}$$

with

$$\begin{cases} A_+\Phi_{\lambda,\mu_{\max}} = 0 \quad \text{———} \quad (\text{v}), \\ A_-\Phi_{\lambda,\mu_{\min}} = 0 \quad \text{———} \quad (\text{vi}). \end{cases}$$

From (iv) and (v),

$$\underbrace{A^2}_{\to\lambda}\Phi_{\lambda,\mu_{\max}} = \Big(\underbrace{A_3(A_3+1)}_{\to\mu_{\max}(\mu_{\max}+1)} + A_-\underbrace{A_+}_{\to 0}\Big)\Phi_{\lambda,\mu_{\max}}$$

$$\therefore\quad \lambda = \mu_{\max}(\mu_{\max}+1) \quad \text{———} \quad (\text{vii}),$$

Similarly from (iv) and (vi),

$$\lambda = \mu_{\min}(\mu_{\min}-1) \quad \text{———} \quad (\text{viii}),$$

[2](At this stage, since A^2 and A_3 are not Hermitian, the eigenvalues λ and μ are not necessarily real numbers, but we will see they are real.)

From (vii)−(viii),

$$(\mu_{\text{max}} + \mu_{\text{min}})(\mu_{\text{max}} - \mu_{\text{min}} + 1) = 0$$

Since $\mu_{\text{max}} - \mu_{\text{min}} = n_+ + n_- \equiv n =$ non-negative integer (and therefore real), $\mu_{\text{max}} - \mu_{\text{min}} + 1 > 0$, and we have $\underline{\mu_{\text{max}} = -\mu_{\text{min}}}$. Together with $\mu_{\text{max}} - \mu_{\text{min}} = n$, we thus have

$$\begin{cases} \mu_{\text{max}} = \dfrac{n}{2} = -\mu_{\text{min}} \\ \\ \lambda = \dfrac{n}{2}\left(\dfrac{n}{2} + 1\right) \end{cases}$$

We have obtained the irreducible representation of the "A-spin".
(Rewriting $A_i \to D(A_i)$, $n/2 \to A$, $\mu \to a$, and defining $D(A^2) = \sum_{i=1}^{3} D(A_i)^2$)

$$\begin{cases} D(A^2)\Phi_a^{(A)} = A(A+1)\ \Phi_a^{(A)}, \\ D(A_3)\Phi_a^{(A)} = a\ \Phi_a^{(A)} \end{cases}$$

where

$$A = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$$

$$a = \underbrace{-A, -A+1, \cdots A-1, A}_{(2A+1)\text{ components}}$$

§ 3.4.4 Irreducible representations of Lorentz group

- Since we have two SU(2)s, $D(A_i)$ and $D(B_i)$, any irreducible representations of Lorentz group are parametrized by a set of two numbers:

$$(A, B) \qquad A, B = \text{integer or half-intetger}$$

- The corresponding field is

$$\Phi_{a,b}^{(A,B)} \qquad (2A+1)(2B+1)\text{ components}$$
$$\begin{cases} a = -A, -A+1, \cdots A-1, A \\ b = -B, -B+1, \cdots B-1, B \end{cases}$$

transforming as

$$\begin{cases} D(A^2)\Phi_{a,b}^{(A,B)} = A(A+1)\Phi_{a,b}^{(A,B)} \\ D(A_3)\Phi_{a,b}^{(A,B)} = a\Phi_{a,b}^{(A,B)} \\ D(B^2)\Phi_{a,b}^{(A,B)} = B(B+1)\Phi_{a,b}^{(A,B)} \\ D(B_3)\Phi_{a,b}^{(A,B)} = b\Phi_{a,b}^{(A,B)} \end{cases} \qquad \text{———(7)}$$

▶ The three kinds of fields introduced in § 1.4 are:

$$(A, B) = (0, 0) \qquad \text{scalar fields} \to \S\, 2 \text{ and } \S\, 3.3$$
$$(A, B) = \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right) \qquad \text{spinor fields} \to \S\, 3.5$$
$$(A, B) = \left(\frac{1}{2}, \frac{1}{2}\right) \qquad \text{vector fields}$$

▶ For scalar fields,

$$A = B = 0,$$
$$a = b = 0$$
$$\underline{\Phi^{(0,0)} \qquad \text{1-components}}$$
$$D(A_i) = D(B_i) = 0$$
$$\therefore D(\Lambda) = I + \frac{i}{2}\omega_{\mu\nu} \underbrace{M^{\mu\nu}}_{=0} = I$$
$$\therefore \Phi'(x') = D(\Lambda)\Phi(x) = \Phi(x)$$

——— on May 21, up to here. ———

<u>A comment after the lecture:</u>

Q: (The same student as the last week) I still think that the LT of fields should be

$$\Phi'(x) = U(\Lambda)^{-1}\Phi(x)U(\Lambda),$$
$$\text{not} \qquad \Phi'(x) = U(\Lambda)\Phi(x)U(\Lambda)^{-1}.$$

It is not a matter of convention. The latter convention, which would lead to $U(\Lambda)|p\rangle = |\Lambda^{-1}p\rangle$, is inconsistent with $U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2)$.

A: (After some discussion with the student. . .) You are right! I was wrong!

[Note added on May 22] Let me clarify the point. Let's consider the scalar field, for simplicity. We want to keep the relations

$$U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2),$$
$$\text{as well as} \quad \phi'(x) = \phi(\Lambda^{-1}x), \text{ or } \phi'(x') = \phi'(\Lambda x) = \phi(x).$$

Then

$$\text{Wrong}: \phi(\Lambda^{-1}x) = \phi'(x) = U(\Lambda)\phi(x)U(\Lambda)^{-1}$$
$$\to \quad U(\Lambda_2)U(\Lambda_1)\phi(x)U(\Lambda_1)^{-1}U(\Lambda_2)^{-1} = U(\Lambda_2\Lambda_1)\phi(x)U(\Lambda_2\Lambda_1)^{-1}$$
$$\text{LHS} = U(\Lambda_2)\phi(\Lambda_1^{-1}x)U(\Lambda_2)^{-1} = \phi(\Lambda_2^{-1}\Lambda_1^{-1}x)$$
$$\text{RHS} = \phi((\Lambda_2\Lambda_1)^{-1}x) = \phi(\Lambda_1^{-1}\Lambda_2^{-1}x) \quad \text{Inconsistent!}$$
$$\text{Correct}: \phi(\Lambda^{-1}x) = \phi'(x) = U(\Lambda)^{-1}\phi(x)U(\Lambda)$$
$$\to \quad U(\Lambda_2)^{-1}U(\Lambda_1)^{-1}\phi(x)U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2)^{-1}\phi(x)U(\Lambda_1\Lambda_2)$$
$$\text{LHS} = U(\Lambda_2)^{-1}\phi(\Lambda_1^{-1}x)U(\Lambda_2) = \phi(\Lambda_2^{-1}\Lambda_1^{-1}x)$$
$$\text{RHS} = \phi((\Lambda_1\Lambda_2)^{-1}x) = \phi(\Lambda_2^{-1}\Lambda_1^{-1}x) \quad \text{Consistent.}$$

——————— May 28, from here. ———————

Correction

$$\begin{array}{rl} \text{Wrong} & \Phi'(x) = U(\Lambda)\Phi(x)U(\Lambda)^{-1}. \\ \text{Correct} & \Phi'(x) = U(\Lambda)^{-1}\Phi(x)U(\Lambda). \end{array}$$

The latter is consistent with $U(\Lambda_2)U(\Lambda_1) = U(\Lambda_2\Lambda_1)$, but the former is not. There are corrections in § 3.3 of the lecture note. Please see the PDF file.

Where were we?

§ 3 Lorentz...
§ 3.1 $x \to x' = \Lambda x$.
§ 3.2 $\Lambda = I + \omega$.
§ 3.3 $\phi'(x') = \phi(x)$.
§ 3.4 $\Phi'(x') = D(\Lambda)\Phi(x)$.
(today $\to$)

§ 3.5 Spinor Fields

- Consider fields with $\boxed{(A, B) = \left(0, \frac{1}{2}\right)}$.

$$\begin{aligned} &(2A+1)(2B+1) = 1 \times 2 = 2 \text{ components} \\ &\Phi_b^{(0,1/2)}, \qquad b = -1/2, 1/2. \end{aligned}$$

Thus, $D(A_i), D(B_i) \Longleftrightarrow D(J_i), D(K_i) \Longleftrightarrow M^{\mu\nu}$ and $D(\Lambda)$ are 2×2 matrices.
From (6) and (7),

$$\begin{cases} D(A_i) = 0 \qquad (2 \times 2), \\ D(B_i) = \dfrac{1}{2}\sigma_i \end{cases} \qquad \text{———} \ (8).$$

σ_i : Pauli matrices $\quad \sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$.

$$\left(\begin{aligned} &\because \text{For } \Phi = \begin{pmatrix} \Phi_{1/2} \\ \Phi_{-1/2} \end{pmatrix}, \\ &D(B_3)\Phi = \begin{pmatrix} 1/2 & \\ & -1/2 \end{pmatrix}\Phi, \ D(B_+)\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Phi, \ D(B_-)\Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\Phi. \end{aligned} \right)$$

Thus,

$$(8) \Longleftrightarrow \begin{cases} D(J_i) = D(A_i) + D(B_i) = \frac{1}{2}\sigma_i \\ D(K_i) = -iD(A_i) + iD(B_i) = i\frac{1}{2}\sigma_i \end{cases}$$

$$\Longleftrightarrow \begin{cases} M^{ij} = -\frac{1}{2}\epsilon_{ijk}\sigma_k \\ M^{0i} = i\frac{1}{2}\sigma_i \end{cases}$$

► Denoting the 2-component field $\Phi_b^{(0,1/2)}$ as

$$\left(0, \frac{1}{2}\right)\text{-field: } \boxed{\psi_{L\alpha}(x) \quad \alpha = 1, 2},$$

its Lorentz transformation is given by

$$\begin{aligned} \psi'_{L\alpha}(x') &= D(\Lambda)_\alpha{}^\beta \psi_{L\beta}(x), \\ D(\Lambda)_\alpha{}^\beta &= \exp\Big(i\theta_i D(J_i) + i\eta_i D(K_i)\Big)_\alpha{}^\beta \\ &= \exp\left(i\theta_i \frac{1}{2}\sigma_i - \eta_i \frac{1}{2}\sigma_i\right)_\alpha{}^\beta \end{aligned}$$

► $\boxed{\text{Example}}$:

- $D(\Lambda)$ for a rotation around the z-axis

$$D(\Lambda) = \exp\left(i\theta_3 \frac{1}{2}\sigma_3\right) = \exp\begin{pmatrix} \frac{i}{2}\theta_3 & \\ & -\frac{i}{2}\theta_3 \end{pmatrix} = \begin{pmatrix} e^{\frac{i}{2}\theta_3} & \\ & e^{-\frac{i}{2}\theta_3} \end{pmatrix} \quad \text{——(i)}$$

- $D(\Lambda)$ for a boost in the z-direction

$$D(\Lambda) = \exp\left(-\eta_3 \frac{1}{2}\sigma_3\right) = \exp\begin{pmatrix} -\frac{1}{2}\eta_3 & \\ & \frac{1}{2}\eta_3 \end{pmatrix} = \begin{pmatrix} e^{-\frac{1}{2}\eta_3} & \\ & e^{\frac{1}{2}\eta_3} \end{pmatrix} \quad \text{——(ii)}$$

Comment on the unitarity.

$D(J_i) = \frac{1}{2}\sigma_i$ are Hermitian, $D(J_i)^\dagger = D(J_i)$,
but $D(K_i) = i\frac{1}{2}\sigma_i$ are anti-Hermitian, $D(K_i)^\dagger = -D(K_i)$.
Thus, the spinor representation of the Lorentz group $D(\Lambda)$ is NOT unitary in general.
(For instance, the rotation (i) is unitary, $D(\Lambda)^\dagger D(\Lambda) = I$,
but the boost (ii) is not unitary, $D(\Lambda)^\dagger D(\Lambda) \neq I$.)

In general, there are no non-trivial finite dimensional unitary representation of the Lorentz group.

► Similarly, for spinor fields with $\boxed{(A,B)=\left(\frac{1}{2},0\right)}$, $\Phi_a^{(1/2,0)}$, from (6) and (7),

$$\begin{cases} D(A_i)=\frac{1}{2}\sigma_i \\ D(B_i)=0 \qquad (2\times 2) \end{cases} \Longleftrightarrow \begin{cases} D(J_i)=\frac{1}{2}\sigma_i \\ D(K_i)=-i\frac{1}{2}\sigma_i \end{cases} \Longleftrightarrow \begin{cases} M^{ij}=-\frac{1}{2}\epsilon_{ijk}\sigma_k \\ M^{0i}=-i\frac{1}{2}\sigma_i \end{cases}$$

► To summarize, there are two kinds of 2-component spinor fields with $(A,B)=(0,1/2)$ and $(1/2,0)$, and their Lorentz transformations are given by

$$\begin{aligned} &(A,B) \\ &\left(0,\frac{1}{2}\right):\ \psi_L\to D_L(\Lambda)\psi_L=\exp\left(i\theta_i\frac{1}{2}\sigma_i-\eta_i\frac{1}{2}\sigma_i\right)\psi_L=\left(I+i\theta_i\frac{1}{2}\sigma_i-\eta_i\frac{1}{2}\sigma_i+\cdots\right)\psi_L \\ &\left(\frac{1}{2},0\right):\ \psi_R\to D_R(\Lambda)\psi_R=\exp\left(i\theta_i\frac{1}{2}\sigma_i+\eta_i\frac{1}{2}\sigma_i\right)\psi_R=\left(I+i\theta_i\frac{1}{2}\sigma_i+\eta_i\frac{1}{2}\sigma_i+\cdots\right)\psi_R \end{aligned}$$

(We omit the argument x and $x'=\Lambda x$ for simplicity.)

Their infinitesimal transformations are

$$\begin{cases} \delta\psi_L=\frac{1}{2}(i\theta_i-\eta_i)\sigma_i\psi_L \\ \\ \delta\psi_R=\frac{1}{2}(i\theta_i+\eta_i)\sigma_i\psi_R \end{cases} \qquad \text{——— (9).}$$

► Note that

$$\begin{aligned} &\psi_L\sim\left(0,\frac{1}{2}\right) \qquad \psi_R\sim\left(\frac{1}{2},0\right) \\ \longleftrightarrow\quad &\psi_L^*\sim\left(\frac{1}{2},0\right) \qquad \psi_R^*\sim\left(0,\frac{1}{2}\right) \end{aligned}$$

From (9), $\delta\psi_L^*=\frac{1}{2}(-i\theta_i-\eta_i)\sigma_i^*\psi_L^*$

by using $\epsilon\sigma_i=-\sigma_i^*\epsilon$ where $\epsilon=i\sigma_2=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$,

$$\epsilon\delta\psi_L^*=\frac{1}{2}(-i\theta_i-\eta_i)\epsilon\sigma_i^*\psi_L^*$$

$$\delta(\epsilon\psi_L^*)=\frac{1}{2}(i\theta_i+\eta_i)\sigma_i(\epsilon\psi_L^*)$$

Thus, $\epsilon\psi_L^*$ transforms in the same way as ψ_R in (9).

▶ **Comment on spinor indices**

The indices of 2-component spinors are often denoted by undotted and dotted labels:

$$(\psi_L)_\alpha, \qquad (\psi_R)_{\dot\alpha}$$

together with invariant tensors $\epsilon^{\alpha\beta}$, $\epsilon^{\dot\alpha\dot\beta}$, and extended Pauli matrices $\sigma^\mu_{\alpha\dot\beta}$ (see below). In particular, the spinor contraction such as $\psi\xi \equiv \psi_\alpha \xi^\alpha = \psi_\alpha \epsilon^{\alpha\beta}\xi_\beta$ are very convenient (once you get used to it), but in this lecture, we do not use them.

§ 3.6 Spinor bilinears

▶ We have seen the Lorentz transformations of spinor fields ψ_L and ψ_R. In order to construct a Lorentz invariant Lagrangian from them, let's consider Lorentz transformations of spinor bilinears, such as

$$\psi_L^\dagger \psi_R = (\psi_{L1}^*, \psi_{L2}^*)\begin{pmatrix}\psi_{R1}\\ \psi_{R2}\end{pmatrix} = \psi_{L1}^*\psi_{R1} + \psi_{L2}^*\psi_{R2}.$$

$$\psi_L^\dagger \sigma_3 \psi_L = (\psi_{L1}^*, \psi_{L2}^*)\sigma_3\begin{pmatrix}\psi_{L1}\\ \psi_{L2}\end{pmatrix} = \psi_{L1}^*\psi_{L1} - \psi_{L2}^*\psi_{L2}.$$

In general, we can think of various combinations

$$\left\{\psi_L^T, \psi_R^T, \psi_L^\dagger, \psi_R^\dagger\right\} \times (2\times 2 \text{ matrix}) \times \{\psi_L, \psi_R, \psi_L^*, \psi_R^*\}.$$

They can be classified according to SU(2)×SU(2).

$$\psi_L, \psi_R^* \cdots \left(0, \frac{1}{2}\right)$$

$$\psi_R, \psi_L^* \cdots \left(\frac{1}{2}, 0\right)$$

§ 3.6.1 $(0, 1/2) \otimes (0, 1/2)$

▶ If there is only ψ_L field, the possible bilinear terms transforming as $\left(0, \frac{1}{2}\right) \otimes \left(0, \frac{1}{2}\right)$ are

$$\psi_L^T \cdot (2\times 2 \text{ matrics}) \cdot \psi_L.$$

Among them, $\psi_L^T \epsilon\, \psi_L$ is Lorentz invariant.

$$\begin{aligned}
\because\ \delta(\psi_L^T \epsilon\, \psi_L) &= (\delta\psi_L^T)\epsilon\psi_L + \psi_L^T\epsilon(\delta\psi_L)\\
&= \left(\frac{1}{2}(i\theta_k - \eta_k)\psi_L^T\sigma_k^T\right)\epsilon\psi_L + \psi_L^T\epsilon\left(\frac{1}{2}(i\theta_k - \eta_k)\sigma_k\psi_L\right) \quad (\because\ (9))\\
&= \frac{1}{2}(i\theta_k - \eta_k)\psi_L^T(\underbrace{\sigma_k^T\epsilon + \epsilon\sigma_k}_{=0})\psi_L\\
&= 0
\end{aligned}$$

- If there is only ψ_R field, similarly, $\psi_R^\dagger \epsilon\, \psi_R^*$ is Lorentz invariant.
- If there are both ψ_L and ψ_R field, we can also think

$$\psi_R^\dagger \cdot (2 \times 2 \text{ matrics}) \cdot \psi_L.$$

Among them, $\psi_R^\dagger \psi_L$ is Lorentz invariant.

$$\begin{aligned}
\because\ \delta(\psi_R^\dagger \psi_L) &= (\delta\psi_R^\dagger)\psi_L + \psi_R^\dagger(\delta\psi_L) \\
&= \left(\frac{1}{2}(-i\theta_k + \eta_k)\psi_R^\dagger \sigma_k\right)\psi_L + \psi_R^\dagger\left(\frac{1}{2}(i\theta_k - \eta_k)\sigma_k \psi_L\right) \quad (\because\ (9)) \\
&= 0
\end{aligned}$$

Comments

(i) In terms of SU(2)×SU(2), the above terms, $\psi_L^T \epsilon\, \psi_L$, $\psi_R^\dagger \epsilon\, \psi_R^*$ and $\psi_R^\dagger \psi_L$ correspond to

$$\left(0, \frac{1}{2}\right) \otimes \left(0, \frac{1}{2}\right) = \underbrace{(0,0)}_{\text{this part.}} \oplus (0,1)$$

(ii) One might think that

$$\psi_L^T \epsilon \psi_L = (\psi_1, \psi_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_1\psi_2 - \psi_2\psi_1$$

vanishes. However, if ψ_i are anti-commuting (as in quantized fermion field), $\psi_1\psi_2 = -\psi_2\psi_1$ and hence $\psi_L^T \epsilon \psi_L$ does not vanish.

(iii) $\psi_L^T \epsilon\, \psi_L$ and $\psi_R^\dagger \epsilon\, \psi_R^*$ terms correspond to Majorana mass terms, and $\psi_R^\dagger \psi_L$ corresponds to Dirac mass term.
If we consider a charged fermion (such as electron and positron), only the Dirac mass term is allowed.

$$\left(\begin{array}{l} \text{Field } \Phi \text{ is charged (under conserved symmetry)} \\ \Longleftrightarrow \text{Lagrangian is invariant under } \Phi \to e^{i\alpha}\Phi. \\ \psi_L^T \epsilon\, \psi_L \text{ is not invariant under } \psi_L \to e^{i\alpha}\psi_L, \\ \text{while } \psi_R^\dagger \psi_L \text{ is invariant under } \psi_L \to e^{i\alpha}\psi_L, \psi_R \to e^{i\alpha}\psi_R. \end{array}\right)$$

In the following we consider a charged fermion and hence only the Dirac mass term $\psi_R^\dagger \psi_L$.
(Neutrinos may have Majorana mass term (maybe Majorana fermion). Still unknown.)

§ 3.6.2 $(1/2, 0) \otimes (1/2, 0)$

► Similarly,

$$\psi_L^\dagger \psi_R, \quad \psi_R^T \epsilon \psi_R, \quad \psi_L^\dagger \epsilon \psi_L^*$$

are Lorentz invariant, corresponding to

$$\left(\frac{1}{2}, 0\right) \otimes \left(\frac{1}{2}, 0\right) = \underbrace{(0,0)}_{\text{this term.}} \oplus (1,0)$$

We only consider the Dirac mass term $\psi_L^\dagger \psi_R$.

§ 3.6.3 $(0, 1/2) \otimes (1/2, 0)$

► Consider

$$\psi_R^\dagger \cdot (2 \times 2 \text{ matrics}) \cdot \psi_R.$$

There are 4 independent combinations, which can be taken as

$$\psi_R^\dagger \psi_R, \qquad \psi_R^\dagger \sigma_i \psi_R \quad (i = 1, 2, 3).$$

They transform as

$$\begin{aligned} \delta(\psi_R^\dagger \psi_R) &= \eta_k (\psi_R^\dagger \sigma_k \psi_R), \\ \delta(\psi_R^\dagger \sigma_i \psi_R) &= \eta_i (\psi_R^\dagger \psi_R) + \epsilon_{ijk} \theta_k (\psi_R^\dagger \sigma_j \psi_R). \end{aligned}$$

Problem

(b-17) Show them.

Combining them,

$$\delta \begin{pmatrix} \psi_R^\dagger \psi_R \\ \psi_R^\dagger \sigma_1 \psi_R \\ \psi_R^\dagger \sigma_2 \psi_R \\ \psi_R^\dagger \sigma_3 \psi_R \end{pmatrix} = \begin{pmatrix} 0 & \eta_1 & \eta_2 & \eta_3 \\ \eta_1 & 0 & \theta_3 & -\theta_2 \\ \eta_2 & -\theta_3 & 0 & \theta_1 \\ \eta_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix} \begin{pmatrix} \psi_R^\dagger \psi_R \\ \psi_R^\dagger \sigma_1 \psi_R \\ \psi_R^\dagger \sigma_2 \psi_R \\ \psi_R^\dagger \sigma_3 \psi_R \end{pmatrix}.$$

This is nothing but the transformation of Lorentz 4-vector! (See the equation in § 3.2.) Defining

$$\boxed{\sigma^\mu = (I, \sigma_i) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

the above equation can be written as

$$\underline{\delta(\psi_R^\dagger \sigma^\mu \psi_R) = \omega^\mu{}_\nu (\psi_R^\dagger \sigma^\nu \psi_R)}$$

—————— on May 28, up to here. ——————

—————— June 4, from here. ——————

$\boxed{\text{12:05-12:10 J-PARC 見学会の宣伝}}$

Where were we?

§3 Lorentz...

...

§3.5 ψ_L $(0,1/2)$ and ψ_R $(1/2,0)$.

§3.6 Spinor bilinears

§3.6.3 $(0,1/2)\otimes(1/2,0)$

$$\underline{\delta(\psi_R^\dagger \sigma^\mu \psi_R) = \omega^\mu{}_\nu (\psi_R^\dagger \sigma^\nu \psi_R)}$$

(← last week)

(today →)

- Similarly, defining

$$\boxed{\bar\sigma^\mu = (I, -\sigma_i)}$$

One can show

$$\underline{\delta(\psi_L^\dagger \bar\sigma^\mu \psi_L) = \omega^\mu{}_\nu (\psi_L^\dagger \bar\sigma^\nu \psi_L)}$$

Comments

(i) In terms of SU(2)×SU(2), this means

$$\left(0,\frac{1}{2}\right)\otimes\left(\frac{1}{2},0\right) = \left(\frac{1}{2},\frac{1}{2}\right)$$

is a Lorentz 4-vector.

(ii) For finite Lorentz transformation, they transform as

$$\begin{aligned}{\psi'_R}^\dagger(x')\sigma^\mu\psi'_R(x') &= \psi_R^\dagger(x) D_R(\Lambda)^\dagger \sigma^\mu D_R(\Lambda)\psi_R(x) \qquad (\because \psi'_R(x') = D_R(\Lambda)\psi_R(x))\\ &= \Lambda^\mu{}_\nu \psi_R^\dagger(x)\sigma^\nu\psi_R(x),\end{aligned}$$

namely

$$D_R(\Lambda)^\dagger \sigma^\mu D_R(\Lambda) = \Lambda^\mu{}_\nu \sigma^\nu\,,$$

where

$$\begin{aligned}D_R(\Lambda) &= \exp\left(i\theta_k \frac{1}{2}\sigma_k + \eta_k \frac{1}{2}\sigma_k\right),\\ \Lambda^\mu{}_\nu &= \exp\left(i\theta_i J_i + i\eta_i K_i\right)^\mu{}_\nu. \qquad [(J_i)^\mu{}_\nu,\ (K_i)^\mu{}_\nu \ \to\ §3.2]\end{aligned}$$

Similarly,

$$\psi_L'^{\dagger}(x')\bar{\sigma}^\mu\psi_L'(x') = \psi_L^\dagger(x)\underbrace{D_L(\Lambda)^\dagger\bar{\sigma}^\mu D_L(\Lambda)}_{\Lambda^\mu{}_\nu\bar{\sigma}^\nu}\psi_L(x)$$
$$= \Lambda^\mu{}_\nu\psi_L^\dagger(x)\bar{\sigma}^\nu\psi_L(x).$$

Problem

(b-18) Show $\begin{cases} D_R(\Lambda)^\dagger\sigma^\mu D_R(\Lambda) = \Lambda^\mu{}_\nu\sigma^\nu \\ D_L(\Lambda)^\dagger\bar{\sigma}^\mu D_L(\Lambda) = \Lambda^\mu{}_\nu\bar{\sigma}^\nu \end{cases}$. [Hint: ($\rightarrow$ see the PDF file)]

One can prove it, for instance, in the following way.

1) Define

$$\hat{D}_R(s) = \exp\left[s\left(i\theta_k\frac{1}{2}\sigma_k + \eta_k\frac{1}{2}\sigma_k\right)\right], \quad X^\mu(s) = \hat{D}_R^\dagger(s)\sigma^\mu\hat{D}_R(s)$$
$$\hat{\Lambda}^\mu{}_\nu(s) = \exp\left[s\left(i\theta_i J_i + i\eta_i K_i\right)\right]^\mu{}_\nu, \quad Y^\mu(s) = \hat{\Lambda}^\mu{}_\nu(s)\sigma^\nu.$$

2) Show $X^\mu(0) = Y^\mu(0)$.

3) Show $\dfrac{\partial}{\partial s}Y^\mu(s) = (i\theta_i J_i + i\eta_i K_i)^\mu{}_\nu Y^\nu(s)$.

4) Show $\dfrac{\partial}{\partial s}X^\mu(s) = (i\theta_i J_i + i\eta_i K_i)^\mu{}_\nu X^\nu(s)$.

5) From 2), 3), 4), $X^\mu(s) = Y^\mu(s)$, and therefore $X^\mu(1) = Y^\mu(1)$.

(iii) The other combinations, $\psi_L^T\epsilon\sigma^\mu\psi_R$ and $\psi_R^\dagger\bar{\sigma}^\mu\epsilon\psi_L^*$, also transform as Lorentz 4-vector, but we do not consider them. (They are not invariant under $\psi_L \rightarrow e^{i\alpha}\psi_L, \psi_R \rightarrow e^{i\alpha}\psi_R$.)

► (Now we have obtained Lorentz scalars and vectors from spinor bilinears, ready to construct the Dirac Lagrangian.)

§ 4 Free Fermion (spin $1/2$) Field

$\boxed{\text{Goal}}$: To construct the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi,$$

solve the EOM (Dirac equation)

$$\mathcal{L} = (i\gamma^\mu \partial_\mu - m)\psi = 0,$$

and quantize the Dirac field.

§ 4.1 Lagrangian

§ 4.1.1 $\mathcal{L}$ in 2-component field

- In § 3.6, we have seen $\underline{\psi_R^\dagger \psi_L,\ \psi_L^\dagger \psi_R}$ are Lorentz invariant. They can be the Lagrangian terms.
- On the other hand, $\underline{\psi_R^\dagger \sigma^\mu \psi_R,\ \psi_L^\dagger \bar{\sigma}^\mu \psi_L}$ are Lorentz vector. They can be combined with ∂_μ to make the action Lorentz invariant.

  For instance, $\displaystyle\int d^4x\ \psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$ is Lorentz invariant:

$$\begin{aligned} \int d^4x\ \psi_R^\dagger(x)\sigma^\mu \partial_\mu \psi_R(x) \to & \int d^4x\ {\psi'_R}^\dagger(x)\sigma^\mu \partial_\mu \psi'_R(x) \qquad \big(\psi'_R(x) = D_R(\Lambda)\psi_R(\Lambda^{-1}x)\big) \\ &= \cdots \\ &= \int d^4x\ {\psi_R}^\dagger(x)\sigma^\nu \partial_\nu \psi_R(x). \end{aligned}$$

> Problem
>
> **(b-19)** Show it.

- There are other combinations with ∂_μ, but
  - $\partial_\mu(\psi_R^\dagger \sigma^\mu \psi_R)$: total derivative and not a viable Lagrangian term.
  - $(\partial_\mu \psi_R^\dagger)\sigma^\mu \psi_R$: equivalent to $\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$ up to total derivative.
- Similarly, $\displaystyle\int d^4x\ \psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L$ is Lorentz invariant.
- Combining them all, we obtain the Lagrangian of the free Dirac field:

$$\boxed{\mathcal{L} = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L - m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R).}$$

Comments

(i) The factor i is to make the Lagrangian Hermitian:

$$\begin{aligned}(i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R)^\dagger &= -i(\partial_\mu \psi_R)^\dagger \sigma^\mu \psi_R \\ &= i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + \text{total derivative}\end{aligned}$$

(ii) $\psi_{L/R}$ is has a mass dimension 3/2. $([\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R] = 2\times[\psi_R]+[\partial] = 2\times 3/2+1 = 4.)$

(iii) m is a real positive parameter $((\psi_R^\dagger \psi_L)^\dagger = \psi_L^\dagger \psi_R)$ with mass dimension 1.

§ 4.1.2 4-component Dirac field and γ matrices

- The above Lagrangian can be written in terms of **four-component Dirac spinor** and **gamma matrices (Dirac matrices)**

$$\begin{aligned}&\boxed{\mathcal{L} = \overline{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi} \\ &= (\psi_R^\dagger, \psi_L^\dagger)\left[\begin{pmatrix} 0 & i\sigma^\mu \partial_\mu \\ i\bar{\sigma}^\mu \partial_\mu & 0 \end{pmatrix} - \begin{pmatrix} mI & \\ & mI \end{pmatrix}\right]\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}\end{aligned}$$

where

$$\text{Dirac Spinor: } \Psi \equiv \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$$

$$\text{gamma matrices: } \gamma^\mu \equiv \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}$$

$$\overline{\Psi} \equiv \Psi^\dagger \gamma^0 = (\psi_L^\dagger, \psi_R^\dagger)\begin{pmatrix} & I \\ I & \end{pmatrix} = (\psi_R^\dagger, \psi_L^\dagger).$$

- The γ matrices

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} = \left\{\begin{pmatrix} & I_{2\times 2} \\ I_{2\times 2} & \end{pmatrix}, \begin{pmatrix} & \sigma_1 \\ -\sigma_1 & \end{pmatrix}, \begin{pmatrix} & \sigma_2 \\ -\sigma_2 & \end{pmatrix}, \begin{pmatrix} & \sigma_3 \\ -\sigma_3 & \end{pmatrix}\right\}$$

satisfy

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_{4\times 4}} \qquad \text{Clifford algebra in 4d}$$

Comments:

(i) $\{A, B\} = AB + BA$.

(ii) There are other representations (bases) of γ matrices which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$.

$\left(\text{e.g., Dirac rep. } \gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix}, \gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix}\right)$

The above rep. $\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}$ is called Weyl (chiral) rep..

(iii) We sometimes use a notation "Feynman slash":

$$\not{a} = \gamma^\mu a_\mu,$$

for a four vector a^μ. The Dirac Lagrangian is written as

$$\boxed{\mathcal{L} = \overline{\Psi}(i\not{\partial} - m)\Psi.}$$

(iv) A convenient identity

$$\begin{aligned} \not{a}\not{a} &= \gamma^\mu a_\mu \gamma^\nu a_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} a_\mu a_\nu \\ &= g^{\mu\nu} I \, a_\mu a_\nu = a^2 I \end{aligned}$$

► The Lorentz transformation of the 4-component Dirac field is given by

$$\begin{aligned} \Psi'(x') &= \begin{pmatrix} \psi'_L(x') \\ \psi'_R(x') \end{pmatrix} \\ &= \begin{pmatrix} D_L(\Lambda) & \\ & D_R(\Lambda) \end{pmatrix} \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \\ &\equiv \exp\left(\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right) \Psi(x) \end{aligned}$$

where, from $D_{L/R}(\Lambda)$ in § 3.5, the generators $S^{\mu\nu}$ are given by

$$\begin{cases} S^{0i} = \begin{pmatrix} \frac{i}{2}\sigma_i & \\ & -\frac{i}{2}\sigma_i \end{pmatrix} \\ S^{ij} = \begin{pmatrix} -\frac{1}{2}\epsilon_{ijk}\sigma_k & \\ & -\frac{1}{2}\epsilon_{ijk}\sigma_k \end{pmatrix} \end{cases} \qquad \text{(block-diagonal: reducible rep.)}$$

They can be written as

$$S^{\mu\nu} = \frac{-i}{4}[\gamma^\mu, \gamma^\nu]$$

and satisfy the commutation relation of the Lorentz algebra,

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\mu\rho} S^{\nu\sigma} - g^{\nu\rho} S^{\mu\sigma} - g^{\mu\sigma} S^{\nu\rho} + g^{\nu\sigma} S^{\mu\rho}\right).$$

► Note that $\Psi^\dagger$ and $\overline{\Psi}$ transform as

$$\begin{aligned} \Psi'^\dagger(x') &= \Psi^\dagger(x) \exp\left(-\frac{i}{2}\omega_{\mu\nu} S^{\dagger\mu\nu}\right). \\ \overline{\Psi}'(x') &= \Psi^\dagger(x) \exp\left(-\frac{i}{2}\omega_{\mu\nu} S^{\dagger\mu\nu}\right) \gamma^0 \\ &= \Psi^\dagger(x)\gamma^0 \exp\left(-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right) \qquad (\because S^{\dagger\mu\nu}\gamma^0 = \gamma^0 S^{\mu\nu}) \\ &= \overline{\Psi}(x)\left(-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right) \end{aligned}$$

Thus, $\Psi^\dagger\Psi$ is NOT Lorentz invariant (note that $S^{\dagger\mu\nu} \neq S^{\mu\nu}$), but $\overline{\Psi}\Psi$ is Lorentz invariant.

§ 4.2 Dirac equation and its solution

► From the Lagrangian $\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi$, the EOM (Euler-Lagrange eq.) is

$$\begin{aligned} 0 &= \partial_\mu \left(\frac{\delta}{\delta(\partial_\mu \Psi_a^\dagger)} \mathcal{L} \right) - \frac{\delta}{\delta \Psi_a^\dagger} \mathcal{L} \\ &= 0 - \left[\gamma^0 (i\partial\!\!\!/ - mI) \right]_{ab} \Psi_b . \end{aligned}$$

$$\therefore \boxed{(i\partial\!\!\!/ - m)\Psi(x) = 0 \qquad \textbf{Dirac equiation}}$$

<u>**Commnets**</u>

(i) The other Euler-Lagrange eq. $0 = \partial_\mu \left(\frac{\delta}{\delta(\partial_\mu \Psi_a)} \mathcal{L} \right) - \frac{\delta}{\delta \Psi_a} \mathcal{L}$ gives the same eq.

(ii) In terms of 2-component spinors, it is

$$\begin{pmatrix} -mI & i\sigma^\mu \partial_\mu \\ i\bar{\sigma}^\mu \partial_\mu & -mI \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

(The mass term mixes left- and right-handed spinors. For massless fermion, ψ_L and ψ_R are different particles.)

► Let's solve it. First of all, if $\Psi(x)$ is a solution of Dirac eq., then it also satisfies the Klein-Gordon eq.

$$\begin{aligned} 0 &= (-i\partial\!\!\!/ - mI)_{ab}(i\partial\!\!\!/ - mI)_{bc}\Psi_c \\ &= (\underbrace{\partial\!\!\!/\partial\!\!\!/}_{=\partial^\mu\partial_\mu I} + m^2 I)_{ac}\Psi_c \quad \text{(sign corrected after the lecture)} \\ &= (\Box + m^2)\Psi_a . \end{aligned}$$

► As in § 2.3, consider Fourier transform of $\Psi(x)$ with respect to $\vec{x}$,

$$\Psi_a(\vec{x}, t) = \int d^3p \, \widetilde{\Psi}_a(\vec{p}, t) e^{i\vec{p}\cdot\vec{x}}$$

Then

$$0 = (\Box + m^2)\Psi_a(x) = \int d^3p \left(\ddot{\widetilde{\Psi}}_a(\vec{p}, t) + \widetilde{\Psi}_a(\vec{p}, t) \underbrace{(\vec{p}^2 + m^2)}_{E_p^2} \right) e^{i\vec{p}\cdot\vec{x}} .$$

$$\begin{aligned} &\text{(inverse FT)} \to \quad \ddot{\widetilde{\Psi}}_a(\vec{p}, t) + E_p^2 \widetilde{\Psi}_a(\vec{p}, t) = 0 \\ &\therefore \; \widetilde{\Psi}_a(\vec{p}, t) = u_a(\vec{p}) e^{-iE_p t} + w_a(\vec{p}) e^{+iE_p t} . \qquad (u_a(\vec{p}), w_a(\vec{p}) : \text{4-component spinor}) \\ &\therefore \Psi_a(\vec{x}, t) = \int d^3p \, \left(u_a(\vec{p}) e^{-iE_p t} + w_a(\vec{p}) e^{+iE_p t} \right) e^{i\vec{p}\cdot\vec{x}} \\ &\qquad = \int d^3p \left(u_a(\vec{p}) e^{-ip\cdot x} + \underbrace{w_a(-\vec{p})}_{\equiv v_a(\vec{p})} e^{+ip\cdot x} \right)_{p^0 = E_p} \qquad \text{———— (1)} \end{aligned}$$

——————— on June 4, up to here. ———————

——————— June 11, from here. ———————

Where were we?

§ 4 Free Fermion

§ 4.1 $\mathcal{L}$

§ 4.2 Dirac eq.

$$\Psi_a(\vec{x},t) = \int d^3p \left(u_a(\vec{p}) e^{-ip\cdot x} + \underbrace{w_a(-\vec{p})}_{\equiv v_a(\vec{p})} e^{+ip\cdot x} \right)_{p^0=E_p} \qquad \text{———— (1)}$$

(← last week)

(today →)

- Eq. (1) satisfies the necessary condition $(\Box + m^2)\Psi_a(x) = 0$, but not sufficient. From Dirac eq.

$$\begin{aligned} 0 &= (i\not{\partial} - m)_{ab}\Psi_b(x) \\ &= \int d^3p \left((\not{p} - m)_{ab} u_b(\vec{p}) e^{-ip\cdot x} + (-\not{p} - m)_{ab} v_b(\vec{p}) e^{+ip\cdot x} \right)_{p^0=E_p} . \\ \text{(inverse FT)} \rightarrow \quad 0 &= (\not{p} - m)_{ab} u_b(\vec{p}) e^{-iE_pt} + (-\gamma^0 p_0 - \gamma^i(-p_i) - m)_{ab} v_b(-\vec{p}) e^{+iE_pt} \end{aligned}$$

This should be satisfied for any t. Thus,

$$\begin{cases} (\not{p} - m)_{ab} u_b(\vec{p}) = 0 \\ (-\not{p} - m)_{ab} v_b(\vec{p}) = 0 \end{cases} \qquad (p^0 = E_p),$$

i.e., $u(\vec{p})$ and $v(\vec{p})$ are eigenvectors of $\not{p}$ with eigenvalues m and $-m$ respectively.

$$\begin{aligned} \begin{pmatrix} & & \\ & \not{p} & \\ & & \end{pmatrix} \begin{pmatrix} \\ u(\vec{p}) \\ \\ \end{pmatrix} &= m \begin{pmatrix} \\ u(\vec{p}) \\ \\ \end{pmatrix}, \\ \begin{pmatrix} & & \\ & \not{p} & \\ & & \end{pmatrix} \begin{pmatrix} \\ v(\vec{p}) \\ \\ \end{pmatrix} &= -m \begin{pmatrix} \\ v(\vec{p}) \\ \\ \end{pmatrix}. \qquad \text{———— (2)} \end{aligned}$$

- In fact, the eigenvalues of the matrix $\not{p}$ are

$$\begin{aligned} \det(\not{p} - xI) &= \cdots = (x^2 - m^2)^2 \\ \rightarrow x &= m, m, -m, -m, \end{aligned}$$

corresponding to two independent $u(\vec{p})$ and two independent $v(\vec{p})$, satisfying (2).

► We can also think the "helicity" (= projection of the spin onto the direction of momentum):

$$h(p) = \frac{\vec{p}}{|\vec{p}|}\cdot\vec{S}, \qquad \text{where } \; S_i = -\frac{1}{2}\epsilon_{ijk}S^{ik}$$
$$= \frac{1}{2}\begin{pmatrix}\sigma_i & \\ & \sigma_i\end{pmatrix} = \begin{pmatrix}D_L(J_i) & \\ & D_R(J_i)\end{pmatrix}. \qquad \text{(See § 3.4, § 3.5, § 4.1.)}$$

whose eigenvalues are $\pm 1/2$. Since $[\not{p}, h(p)] = 0$, simultaneous eigenvectors of $\not{p}$ and $h(p)$ can be taken:

	$u_+(p)$	$u_-(p)$	$v_+(p)$	$v_-(p)$
$\not{p}$	m	m	$-m$	$-m$
$h(p)$	$1/2$	$-1/2$	$-1/2$	$1/2$

e.g., $\begin{cases}\not{p}u_+(p) = mu_+(p)\\ h(p)u_+(p) = \frac{1}{2}u_+(p)\end{cases}$

► The explicit form of $u_\pm(\vec{p})$ and $v_\pm(\vec{p})$ can be written as

$$u_\pm(p) = \begin{pmatrix}\sqrt{p^0 \mp |\vec{p}|}\;\eta_\pm\\ \sqrt{p^0 \pm |\vec{p}|}\;\eta_\pm\end{pmatrix}, \qquad v_\pm(p) = \begin{pmatrix}\sqrt{p^0 \pm |\vec{p}|}\;\epsilon\eta_\pm^*\\ -\sqrt{p^0 \mp |\vec{p}|}\;\epsilon\eta_\pm^*\end{pmatrix},$$

with

$$\begin{cases}\eta_+ = \dfrac{1}{\sqrt{2(1-\hat{p}^3)}}\begin{pmatrix}\hat{p}^1 - i\hat{p}^2\\ 1-\hat{p}^3\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi}\\ \sin\frac{\theta}{2}\end{pmatrix}\\ \eta_- = \dfrac{1}{\sqrt{2(1-\hat{p}^3)}}\begin{pmatrix}1-\hat{p}^3\\ -\hat{p}^1 - i\hat{p}^2\end{pmatrix} = \begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}e^{+i\phi}\end{pmatrix}\end{cases} \qquad \begin{cases}\hat{p}^1 = p^1/|\vec{p}| = \sin\theta\cos\phi\\ \hat{p}^2 = p^2/|\vec{p}| = \sin\theta\sin\phi\\ \hat{p}^3 = p^3/|\vec{p}| = \cos\theta\end{cases}$$

satisfying $(\vec{p}\cdot\vec{\sigma})\eta_\pm = \pm|\vec{p}|\eta_\pm, \quad \eta_s^\dagger\eta_{s'} = \delta_{ss'}$

Problem

(b-20) Show that the above $u_\pm(p)$ and $v_\pm(p)$ are indeed the eigenvectors of $\not{p}$ and $h(p)$.

(b-21) Show $\begin{cases}\bar{u}_s(p)u_{s'}(p) = 2m\delta_{ss'}\\ \bar{v}_s(p)v_{s'}(p) = -2m\delta_{ss'}\\ \bar{u}_s(p)v_{s'}(p) = 0\end{cases}$

(b-22) Show $\begin{cases}u_r^\dagger(\vec{p})u_s(\vec{p}) = 2E_p\delta_{rs}\\ v_r^\dagger(\vec{p})v_s(\vec{p}) = 2E_p\delta_{rs}\\ u_r^\dagger(\vec{p})v_s(-\vec{p}) = 0\\ v_r^\dagger(\vec{p})u_s(-\vec{p}) = 0\end{cases}$

(b-23) Show $\begin{cases}\sum_{s=\pm} u_{s,a}(p)\bar{u}_{s,b}(p) &= (\not{p}+m)_{ab}\\ \sum_{s=\pm} v_{s,a}(p)\bar{v}_{s,b}(p) &= (\not{p}-m)_{ab}\end{cases}.$

▶ To summarize, there are four independent solutions to the Dirac equation,

$$\Psi(x) = \int d^3p \left(u_+(p)e^{-ip\cdot x},\ u_-(p)e^{-ip\cdot x},\ v_+(p)e^{+ip\cdot x},\ v_-(p)e^{+ip\cdot x}\right)_{p^0=E_p}.$$

▶ The following is a technical comment for those who read Peskin [2]. You can check

$$p\cdot\sigma \equiv p^0 I - \vec{p}\cdot\vec{\sigma} = U\begin{pmatrix} p^0-|\vec{p}| & 0 \\ 0 & p^0+|\vec{p}| \end{pmatrix}U^\dagger, \qquad U\equiv(\eta_+,\eta_-)$$

Thus, the matrix $\sqrt{p\cdot\sigma}$ in [2] can be explicitly written as

$$\sqrt{p\cdot\sigma} \equiv U\begin{pmatrix} \sqrt{p^0-|\vec{p}|} & 0 \\ 0 & \sqrt{p^0+|\vec{p}|} \end{pmatrix}U^\dagger$$

$$\text{then,}\quad \sqrt{p\cdot\sigma}\ \eta_\pm = \sqrt{p^0\mp|\vec{p}|}\ \eta_\pm$$

Similar expression holds for $p\cdot\bar{\sigma}$ and $\sqrt{p\cdot\bar{\sigma}}$ as well.

§ 4.3 Quantization of Dirac field

$$\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi.$$

$$\Psi_a \overset{\text{conjugate}}{\longleftrightarrow} \Pi_{\Psi a} = \frac{\delta\mathcal{L}}{\delta\dot{\Psi}_a} = (\overline{\Psi}i\gamma^0)_a = i\Psi_a^\dagger\ (= i\Psi_a^*)$$

Comments

(i) $\Psi_a \longleftrightarrow \Pi_{\Psi a} = i\Psi_a^*$

but then, $\Psi_a^* \longleftrightarrow\ ?\qquad \left(\Pi_{\Psi^* a} = \dfrac{\delta\mathcal{L}}{\delta\dot{\Psi}_a^*} = 0\ ???\right)$

One should do the quantization of constrained sysytem with "Dirac bracket".

In general, if $\det\left(\dfrac{\partial^2\mathcal{L}(q,\dot{q})}{\partial\dot{q}_i\partial\dot{q}_j}\right) = 0$, p_i and q_i are not independent.

In such a case,

$$\begin{tikzcd} \text{Poisson bracket} \arrow[rr, "\times" marking] \arrow[dr] & & \text{quantization} \\ & \text{Dirac bracket} \arrow[ur] & \end{tikzcd}$$

Here, we skip the details and do naive quantization with Ψ_a and $\Pi_{\Psi a}$.

(ii) When Ψ_a and $\Pi_{\Psi a}$ are anti-commuting, right-derivative and left-derivative gives opposite sign. Here, $\Pi_{\Psi a}$ is defined with right-derivative.

(If A and B are anti-commuting, $(BA)\dfrac{\overleftarrow{\partial}}{\partial A} = B$, while $\dfrac{\overrightarrow{\partial}}{\partial A}(BA) = -B$.

§ 4.3.1 Quantization with Commutation relation vs Anti-commutation relation

- Quantization with equal-time commutation relation

$$[\Psi_a(x), \Pi_{\Psi b}(y)]_{x^0=y^0} = [\Psi_a(x), i\Psi_b^\dagger(y)]_{x^0=y^0} = i\delta^{(3)}(\vec{x}-\vec{y})\delta_{ab}$$

does NOT work. Instead, quantization with equal-time anti-commutation relation

$$\{\Psi_a(x), \Pi_{\Psi b}(y)\}_{x^0=y^0} = \{\Psi_a(x), i\Psi_b^\dagger(y)\}_{x^0=y^0} = i\delta^{(3)}(\vec{x}-\vec{y})\delta_{ab}$$

works. Let's see it.

- First of all, expand the $\Psi(x)$ with the 4 independent solutions of Dirac eq.

$$u_\pm(p)e^{-ip\cdot x}, \quad v_\pm(p)e^{+ip\cdot x}$$

$$\text{as} \quad \Psi_a(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=\pm} \left(a_s(\vec{p})u_{s,a}(p)e^{-ip\cdot x} + d_s(\vec{p})v_{s,a}(p)e^{+ip\cdot x}\right)_{p^0=E_p}.$$

$$\text{then} \quad \Psi_a^\dagger(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=\pm} \left(a_s^\dagger(\vec{p})u_{s,a}^\dagger(p)e^{+ip\cdot x} + d_s^\dagger(\vec{p})v_{s,a}^\dagger(p)e^{-ip\cdot x}\right)_{p^0=E_p}.$$

Here, $\Psi(x)$, $a_s(\vec{p})$, and $d_s(\vec{p})$ are the quantum operators. At this moment $a_s(\vec{p})$ and $d_s(\vec{p})$ are just expansion coefficients.

- The following can be shown:

$$\begin{cases} [\Psi_a(x), \Psi_b^\dagger(y)]_{x^0=y^0} = \delta^{(3)}(\vec{x}-\vec{y})\delta_{ab} \\ \text{others} = 0 \end{cases} \quad \text{——— (1)'}$$

$$\Updownarrow$$

$$\begin{cases} [a_r(\vec{p}), a_s^\dagger(\vec{q})] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs} \\ [d_r(\vec{p}), d_s^\dagger(\vec{q})] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs} \\ \text{others} = 0 \end{cases} \quad \text{——— (2)'} \dashrightarrow \text{(problematic)}$$

$$\begin{cases} \{\Psi_a(x), \Psi_b^\dagger(y)\}_{x^0=y^0} = \delta^{(3)}(\vec{x}-\vec{y})\delta_{ab} \\ \text{others} = 0 \end{cases} \quad \text{——— (1)}$$

$$\Updownarrow$$

$$\begin{cases} \{a_r(\vec{p}), a_s^\dagger(\vec{q})\} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs} \\ \{d_r(\vec{p}), d_s^\dagger(\vec{q})\} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs} \\ \text{others} = 0 \end{cases} \quad \text{——— (2)} \dashrightarrow \text{(OK)}$$

▶ Let's first show (2)' $\Longrightarrow$ (1)' and (2) $\Longrightarrow$ (1). Hereafter, we use a notation

$$[A, B\} = \begin{cases} [A, B] = AB - BA \\ \{A, B\} = AB + BA \end{cases}$$

to discuss the two cases simultaneously.

▶ First of all, from (2)' (2), $[a, a\} = [d, d\} = [a, d\} = 0$, and therefore $[\Psi, \Psi\} = 0$. Similarly, $[\Psi^\dagger, \Psi^\dagger\} = 0$.

▶ The remaining is $[\Psi, \Psi^\dagger\}$, and

$$\begin{aligned}
[\Psi_a(t, \vec{x}), \Psi_b^\dagger(t, \vec{y})\} &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}} \\
&\times \sum_{r=\pm}\sum_{s=\pm} \Big([a_r(\vec{p}), a_s^\dagger(\vec{q})\}\ u_{a,r}(p) u_{b,s}^\dagger(q) e^{-ip\cdot x} e^{iq\cdot y} \\
&\qquad\qquad + [d_r(\vec{p}), d_s^\dagger(\vec{q})\}\ v_{a,r}(p) v_{b,s}^\dagger(q) e^{ip\cdot x} e^{-iq\cdot y} \Big)_{x^0=y^0=t} \\
&= \cdots \\
&= \delta^{(3)}(\vec{x} - \vec{y})\delta_{ab} \qquad \blacksquare
\end{aligned}$$

Problem

(b-24) Show it, by using (2)(2)' and (b-23).

▶ Thus, (2)' $\Longrightarrow$ (1)' and (2) $\Longrightarrow$ (1).

▶ (1)' $\Longrightarrow$ (2)' and (1) $\Longrightarrow$ (2) can also be shown.

Problem

(b-25) Using (b-22), show that

$$\begin{cases} a_s(\vec{p}) = \dfrac{1}{\sqrt{2E_p}} \displaystyle\sum_a u_{s,a}^\dagger(\vec{p}) \int d^3x e^{ip\cdot x} \Psi_a(x) \Big|_{p^0=E_p} \\ d_s(\vec{p}) = \dfrac{1}{\sqrt{2E_p}} \displaystyle\sum_a v_{s,a}^\dagger(\vec{p}) \int d^3x e^{-ip\cdot x} \Psi_a(x) \Big|_{p^0=E_p} \end{cases}$$

(One can also show that the RHS is independent of x^0.)

(b-26) Using (b-22) and (b-25), show (1)' $\Longrightarrow$ (2)' and (1) $\Longrightarrow$ (2).

▶ Therefore, (1)' $\Longleftrightarrow$ (2)' and (1) $\Longleftrightarrow$ (2).

► On the other hand, the Hamiltonian is given by

$$H = \int d^3x \left(\Pi_\Psi \dot{\Psi} - \mathcal{L}\right)$$

(note that Π_Ψ is defined with right-derivative, and hence $H\frac{\overleftarrow{\partial}}{\partial\dot{\Psi}} = 0$.)

$$= \int d^3x \Big(i\Psi^\dagger\dot{\Psi} - \overline{\Psi}\underbrace{(i\partial\!\!\!/ - m)\Psi}_{=0}\Big)$$

$$= \cdots \text{(using (b-22))}$$

$$= \underline{\int \frac{d^3p}{(2\pi)^3} E_p \sum_{s=\pm}\Big(a_s^\dagger(\vec{p})a_s(\vec{p}) - d_s^\dagger(\vec{p})d_s(\vec{p})\Big)} \quad \text{———— (3).}$$

► From (3) and (2)'(2), one can show:

$$\begin{cases} [H, a_s^\dagger(\vec{p})] = E_p a_s^\dagger(\vec{p}) \\ [H, a_s(\vec{p})] = -E_p a_s(\vec{p}) \\ [H, d_s^\dagger(\vec{p})] = -E_p d_s^\dagger(\vec{p}) \\ [H, d_s(\vec{p})] = E_p d_s(\vec{p}) \end{cases} \quad \text{———— (4).}$$

Note that (3) and (4) are true for each of the $[\bullet, \bullet]$ quantization and $\{\bullet, \bullet\}$ quantization.

———— on June 11, up to here. ————

———— June 18, from here. ————

Where were we?

§ 4 Free Fermion

§ 4.3 $[\bullet, \bullet]$ vs $\{\bullet, \bullet\}$.

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_{s=\pm}\Big(a_s^\dagger(\vec{p})a_s(\vec{p}) - d_s^\dagger(\vec{p})d_s(\vec{p})\Big) \quad \text{———— (3).}$$

$$\begin{cases} [H, a_s^\dagger(\vec{p})] = E_p a_s^\dagger(\vec{p}) \\ [H, a_s(\vec{p})] = -E_p a_s(\vec{p}) \\ [H, d_s^\dagger(\vec{p})] = -E_p d_s^\dagger(\vec{p}) \\ [H, d_s(\vec{p})] = E_p d_s(\vec{p}) \end{cases} \quad \text{———— (4).}$$

(← last week)

(today →)

► **[wrong quantization →]** Now, if we would quantize with $[\bullet, \bullet]$, (1)'$\Longleftrightarrow$(2)', then $d_s^\dagger(\vec{p})$ would decrease energy.

$$H\Big(d_s^\dagger(\vec{p})\,|X\rangle\Big) = \Big(d_s^\dagger(\vec{p})H + [H, d_s^\dagger(\vec{p})]\Big)\,|X\rangle$$
$$= (E_X - E_p)\Big(d_s^\dagger(\vec{p})\,|X\rangle\Big),$$

and one could construct a state with infinitely negative energy.

$$H\left(d_1^\dagger d_2^\dagger \cdots |X\rangle\right) = \underbrace{(E_X - E_1 - E_2 - \cdots)}_{\to -\infty}\left(d_1^\dagger d_2^\dagger \cdots |X\rangle\right).$$

Note that, we cannot change the roles of d and $d^\dagger$, because

$$\left[d(\vec{p}), d^\dagger(\vec{q})\right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}),$$

fixes that $d^\dagger$ (d) is the creation (annihilation):

$$\begin{aligned} & d(\vec{p})d^\dagger(\vec{p}) - d^\dagger(\vec{p})d(\vec{p}) = (2\pi)^3 \delta^{(3)}(0), \\ \therefore \quad & \left|\left|d^\dagger(\vec{p})\,|X\rangle\right|\right|^2 - \left|\left|d(\vec{p})\,|X\rangle\right|\right|^2 = (2\pi)^3 \delta^{(3)}(0)\langle X|X\rangle \geq 0. \end{aligned}$$

(If we would define $\widetilde{d} = d^\dagger$, $\widetilde{d}^\dagger = d$, and define the vacuum by $\widetilde{d}\,|0\rangle = 0$, then $-||\widetilde{d}^\dagger(\vec{p})\,|X\rangle\,||^2 \geq 0$, inconsistent!) $\leftarrow$ **[wrong quantization]**

- On the other hand, if we quantize with $\{\bullet, \bullet\}$, (1)$\Longleftrightarrow$(2), we still have

$$\begin{cases} [H, d_s^\dagger(\vec{p})] = -E_p d_s^\dagger(\vec{p}) \\ [H, d_s(\vec{p})] = E_p d_s(\vec{p}) \end{cases}$$

but now we can exchange the roles of creation and annihilation operator.

$$\begin{aligned} b^\dagger(\vec{p}) &\equiv d(\vec{p}) \\ b(\vec{p}) &\equiv d^\dagger(\vec{p}) \end{aligned}$$

because

$$\begin{aligned} & \left\{d(\vec{p}), d^\dagger(\vec{q})\right\} = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \\ &= dd^\dagger + d^\dagger d \\ &= b^\dagger b + bb^\dagger \\ &= \left\{b(\vec{q}), b^\dagger(\vec{p})\right\} \end{aligned}$$

and also we can define the vacuum state by $b\,|0\rangle = 0$.

Comment

$b(\vec{p})\,|0\rangle = 0$ means that, in terms of original d and $d^\dagger$, $d^\dagger(\vec{p})\,|0\rangle = 0$.
In terms of the original "vacuum" $|0_d\rangle$ with $d(\vec{p})\,|0_d\rangle = 0$, the vacuum $|0\rangle$ can be understood as

$$|0\rangle \propto \prod_{\text{all } \vec{p}} d^\dagger(\vec{p})\,|0_d\rangle\,,$$

which leads to $d^\dagger(\vec{p})\,|0\rangle = 0$ because $d^\dagger(\vec{p})^2 = 0$. This is related to the idea of the "Dirac sea".

- The Hamiltonian then becomes

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_{s=\pm} \Big(a_s^\dagger(\vec{p}) a_s(\vec{p}) \underbrace{-d_s^\dagger(\vec{p}) d_s(\vec{p})}_{\substack{= -b_s(\vec{p}) b_s^\dagger(\vec{p}) \\ = +b_s^\dagger(\vec{p}) b_s(\vec{p}) - (2\pi)^3 \delta^{(3)}(0)}} \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} E_p \sum_{s=\pm} \Big(a_s^\dagger(\vec{p}) a_s(\vec{p}) + b_s^\dagger(\vec{p}) b_s(\vec{p}) \Big) - \int d^3p \; E_p \delta^{(3)}(0)$$

We neglect the (infinite) constant term, as in the scalar case.

- To summarize, quantization with anti-commutation works, and we have

$$\Psi_a(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \sum_{s=\pm} \left(a_s(\vec{p}) u_{s,a}(p) e^{-ip\cdot x} + b_s^\dagger(\vec{p}) v_{s,a}(p) e^{+ip\cdot x} \right)_{p^0=E_p}$$

$$\begin{cases} \{\Psi_a(x), \Psi_b^\dagger(y)\}_{x^0=y^0} & = \delta^{(3)}(\vec{x}-\vec{y})\delta_{ab} \\ \text{others} & = 0 \end{cases} \iff \begin{cases} \{a_r(\vec{p}), a_s^\dagger(\vec{q})\} & = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\delta_{rs} \\ \{b_r(\vec{p}), b_s^\dagger(\vec{q})\} & = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\delta_{rs} \\ \text{others} & = 0 \end{cases}$$

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_{s=\pm} \Big(a_s^\dagger(\vec{p}) a_s(\vec{p}) + b_s^\dagger(\vec{p}) b_s(\vec{p}) \Big)$$

The anti-commutation relation implies Fermi-Dirac statistics;

$$a_r^\dagger(\vec{p}) a_s^\dagger(\vec{q}) = -a_s^\dagger(\vec{q}) a_r^\dagger(\vec{p}), \quad \text{in particular} \quad \Big(a_r^\dagger(\vec{p}) \Big)^2 = 0 \quad \boxed{\text{Pauli blocking}}$$

§ 4.3.2 Particle and Anti-particle

U(1) Symmetry: $\Psi \to \Psi e^{i\alpha}$

$\to$ Current: $j^\mu = \overline{\Psi} \gamma^\mu \Psi$

$$\to \text{Charge: } Q = \int d^3x j^0$$

$$= \cdots$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a_s^\dagger(\vec{p}) a(\vec{p}) - b_s^\dagger(\vec{p}) b(\vec{p}) \right) \quad (+\text{constant})$$

and hence

$$\begin{cases} [Q, a_s^\dagger(\vec{p})] = +a_s^\dagger(\vec{p}) \\ [Q, b_s^\dagger(\vec{p})] = -b_s^\dagger(\vec{p}) \end{cases}$$

Namely,

- $a_s^\dagger(\vec{p})$ increases the charge by one. (Particle creation)
- $b_s^\dagger(\vec{p})$ decreases the charge by one. (Anti-particle creation)

§ 4.3.3 One particle states

$$\begin{aligned}
|\psi;\vec{p},r\rangle &= \sqrt{2E_p}a_r^\dagger(\vec{p})\,|0\rangle \text{ : particle} \\
\left|\bar{\psi};\vec{p},r\right\rangle &= \sqrt{2E_p}b_r^\dagger(\vec{p})\,|0\rangle \text{ : anti-particle}
\end{aligned}$$

Normalization:

$$\begin{aligned}
\langle\psi;\vec{p},r|\psi;\vec{q},s\rangle &= \sqrt{2E_p}\sqrt{2E_q}\langle 0|a_r(\vec{p})a_s^\dagger(\vec{q})\rangle 0 \\
&= \sqrt{2E_p}\sqrt{2E_q}\langle 0|\ \underbrace{\{a_r(\vec{p})a_s^\dagger(\vec{q})\}}_{(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs}}\ -a_s^\dagger(\vec{p})a_r(\vec{p})\rangle 0 \\
&= (2\pi)^3 2E_p\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs}. \\
\text{Similarly } \langle\bar{\psi};\vec{p},r|\bar{\psi};\vec{q},s\rangle &= (2\pi)^3 2E_p\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs}.
\end{aligned}$$

§ 5 Interacting Scalar Field

§ 5.1 Outline: what we will learn

1 Lagrangian → quantization

2 ↓ • operator • path integral (2 ways)

• $\langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle$

3 ↓ • LSZ reduction

• S-matrix, amplitude $\mathcal{M}$

4 ↓

• observables (cross section σ and decay rate Γ)

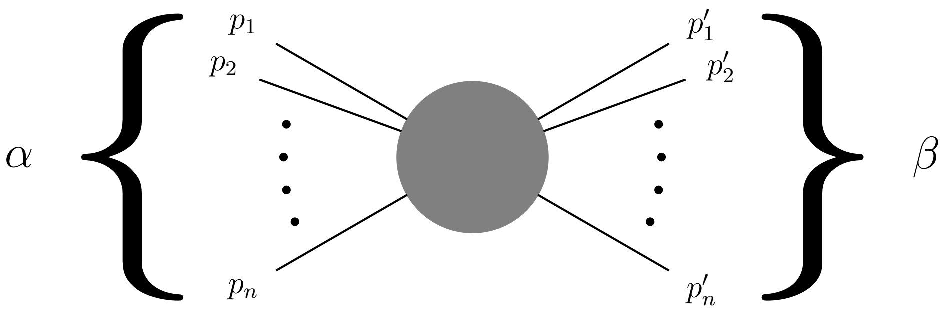
short cut

$\boxed{\textbf{Feynman rule}}$

A long way to go,... let's start from 4.

§ 5.2 S-matrix, amplitude $\mathcal{M} \Longrightarrow$ observables (σ and Γ)

Let's consider the probability of the following process, $P(\alpha \to \beta)$.



► If the initial and final states are normalized as $\langle\alpha|\beta\rangle = \delta_{\alpha\beta}$, then

$$P(\alpha \to \beta) = |\langle\beta, \mathrm{out}|\alpha, \mathrm{in}\rangle|^2$$

(The meaning of "in" and "out" will be explained later.)

- ▶ However, we are interested in states like

$$|\alpha\rangle = |h_1, \vec{p}_1, h_2, \vec{p}_2, \cdots h_n, \vec{p}_n\rangle \ ,$$

(h_i ; helicities and other quantum numbers of the particle i)

with a normalization (see § 2.7 and § 4.3.3)

$$\langle h, \vec{p}|h', \vec{q}\rangle = (2\pi)^3 2E_p \delta^{(3)}(\vec{p}-\vec{q})\delta_{hh'} \ . \qquad \text{———}(1)$$

§ 5.2.1 S-matrix

- ▶ The transition amplitude

$$\boxed{\langle h'_1, \vec{p'}_1, \cdots h'_m, \vec{p'}_m \,; \text{out}|h_1, \vec{p}_1, \cdots h_n, \vec{p}_n \,; \text{in}\rangle = \langle h'_1, \vec{p'}_1, \cdots h'_m, \vec{p'}_m|S|h_1, \vec{p}_1, \cdots h_n, \vec{p}_n\rangle}$$

is called S-matrix.

Comments

(i) The definition of in and out-states will be given later. (→§ 5.5.)

(ii) S-matrix is Lorentz invariant. (→§ 5.5.)

(iii) In the following, we omit the label h_i and h'_i.

§ 5.2.2 invariant matrix element, or scattering amplitude, $\mathcal{M}$

- ▶ As long as the total energy and momentum are conserved,

$$\begin{aligned}\langle \vec{p'}_1 \cdots \vec{p'}_m|S|\vec{p}_1 \cdots \vec{p}_n\rangle \ \propto \ & \delta(\underbrace{\sum_f E'_f}_{\text{final}} - \underbrace{\sum_i E'_i}_{\text{initial}}) \times \delta^{(3)}(\sum_f \vec{p'}_f - \sum_i \vec{p'}_i) \\ = \ & \delta^{(4)}(\sum_f p'_f - \sum_i p_i)\end{aligned}$$

We define the invariant matrix element, or scattering amplitude, $\mathcal{M}$ as

$$\boxed{\langle \vec{p'}_1 \cdots \vec{p'}_m|S|\vec{p}_1 \cdots \vec{p}_n\rangle = (2\pi)^4 \delta^{(4)}(\sum_f p'_f - \sum_i p_i) \cdot i\mathcal{M}(\vec{p}_1 \cdots \vec{p}_n \to \vec{p'}_1 \cdots \vec{p'}_m) \qquad \text{———}(2)}$$

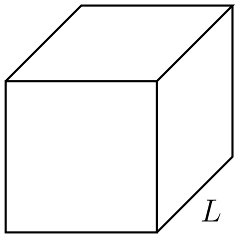
Comments

(i) Since the S-matrix is Lorentz invariant, the amplitude $\mathcal{M}$ is also Lorentz invariant.

(ii) The amplitude $\mathcal{M}$ can be calculated by the Feynman rule. (→ goal of § 5.)

§ 5.2.3 transition probability: general case

► Let's normalize the system with a box.



$$V = L^3, \qquad \vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

Then, from (1),

$$\begin{aligned}\langle \vec{q}|\vec{p}\rangle &= 2E_p(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \\ &= 2E_p\int d^3x e^{i(\vec{p}-\vec{q})\cdot\vec{x}} \\ &= 2E_p V \underbrace{\delta_{\vec{p},\vec{q}}}_{\text{discrete}} . \text{———————}(3)\end{aligned}$$

Define

$$|\vec{p}\rangle_{\text{Box}} = \frac{1}{\sqrt{2E_pV}}|\vec{p}\rangle . \text{———————}(4)$$

Then, from (3)(4),

$${}_{\text{Box}}\langle\vec{q}|\vec{p}\rangle_{\text{Box}} = \delta_{\vec{p},\vec{q}}.$$

Therefore, $|\vec{p}\rangle_{\text{Box}}$ has the correct normalization. For instance, if there is no interaction,

$$P(\vec{p}\to\vec{p'}) = \left|{}_{\text{Box}}\langle\vec{p'}|\vec{p}\rangle_{\text{Box}}\right|^2 = \delta_{\vec{p'},\vec{p}} = \begin{cases}1 & (\vec{p'}=\vec{p})\\ 0 & (\vec{p'}\neq\vec{p})\end{cases}$$

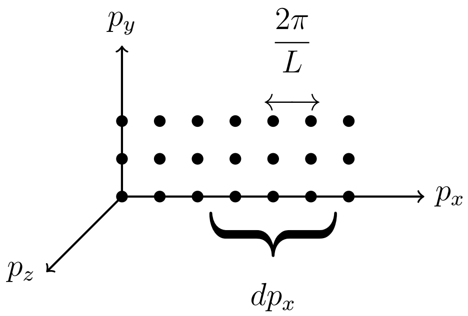
► Thus,

$$\begin{aligned}&\text{Probability } P(\vec{p}_1\cdots\vec{p}_n\to\vec{p'}_1\cdots\vec{p'}_m) \\ &= \left|{}_{\text{Box}}\langle\vec{p'}_1\cdots\vec{p'}_m|S|\vec{p}_1\cdots\vec{p}_n\rangle_{\text{Box}}\right|^2 \\ &= \left(\prod_{f=1}^{m}\frac{1}{2E'_f}\right)\left(\prod_{i=1}^{n}\frac{1}{2E_i}\right)\left(\frac{1}{V}\right)^{n+m}\left|\langle\vec{p'}_1\cdots\vec{p'}_m|S|\vec{p}_1\cdots\vec{p}_n\rangle\right|^2 \quad \because (4)\end{aligned}$$

———————(5)

► But this becomes zero for $V\to\infty$.
<u>What is the (differential) probability that the final state is within $[\vec{p'}_f, \vec{p'}_f + d\vec{p'}_f]$?</u>

$$dP = P(\vec{p}_1\cdots\vec{p}_n\to\vec{p'}_1\cdots\vec{p'}_m)\times\underbrace{dN}_{\text{number of states within }[\vec{p'}_f,\vec{p'}_f+d\vec{p'}_f]}$$



$$dN = \frac{d^3p}{(2\pi/L)^3} = \frac{d^3p}{(2\pi)^3} \cdot V$$

For the m particle final states, $\vec{p'}_1 \cdots \vec{p'}_m$,

$$dN = \prod_{f=1}^{m} \left(\frac{d^3p'_f}{(2\pi)^3} \cdot V \right) \text{———————(6)}$$

From (5) and (6),

$$\begin{aligned} dP &= P(\vec{p}_1 \cdots \vec{p}_n \to \vec{p'}_1 \cdots \vec{p'}_m) \times dN \\ &= \left(\prod_{f=1}^{m} \frac{d^3p'_f}{(2\pi)^3} \frac{1}{2E'_f} \right) \left(\prod_{i=1}^{n} \frac{1}{2E_i} \right) \left(\frac{1}{V} \right)^n \left| \langle \vec{p'}_1 \cdots \vec{p'}_m | S | \vec{p}_1 \cdots \vec{p}_n \rangle \right|^2 \text{———————(7)} \end{aligned}$$

From (2), $\left| \langle \vec{p'}_1 \cdots \vec{p'}_m | S | \vec{p}_1 \cdots \vec{p}_n \rangle \right|^2$

$$\begin{aligned} &= (2\pi)^4 \delta^{(4)}(\sum p'_f - \sum p'_i) \cdot (2\pi)^4 \delta^{(4)}(\sum p'_f - \sum p'_i) \cdot |\mathcal{M}|^2 \\ &= (2\pi)^4 \delta^{(4)}(\sum p'_f - \sum p'_i) \cdot (2\pi)^4 \delta^{(4)}(0) \cdot |\mathcal{M}|^2 \\ &\quad \left(\delta^{(4)}(0) = \int \frac{d^4x}{(2\pi)^4} e^{i0 \cdot x} = \frac{V \cdot T}{(2\pi)^4} \quad T : \text{time } (\to \infty) \right) \\ &= (2\pi)^4 \delta^{(4)}(\sum p'_f - \sum p'_i) \cdot V \cdot T \cdot |\mathcal{M}|^2 \end{aligned}$$

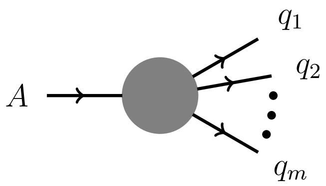
- Thus, dividing Eq.(7) by T, we obtain the differential transition rate

$$\frac{dP}{T} = V^{1-n} \left(\prod_{i=1}^{n} \frac{1}{2E_i} \right) \underbrace{\left(\prod_{f=1}^{m} \frac{d^3p'_f}{(2\pi)^3} \frac{1}{2E'_f} \right) (2\pi)^4 \delta^{(4)}(\sum p'_f - \sum p'_i)}_{\equiv d\Phi_m} \cdot |\mathcal{M}(\vec{p}_1 \cdots \vec{p}_n \to \vec{p'}_1 \cdots \vec{p'}_m)|^2$$

———————(8)

- Now let's discuss the cases $n = 1$ and $n = 2$.

§ 5.2.4 $n = 1$, decay rate

- 

particle decay

From Eq.(8), the probability that the particle A decays into the range of final states $[\vec{p'}_f, \vec{p'}_f + d\vec{p'}_f]$ per unit time is

$$\frac{dP(p_A \to q_1 \cdots q_m)}{T} = \frac{1}{2E_A} d\Phi_m |\mathcal{M}(p_A \to q_1 \cdots q_m)|^2$$

Integrating over the final momenta, we have

Decay Rate

$$\begin{aligned}
&\Gamma(A \to 1, 2, \cdots) \\
&= \frac{1}{2m_A} \int d\Phi_m |\mathcal{M}(p_A \to q_1 \cdots q_m)|^2 \\
&= \underbrace{\frac{1}{2m_A}}_{\text{at rest frame}} \prod_{f=1}^{m} \int \frac{d^3 q_f}{(2\pi)^3 2E_f} (2\pi)^4 \delta^{(4)}(p_A - \sum_f q_f) |\mathcal{M}(p_A \to q_1 \cdots q_m)|^2
\end{aligned}$$

$(\times$ symmetry factor)

——— on June 18, up to here. ———

——— June 25, from here. ———

Where were we?

$\mathcal{L} \to$ quantization

$\downarrow$

• $\langle 0|T[\phi \cdots \phi]|0\rangle$

$\downarrow$ • LSZ

• S-matrix, amplitude $\mathcal{M}$

We are here. $\downarrow$ **§ 5.2**

• observables (σ and Γ)

Feynman rule

Comments

(i) The mass dimension of Γ is $(\text{energy})^{+1} \sim (\text{time})^{-1}$.

Problem

(b-27) Show it.

(ii) If there is more than one decay modes, their sum

$$\Gamma(A \to \text{all}) = \Gamma(A \to 1, 2 \cdots) + \Gamma(A \to 1', 2' \cdots) + \cdots$$

is called the <u>total decay rate</u>, and its inverse $\tau_A = \dfrac{1}{\Gamma(A \to \text{all})}$ gives the lifetime of particle A.

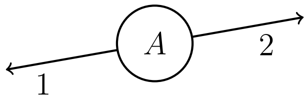
(iii) If not in the rest frame, $E_A = \gamma m_A$, and

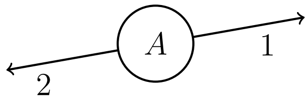
$$\Gamma = \frac{1}{2E_A} \underbrace{\int d\Phi_m |\mathcal{M}|^2}_{\text{Lorentz inv.}} = \frac{1}{\gamma}\Gamma_{\text{rest}} \quad \Longrightarrow \quad \tau = \gamma \tau_{\text{rest}}$$

The lifetime becomes longer by a factor of γ. (consistent with the Special Relativity!)

(iv) If there are identical particles in the final state, one should divide by a symmetry factor.

(Example) If particles 1 and 2 are identical,



and


are indistinguishable.

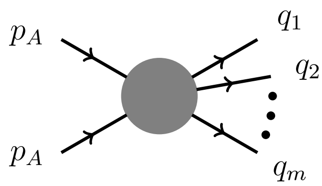
Thus, we should

① reduce the integration range ($\theta = [0, \pi] \to [0, \pi/2]$),

or

② divide by a symmetry factor ($= 2$) after integration.

§ 5.2.5 $n = 2$, cross section

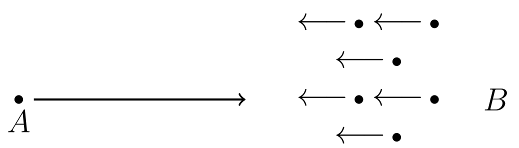


particle scattering

► From Eq.(8), the probability that the final particles are in the range of $[\vec{q}_f, \vec{q}_f + d\vec{q}_f]$ per unit time is

$$\frac{dP(p_A, p_B \to q_1 \cdots q_m)}{T} = \frac{1}{V}\frac{1}{2E_A \cdot 2E_B} d\Phi_m |\mathcal{M}(p_A \to q_1 \cdots q_m)|^2 \qquad \text{———(9)}$$

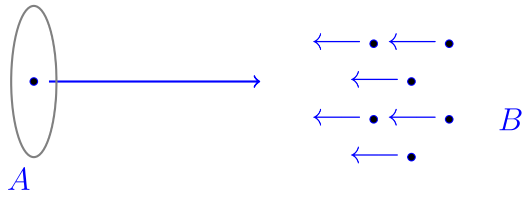
► In this case, we consider a quantity called "scattering cross section" (or just cross section). Suppose that a particle A collides with a bunch of particles B (with number density n_B) with a relative velocity v_{rel}.



The probability that the scattering $A, B \to 1, 2 \cdots$ occurs per unit time is given by

$$\frac{P(p_A, p_B \to 1, 2 \cdots)}{T} = n_B \cdot v_{\text{rel}} \cdot \underbrace{\sigma(p_A, p_B \to 1, 2 \cdots)}_{\text{cross section}} \qquad \text{———————}(10)$$

Why "cross section"?



If we think a disk with an area σ, the number of B particles which goes through this disk within time T is given by

$$N_B = \sigma \cdot v_{\text{rel}} \cdot T \cdot n_B.$$

This is consistent with (10). (For small T, $N_B < 1$ and it gives the probability.)

► In the situation of Eq.(9), there is only one B particle, so $n_B = 1/V$. Thus, the differential cross section that the final state goes within $[\vec{p'}_f, \vec{p'}_f + d\vec{p'}_f]$ is

$$\begin{aligned} d\sigma(p_A, p_B \to 1, 2 \cdots) &= \frac{1}{v_{\text{rel}}} V \frac{dP(p_A, p_B \to 1, 2 \cdots)}{T} \quad [\because (10)] \\ &= \frac{1}{v_{\text{rel}}} \frac{1}{2E_A \cdot 2E_B} d\Phi_m |\mathcal{M}(p_A, p_B \to q_1 \cdots q_m)|^2 \quad [\because (9)] \end{aligned}$$

Integrating over the final momenta,

Cross Section

$$\begin{aligned} &\sigma(p_A, p_B \to 1, 2 \cdots) \\ &= \frac{1}{2E_A \cdot 2E_B \cdot v_{\text{rel}}} \int d\Phi_m |\mathcal{M}(p_A \to q_1 \cdots q_m)|^2 \\ &= \frac{1}{2E_A \cdot 2E_B \cdot v_{\text{rel}}} \prod_{f=1}^{m} \int \frac{d^3 q_f}{(2\pi)^3 2E_f} (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum_f q_f) |\mathcal{M}(p_A, p_B \to q_1 \cdots)|^2 \\ &\qquad (\times \text{ symmetry factor}) \end{aligned}$$

<u>Comments</u>

(i) The mass dimension of σ is $(\text{energy})^{-2} \sim (\text{length})^2 \sim (\text{area})$.

(ii) If there are identical particles, divide by the symmetry factor (same as Γ).

(iii) The relative velocity $v_{\rm rel}$ is given by

$$v_{\rm rel} = \left| \frac{\vec{p}_A}{E_A} - \frac{\vec{p}_B}{E_B} \right|$$

(For a head-on collision with speeds of light, $v_{\rm rel} = 2$.)

(iv) $E_A E_B v_{\rm rel} = |E_B \vec{p}_A - E_A \vec{p}_B|$ is not Lorentz inv., and therefore the above σ in not Lorentz inv. either.
(Lorentz inv. cross section can be defined by replacing as $E_A E_B v_{\rm rel} \to \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}$.)

Problem

(b-28) Show that $\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} = E_A E_B v_{\rm rel}$ for $\vec{p}_A \parallel \vec{p}_B$.

§ 5.3 Interacting Scalar Field: Lagrangian and Quantization

► We consider the so-called ϕ^4-theory.

$$L = \int d^3x \Big(\underbrace{\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2}_{\text{same as free theory}} \underbrace{-\frac{\lambda}{24}\phi^4}_{\text{Interaction}} \Big)$$

(λ : real and positive constant)

$$\begin{aligned} \phi(\vec{x}, t) &\longleftrightarrow \pi(\vec{x}, t) \\ &= \frac{\delta L}{\delta \dot{\phi}(\vec{x}, t)} = \dot{\phi}(\vec{x}, t) \qquad \text{(same as free theory)} \end{aligned}$$

Equal-Time Commutation Relation (ETCR)

$$\begin{aligned} &[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y}) \\ &[\phi(\vec{x}, t), \phi(\vec{y}, t)] = 0 \\ &[\pi(\vec{x}, t), \pi(\vec{y}, t)] = 0 \qquad \text{(same as free theory)} \end{aligned}$$

$$\begin{aligned} \text{Hamiltonian} \quad H &= \int d^3x (\pi\dot{\phi} - \mathcal{L}) \\ &= \int d^3x \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4 \right). \end{aligned}$$

► Why $\mathcal{L}_{\rm int.} = -\lambda/24\phi^4$?

$$\mathcal{L}_{\rm int.} \sim \begin{cases} +\phi^3 \\ -\phi^3 \\ -\phi^4 \end{cases} \Longrightarrow \mathcal{H}_{\rm int.} \sim \begin{cases} -\phi^3 \\ +\phi^3 \\ +\phi^4 \end{cases} \to -\infty \quad \text{for} \quad \phi \to \infty \text{ or } -\infty. \quad \text{(unbounded below)}$$

Thus $\mathcal{L}_{\text{int.}} \sim -\phi^4$ is the simplest possibility.
$1/24 = 1/4!$ is for later convenience (Feynman rule).

- The EOM is

$$\boxed{\left(\Box + m^2\right)\phi(x) = -\frac{\lambda}{6}\phi(x)^3},$$

which can be derived by (i) Euler-Lagrange eq. or (ii) Heisenberg eq. + ETCR.
(see § 2.2 and (b-1))

§ 5.4 What is $\phi(x)$?

- In the case of free theory ($\lambda = 0$),

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(e^{-iE_pt}e^{i\vec{p}\cdot\vec{x}}a(\vec{p}) + e^{iE_pt}e^{-i\vec{p}\cdot\vec{x}}a^\dagger(\vec{p})\right)$$

We could solve the t-dependence from K-G eq. $(\Box + m^2)\phi = 0$.

- What if $\lambda \neq 0$?? Let's try Fourier transform at $t = 0$.

$$\phi(t=0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(a(\vec{p})e^{i\vec{p}\cdot\vec{x}} + \underbrace{a^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}}_{\text{from } \phi=\phi^\dagger}\right). \qquad (1)$$

Then,
$$\begin{aligned}\phi(t,\vec{x}) &= e^{iHt}\phi(0,\vec{x})e^{-iHt} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(e^{iHt}a(\vec{p})e^{-iHt}e^{i\vec{p}\cdot\vec{x}} + e^{iHt}a^\dagger(\vec{p})e^{-iHt}e^{-i\vec{p}\cdot\vec{x}}\right).\end{aligned}$$

- With the interaction term,

$$\begin{aligned}&H = H_0 + \underline{H_{\text{int}}} \\ &\qquad\qquad \sim \phi^4 \sim (a + a^\dagger)^4 \\ \to\ &[H, a(\vec{p})] = -E_p a(\vec{p}) + \mathcal{O}(a^3, a^2a^\dagger, a(a^\dagger)^2, (a^\dagger)^3) \\ \to\ &e^{iHt}a(\vec{p})e^{-iHt} \quad \text{includes many } a \text{ and } a^\dagger. \\ \to\ &\phi(t \neq 0, \vec{x}) \quad \text{also includes many } a \text{ and } a^\dagger.\end{aligned}$$

Thus, $\phi(x)$ cannot be considered as a field to create/annihilate just 1-particle state, but it includes (infinitely many) particle creation/annihilation.

Comment
Here, we have used $[a, a^\dagger]$ etc, but a and $a^\dagger$ are just Fourier coefficients in Eq. (1).
What are these a and $a^\dagger$?

$$\begin{cases} \bullet\ [a, a^\dagger] = ? \\ \bullet\ H = ? \\ \bullet\ [H, a] = ?,\ [H, a^\dagger] = ? \end{cases} \Longrightarrow \S\, 5.6$$

(Skip this part in the lecture) In fact, $a(\vec{p})$ is not uniquely determined by

$$(1) \longleftrightarrow \phi(t=0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a(\vec{p})+a^\dagger(-\vec{p})\right)e^{i\vec{p}\cdot\vec{x}}.$$

For any operator $f(\vec{p})$, replacing

$$a(\vec{p}) \to a(\vec{p}) + i\left(f(\vec{p})+f^\dagger(-\vec{p})\right)$$

does not change the above equation. We will define $a(\vec{p})$ more precisely in § 5.6.

§ 5.5 In/out states and the LSZ reduction formula

- We want to define the in/out states in § 5.2.
- In the free theory, one particle state is (see § 2.7)

$$|p\rangle = \sqrt{2E_p}a^\dagger(\vec{p})\,|0\rangle\,.$$

where (see § 2.3)

$$\begin{aligned} a^\dagger(\vec{p}) &= \frac{1}{\sqrt{2E_p}}\int d^3x\, e^{-ip\cdot x}\left(-i\dot{\phi}(x)+E_p\phi(x)\right) \\ &= \frac{-i}{\sqrt{2E_p}}\int d^3x\, e^{-ip\cdot x}\overleftrightarrow{\partial_0}\phi(x). \end{aligned}$$

$$\left(f\overleftrightarrow{\partial_0}g \equiv f\partial_0 g - (\partial_0 f)g, \quad \partial_0 = \frac{\partial}{\partial t}\right)$$

- We consider the same operator in the interacting theory.

$$a^\dagger(\vec{p},t) = \frac{-i}{\sqrt{2E_p}}\int d^3x\, e^{-ip\cdot x}\overleftrightarrow{\partial_0}\phi(x),$$

which is now time-dependent. ($\frac{\partial}{\partial t}(RHS)\neq 0$ for $\lambda \neq 0$.)

- And we define the in/out states by

$$\begin{aligned} |\vec{p_1}\cdots\vec{p_n};\,\text{in}\rangle &= \sqrt{2E_{p_1}}a^\dagger(\vec{p_1},-\infty)\cdots\sqrt{2E_{p_n}}a^\dagger(\vec{p_n},-\infty)\,|0\rangle \\ |\vec{q_1}\cdots\vec{q_m};\,\text{out}\rangle &= \sqrt{2E_{q_1}}a^\dagger(\vec{q_1},+\infty)\cdots\sqrt{2E_{q_n}}a^\dagger(\vec{q_m},+\infty)\,|0\rangle \end{aligned}$$

where

$$a^\dagger(\vec{p},\mp\infty) = \lim_{x^0\to\mp\infty}\frac{-i}{\sqrt{2E_p}}\int d^3x\, e^{-ip\cdot x}\overleftrightarrow{\partial_0}\phi(x).$$

Comments

(i) One can think of operators with wave-packets:

$$\widetilde{a}^{\dagger}(t) = \int d^3p\, f(\vec{p}) a^{\dagger}(\vec{p}, t)$$

$$\text{with} \quad f(\vec{p}) \sim \exp\left(-\frac{(\vec{p}-\vec{p_1})^2}{4\sigma^2}\right)$$

and then later take $\sigma \to 0$. See the textbooks by Srednicki [1] and/or Peskin [2].

(ii) The vacuum state $|0\rangle$ is the ground state of the full Hamiltonian $H = H_0 + H_{\text{int}}$.

► Then, one can show

LSZ reduction formula

$$\begin{aligned}
&\langle \vec{q_1} \cdots \vec{q_m};\ \text{out}|\vec{p_1} \cdots \vec{p_n};\ \text{in}\rangle \\
&= \prod_{i=1}^{m}\left[i\int d^4x_i e^{+iq_i\cdot x_i}\left(\Box_{x_i} + m^2\right)\right] \prod_{i=1}^{n}\left[i\int d^4y_i e^{-ip_i\cdot y_i}\left(\Box_{y_i} + m^2\right)\right] \\
&\quad \times \langle 0|\mathrm{T}\left(\phi(x_i)\cdots\phi(x_m)\ \phi(y_1)\cdots\phi(y_n)\right)|0\rangle
\end{aligned}$$

——————— on June 25, up to here. ———————

——————— July 2, from here. ———————

Where were we?

$\mathcal{L} \to$ quantization § **5.3**

$\downarrow$

• $\langle 0|T[\phi\cdots\phi]|0\rangle$

$\boxed{\textbf{We are here.}}$ $\downarrow$ • LSZ, § **5.5**

• S-matrix, amplitude $\mathcal{M}$

$\downarrow$ § **5.2**

• observables (σ and Γ)

Feynman rule

LSZ reduction formula

$\cancel{\langle \vec{p_1} \cdots \vec{p_n};\ \text{in} | \vec{q_1} \cdots \vec{q_m};\ \text{out} \rangle}$ $\boxed{\text{Correction}}$

$$\langle \vec{q_1} \cdots \vec{q_m};\ \text{out} | \vec{p_1} \cdots \vec{p_n};\ \text{in} \rangle$$
$$= \prod_{i=1}^{m} \left[i \int d^4 x_i e^{+iq_i \cdot x_i} \left(\Box_{x_i} + m^2 \right) \right] \prod_{i=1}^{n} \left[i \int d^4 y_i e^{-ip_i \cdot y_i} \left(\Box_{y_i} + m^2 \right) \right]$$
$$\times \langle 0 | \mathrm{T} \left(\phi(x_i) \cdots \phi(x_m)\ \phi(y_1) \cdots \phi(y_n) \right) | 0 \rangle$$

$\leftarrow$ last week

$\rightarrow$ today

where

time-ordering T

$$\mathrm{T}\left(\phi(x)\phi(y)\right) = \begin{cases} \phi(x)\phi(y) & \text{for} \quad x^0 > y^0 \\ \phi(y)\phi(x) & \text{for} \quad y^0 > x^0 \end{cases}$$

$$\mathrm{T}\left(\phi(x_1)\phi(x_2)\phi(x_3)\cdots\right) = \begin{cases} \phi(x_{i_1})\phi(x_{i_2})\phi(x_{i_3})\cdots & \text{for} \quad x^0_{i_1} > x^0_{i_2} > x^0_{i_3} > \cdots \\ \cdots \\ \cdots \end{cases}$$

▶ **Proof of the LSZ formula** First, from def. of $|\text{in}\rangle$ and $|\text{out}\rangle$,

(LHS of the LSZ formula)

$$= \sqrt{2E_{p_1}} \cdots \sqrt{2E_{q_1}} \cdots \langle 0 | a(\vec{q_1}, +\infty) \cdots a(\vec{q_m}, +\infty) a^\dagger(\vec{p_1}, -\infty) \cdots a^\dagger(\vec{p_n}, -\infty) | 0 \rangle$$
$$= \sqrt{2E_{p_1}} \cdots \sqrt{2E_{q_1}} \cdots \langle 0 | \mathrm{T} \Big(a(\vec{q_1}, +\infty) \cdots a(\vec{q_m}, +\infty) a^\dagger(\vec{p_1}, -\infty) \cdots a^\dagger(\vec{p_n}, -\infty) \Big) | 0 \rangle$$

($\because$ already time-ordered) ——————(1)

▶ Next,

$$a^\dagger(\vec{p}, +\infty) - a^\dagger(\vec{p}, -\infty) = \int_{-\infty}^{\infty} dt\ \partial_0\ a^\dagger(\vec{p}, t)$$
$$= \int_{-\infty}^{\infty} dt\ \partial_0 \left[\frac{-i}{\sqrt{2E_p}} \int d^3x\ e^{-ip\cdot x} \overleftrightarrow{\partial_0} \phi(x) \right] \quad (\because \text{def. of } a^\dagger(\vec{p}, t))$$
$$= \cdots \boxed{\text{Problem } \textbf{(b-29)}\text{: Fill this gap}} \cdots$$
$$= \frac{-i}{\sqrt{2E_p}} \int d^4x\ e^{-ip\cdot x} (\Box + m^2) \phi(x).$$

$$\text{Thus,}\quad a^\dagger(\vec{p},-\infty) = a^\dagger(\vec{p},+\infty) + \frac{i}{\sqrt{2E_p}}\int d^4x\, e^{-ip\cdot x}(\Box + m^2)\phi(x).$$

$$\text{Similarly,}\quad a(\vec{p},+\infty) = a(\vec{p},-\infty) + \frac{i}{\sqrt{2E_p}}\int d^4x\, e^{ip\cdot x}(\Box + m^2)\phi(x).$$

- Therefore, from (1),

(LHS of the LSZ formula)

$$= \sqrt{2E_{p_1}}\cdots\sqrt{2E_{q_1}}\cdots\langle 0|\mathrm{T}\Big(a(\vec{q}_1,+\infty)\cdots a(\vec{q}_m,+\infty)a^\dagger(\vec{p}_1,-\infty)\cdots a^\dagger(\vec{p}_n,-\infty)\Big)|0\rangle$$

$$a(\vec{q}_1,-\infty) + \frac{i}{\sqrt{2E_{q_1}}}\int\cdots \qquad\qquad a^\dagger(\vec{p}_1,+\infty) + \frac{i}{\sqrt{2E_{p_1}}}\int\cdots$$

time-ordering
$a(\vec{q}_1,-\infty)|0\rangle = 0$

time-ordering
$\langle 0|\,a^\dagger(\vec{p}_1,+\infty) = 0$

$$= \langle 0|\left[\left(i\int d^4x_1\, e^{iq_1\cdot x_1}(\Box_{x_1}+m^2)\phi(x_1)\right)\cdots\left(i\int d^4y_1\, e^{-ip_1\cdot y_1}(\Box_{y_1}+m^2)\phi(y_1)\right)\cdots\right]|0\rangle$$

$= $ (RHS of the LSZ formula) ■

Comments

(i) In the derivation of the LSZ formula, we have used

$$a(\vec{p},\pm\infty)|0\rangle = 0$$

where $|0\rangle$ is the ground state of the full Hamiltonian $H = H_0 + H_{\text{int}}$. There is a subtlety here, but we do not discuss the details in this lecture. (**See also the comments in the pdf note.**)

Under certain assumptions (axioms) on the quantum field theory, such as "spectral conditions" (スペクトル条件), "asymptotic completion" (漸近的完全性), and "LSZ asymptotic condition", one can show the above equation $a(\vec{p},\pm\infty)|0\rangle = 0$.
For instance, the "asymptotic completeness" (漸近的完全性) says that, the Fock space spanned by the "in"-operators:

$$\mathcal{V}^{\text{in}} = \left\{|0\rangle, a^\dagger(\vec{p},-\infty)|0\rangle, a^\dagger(\vec{p},-\infty)a^\dagger(\vec{p}',-\infty)|0\rangle, \cdots\right\}$$

and that by the "out"-operators:

$$\mathcal{V}^{\text{out}} = \left\{|0\rangle, a^\dagger(\vec{p},+\infty)|0\rangle, a^\dagger(\vec{p},+\infty)a^\dagger(\vec{p}',+\infty)|0\rangle, \cdots\right\}$$

are the same as the Fock space spanned by the Heisenberg operator $\phi(x)$, $\mathcal{V}$:

$$\mathcal{V}^{\text{in}} = \mathcal{V}^{\text{out}} = \mathcal{V}.$$

For more details, see e.g.,
Kugo-san's textbook [5] 「ゲージ場の量子論 I」九後汰一郎、培風館
and Sakai-san's textbook [6] 「場の量子論」坂井典佑、裳華房.
They are in Japanese. I have checked several QFT textbooks in English, such as Peskin [2], Schrednicki [1], Weinberg [3], etc, but I couldn't find the corresponding explanation.

(ii) In general, the state

$$\begin{aligned} &\sqrt{2E_p}a^\dagger(\vec{p}, \pm\infty)\,|0\rangle \\ \text{with} \qquad &a^\dagger(\vec{p}, \pm\infty) = \lim_{x^0\to\pm\infty} \frac{-i}{\sqrt{2E_p}} \int d^3x e^{-ip\cdot x} \overleftrightarrow{\partial_0} \phi(x), \end{aligned}$$

does NOT give the correct normalization for the 1-particle state. The normalization receives corrections from the interaction, but we neglect the correction here. (**See also the pdf lecture note.)**

One should either define the operator by

$$a^\dagger(\vec{p}, \pm\infty) = \frac{1}{\sqrt{Z}} \lim_{x^0\to\pm\infty} \frac{-i}{\sqrt{2E_p}} \int d^3x e^{-ip\cdot x} \overleftrightarrow{\partial_0} \phi(x)$$

(see e.g, Kugo-san's and Sakai-san's textbooks [5, 6]),
or rescale the field as

$$\phi(x) = \sqrt{Z}\phi_r(x) \qquad (\phi_r(x) : \text{rescaled, or renormalized field})$$

(see e.g., Srednicki's textbook [1])
where

$$Z = |\langle p|\phi(x)|0\rangle|^2$$

represents how much the state $\phi(x)\,|0\rangle$ overlaps with the 1-particle state $|p\rangle$. (Note that $\langle p|\phi(x)|0\rangle = e^{ip\cdot x}$ and $Z = 1$ for the free theory.)
In this lecture, we do not discuss the renormalization, and take $Z = 1$ as the leading order perturbation.

§ 5.6 Heisenberg field and Interaction picture field

$$\langle 0|\mathrm{T}(\phi(x_1)\cdots\phi(x_n))|0\rangle \longleftarrow \textbf{Now we want to calculate this.}$$

$$\Big\downarrow \ \S\,5.5\text{, LSZ formula}$$

$$\langle \text{out}|\text{in}\rangle$$

$$\Big\downarrow \ \S\,5.2$$

$$\sigma, \Gamma$$

Idea: perturbative expansion in the coupling λ.

There are two ways:

- "interaction picture field". ($\leftarrow$ We choose this in this lecture.)
- "path integral" formalism.

The two ways give the same result for $\langle 0|\mathrm{T}(\phi(x_1)\cdots\phi(x_n))|0\rangle$.

► First,

$$\begin{aligned}\phi(t,\vec{x}) &= e^{iHt}\phi(0,\vec{x})e^{-iHt}\\ \dot{\phi}(t,\vec{x}) &= e^{iHt}(i[H,\phi(0,\vec{x})])e^{-iHt}\\ &= e^{iHt}\dot{\phi}(0,\vec{x})e^{-iHt}\end{aligned}$$

and therefore the Hamiltonian can be written as

$$\begin{aligned}H &= e^{-iHt}He^{iHt}\\ &= e^{-iHt}\int d^3x\left(\frac{1}{2}\dot{\phi}(t,\vec{x})^2+\frac{1}{2}(\vec{\nabla}\phi(t,\vec{x}))^2+\frac{1}{2}m^2\phi(t,\vec{x})^2+\frac{\lambda}{24}\phi(t,\vec{x})^4\right)e^{iHt}\\ &= \underbrace{\int d^3x\left(\frac{1}{2}\dot{\phi}(0,\vec{x})^2+\frac{1}{2}\left(\vec{\nabla}\phi(0,\vec{x})\right)^2+\frac{1}{2}m^2\phi(0,\vec{x})^2\right)}_{H_0}+\underbrace{\int d^3x\left(\frac{\lambda}{24}\phi(0,\vec{x})^4\right)}_{H_{\text{int}}}\end{aligned}$$

Note that H_0 and H_{int} are defined in terms of $\phi(0,\vec{x})$ and $\dot{\phi}(0,\vec{x})$ at $t=0$, and they are time-independent.

► Define

Interaction picture field $\phi_I(t,\vec{x})$

$$\phi_I(t,\vec{x}) = e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t}$$

Problem

(b-30,31) Show the following (i)-(vi) for the interaction picture field $\phi_I(t,\vec{x})$, by using

$$\phi_I(t,\vec{x}) = e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t} \quad \text{and} \quad \begin{cases} [\phi(0,\vec{x}),\dot{\phi}(0,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \\ [\phi(0,\vec{x}),\phi(0,\vec{y})] = 0 \\ [\dot{\phi}(0,\vec{x}),\dot{\phi}(0,\vec{y})] = 0 \end{cases}$$

	Heisenberg field $\phi(t,\vec{x}) = e^{iHt}\phi(0,\vec{x})e^{-iHt}$ (evolved by H)	Interaction picture field $\phi_I(t,\vec{x}) = e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t}$ (evolved by H_0)
Heisenberg eq.	$\begin{cases}\dot{\phi} = i[H,\phi] \\ \ddot{\phi} = i[H,\dot{\phi}]\end{cases}$	$\begin{cases}\dot{\phi}_I = i[H_0,\phi_I] \\ \ddot{\phi}_I = i[H_0,\dot{\phi}_I]\end{cases}$ —— (i)
$\dot{\phi}$ and $\dot{\phi}_I$	$\dot{\phi}(t,\vec{x}) = e^{iHt}\dot{\phi}(0,\vec{x})e^{-iHt}$	$\dot{\phi}_I(t,\vec{x}) = e^{iH_0t}\dot{\phi}_I(0,\vec{x})e^{-iH_0t}$ —— (ii) $\dot{\phi}_I(0,\vec{x}) = \dot{\phi}(0,\vec{x})$ —— (iii) (Note, however, that $\ddot{\phi}_I(0,\vec{x}) \neq \ddot{\phi}(0,\vec{x})$.)
ETCR	$\begin{cases}[\phi(t,\vec{x}),\dot{\phi}(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \\ [\phi(t,\vec{x}),\phi(t,\vec{y})] = 0 \\ [\dot{\phi}(t,\vec{x}),\dot{\phi}(t,\vec{y})] = 0\end{cases}$	$\begin{cases}[\phi_I(t,\vec{x}),\dot{\phi}_I(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \\ [\phi_I(t,\vec{x}),\phi_I(t,\vec{y})] = 0 \\ [\dot{\phi}_I(t,\vec{x}),\dot{\phi}_I(t,\vec{y})] = 0\end{cases}$ —— (iv)
Hamiltonian	$H = \int d^3x \left(\frac{1}{2}\dot{\phi}(t,\vec{x})^2 + \frac{1}{2}(\vec{\nabla}\phi(t,\vec{x}))^2 + \frac{1}{2}m^2\phi(t,\vec{x})^2 + \frac{\lambda}{24}\phi(t,\vec{x})^4\right)$ Each term in the RHS depends on t, but the sum is t-independent.	$H_0 = \int d^3x \left(\frac{1}{2}\dot{\phi}_I(t,\vec{x})^2 + \frac{1}{2}(\vec{\nabla}\phi_I(t,\vec{x}))^2 + \frac{1}{2}m^2\phi_I(t,\vec{x})^2\right)$ —— (v) Each term in the RHS depends on t, but the sum is t-independent.
EOM	$\boxed{(\Box + m^2)\phi = -\frac{\lambda}{6}\phi^3}$	$\boxed{(\Box + m^2)\phi_I = 0}$ —— (vi) ϕ_I is a free field!

§ 5.7 a and $a^\dagger$ (again)

- ϕ_I satisfies $(\Box + m^2)\phi_I = 0$ and therefore it can be solved exactly (see § 2.3).

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x}\right)\Big|_{p^0=E_p}.$$

Note: $a(\vec{p})$ and $a^\dagger(\vec{p})$ are the expansion coefficients of ϕ_I, not ϕ.
(We can also write them as a_I and $a_I^\dagger$.)

Thus,

$$\left.\begin{aligned} \phi(0,\vec{x}) = \phi_I(0,\vec{x}) &= \cdots \\ \dot\phi(0,\vec{x}) = \dot\phi_I(0,\vec{x}) &= \cdots \\ H_0 &= \cdots \\ \xrightarrow[\text{substitute}]{} \quad H_{\text{int}} &= \cdots \end{aligned}\right\} \quad \text{all of them can be written in terms of } a, a^\dagger.$$

▶ From (i)(iv)(v) in § 5.6, one can show (see § 2.4 and § 2.5)

$$\begin{aligned} &[a(\vec{p}), a^\dagger(\vec{q})] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}), \\ &[a,a] = 0, \\ &[a^\dagger, a^\dagger] = 0, \\ &[H_0, a^\dagger(\vec{p})] = E_p a^\dagger(\vec{p}), \\ &[H_0, a(\vec{p})] = -E_p a(\vec{p}). \end{aligned}$$

▶ Define a state annihilated by $a(\vec{p})$:

$$|0\rangle_I : \qquad a(\vec{p})\,|0\rangle_I = 0, \qquad H_0\,|0\rangle_I = 0 \quad (H_0 : \text{ normal ordered})$$

Note that $|0\rangle_I$ is <u>NOT</u> the ground state of the full Hamiltonian:

$$H\,|0\rangle_I = (H_0 + H_{\text{int}})\,|0\rangle_I \neq 0$$
$$\because H_{\text{int}} \sim \phi_I^4 \sim (a+a^\dagger)^4$$
$$\boxed{|0\rangle_I \neq |0\rangle}$$

§ 5.8 $\langle 0|\mathrm{T}\,(\phi(x_1)\cdots\phi(x_n))\,|0\rangle = ?$

We want to express $\langle 0|\mathrm{T}\,(\phi\cdots)\,|0\rangle = ?$ in terms of ϕ_I (a and $a^\dagger$).
Step (i) $\sim$ (vii).

<u>step (i)</u> redefine the space-time points such that $x_1^0 > x_2^0 > \cdots x_n^0$,

$$\langle 0|\mathrm{T}\,(\phi(x_1)\cdots\phi(x_n))\,|0\rangle = \langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle. \qquad \text{———} \quad (1)$$

<u>step (ii)</u> $\phi(x) = ?$

$$\begin{cases} \phi(x) = e^{iHt}\phi(0,\vec{x})e^{-iHt} \\ \phi_I(x) = e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t} \end{cases}$$
$$\to \ \phi(x) = e^{iHt}e^{-iH_0t}\phi_I(x)\underbrace{e^{iH_0t}e^{-iHt}}_{\equiv\, u(t)}$$
$$\phi(x) = u^\dagger(t)\phi_I(x)u(t). \qquad \text{———} \quad (2)$$

step (iii) $|0\rangle = ?$

$$\begin{aligned} {}_I\langle 0|\, u(t) &= {}_I\langle 0|\, e^{iH_0 t} e^{-iHt} \\ &= {}_I\langle 0|\, e^{-iHt} \qquad (\because H_0 |0\rangle_I = 0) \end{aligned}$$

Insert an identity operator:

$$\mathbf{1} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|$$

where $|n\rangle$ represent the eigenstates of H with eigenvalues $E_n > E_0 = 0$. (The summation includes continuous parameter (integral).) Then

$$\begin{aligned} {}_I\langle 0|\, u(t) &= {}_I\langle 0| \left[|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| \right] e^{-iHt} \\ &= {}_I\langle 0|0\rangle \langle 0|\, e^{-iHt} + \sum_{n=1} {}_I\langle 0|n\rangle \langle n|\, e^{-iHt} \\ &= {}_I\langle 0|0\rangle \langle 0| \qquad + \sum_{n=1} {}_I\langle 0|n\rangle \langle n|\, e^{-iE_n t} \end{aligned}$$

The 2nd term oscillates for $t \to \infty$. Thus, for regularization, we take

$$t \to \infty(1 - i\epsilon) \qquad (\epsilon > 0, \epsilon \to 0)$$
$$\text{then, } e^{-iE_n t} \to e^{-iE_n \infty(1-i\epsilon)} \propto e^{-E_n \infty \cdot \epsilon} \to 0$$

Therefore

$$\lim_{t \to \infty(1-i\epsilon)} {}_I\langle 0|\, u(t) = {}_I\langle 0|0\rangle \langle 0|\,.$$

Similarly

$$\lim_{t \to \infty(1-i\epsilon)} u^\dagger(-t)\, |0\rangle_I = |0\rangle \langle 0|0\rangle_I.$$

Thus

$$\begin{aligned} \langle 0|\mathcal{O}|0\rangle &= \frac{{}_I\langle 0|0\rangle\langle 0|\mathcal{O}|0\rangle\langle 0|0\rangle_I}{{}_I\langle 0|0\rangle \underbrace{\langle 0|0\rangle}_{1} \langle 0|0\rangle_I} \\ &= \lim_{t \to \infty(1-i\epsilon)} \frac{{}_I\langle 0|u(t)\mathcal{O}u^\dagger(-t)|0\rangle_I}{{}_I\langle 0|u(t)u^\dagger(-t)|0\rangle_I} \qquad \text{———} \quad (3) \end{aligned}$$

<u>step (iv)</u> Substituting (2) (3) to (1),

$$
\begin{aligned}
&\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle \\
&= \lim_{t\to\infty(1-i\epsilon)} \frac{1}{{}_I\langle 0|u(t)u^\dagger(-t)|0\rangle_I} \\
&\times\ {}_I\langle 0|\,\underbrace{u(t)\cdot u^\dagger(t_1)}\,\phi_I(x_1)\,\underbrace{u(t_1)\cdot u^\dagger(t_2)}\,\phi_I(x_2)\,\underbrace{u(t_2)\cdots}\cdots\underbrace{\cdots u^\dagger(t_n)}\,\phi_I(x_n)\,\underbrace{u(t_n)\cdot u^\dagger(-t)}\,|0\rangle_I \\
&= \lim_{t\to\infty(1-i\epsilon)} \frac{{}_I\langle 0|U(t,t_1)\phi_I(x_1)U(t_1,t_2)\phi_I(x_2)\cdots\phi_I(x_n)U(t_n,-t)|0\rangle_I}{{}_I\langle 0|U(t,-t)|0\rangle_I} \qquad \text{———— (4)}
\end{aligned}
$$

where

$$
\begin{aligned}
U(t_1,t_2) &\equiv u(t_1)u^\dagger(t_2) \qquad (t_1 > t_2) \\
&= e^{iH_0t_1}e^{-iH(t_1-t_2)}e^{-iH_0t_2}
\end{aligned}
$$

———— on July 2, up to here. ————

———— July 9, from here. ————

<u>Where were we?</u>

$\mathcal{L} \to$ quantization § **5.3**

$\downarrow$ § **5.6**—§ **5.8** **We are here.**

• $\langle 0|T[\phi\cdots\phi]|0\rangle$

$\downarrow$ • LSZ, § **5.5**

• S-matrix, amplitude $\mathcal{M}$

$\downarrow$ § **5.2**

• observables (σ and Γ)

Feynman rule

§ 5.8

step (i) - (vii)

step (iv)

$$
\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{{}_I\langle 0|U(t,t_1)\phi_I(x_1)\cdots|0\rangle_I}{{}_I\langle 0|U(t,-t)|0\rangle_I} \qquad \text{———— (4)}
$$

where $U(t_1,t_2) \equiv u(t_1)u^\dagger(t_2)$ $(t_1 > t_2)$ ($\leftarrow$ added)

$= e^{iH_0t_1}e^{-iH(t_1-t_2)}e^{-iH_0t_2}$

$\leftarrow$ last week

$\to$ today

step (v) $U(t_1,t_2)=?$ It satisfies

$$\begin{cases} U(t,t)=0 \\ \dfrac{\partial}{\partial t_1}U(t_1,t_2)=-iH_I(t_1)U(t_1,t_2) \\ \dfrac{\partial}{\partial t_2}U(t_1,t_2)=iU(t_1,t_2)H_I(t_2) \end{cases} \qquad \text{———} \quad (5)$$

$$\text{where} \quad H_I(t)\equiv e^{iH_0t}H_{\text{int}}e^{-iH_0t}=\int d^3x\frac{\lambda}{24}\phi_I(x)^4$$

Problem

(b-32) Show (5).

The solution of (5) is, if $H_I(t)$ at different t commute,

$$\times \quad U(t_1,t_2)=\exp\left(-i\int_{t_2}^{t_1}H_I(t)dt\right),$$

but this is wrong. The correct solution is

$$\begin{aligned} U(t_1,t_2) &= \mathrm{T}\left[\exp\left(-i\int_{t_2}^{t_1}H_I(t)dt\right)\right] \\ &= \mathrm{T}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(-i\int_{t_2}^{t_1}H_I(t)dt\right)^n\right] \qquad \text{———} \quad (6) \end{aligned}$$

Let's show that (6) satisfies (5).

$$\begin{aligned} \frac{\partial}{\partial t_1}U(t_1,t_2) &= \sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{T}\left[\frac{\partial}{\partial t_1}\left(-i\int_{t_2}^{t_1}H_I(t)dt\right)^n\right] \\ &= \sum_{n=1}^{\infty}\frac{1}{n!}\mathrm{T}\left[\sum_{k=1}^{n}\left(-i\int_{t_2}^{t_1}H_I(t)dt\right)^{k-1}\underbrace{(-iH_I(t_1))}\left(-i\int_{t_2}^{t_1}H_I(t)dt\right)^{n-k}\right] \\ &\qquad \text{(Here, } t_1\geq t\geq t_2, \text{and hence } H_I(t_1) \text{ can be moved in front of T)} \\ &= -iH_I(t_1)\sum_{n=1}^{\infty}\frac{1}{n!}\mathrm{T}\left[n\cdot\left(-i\int_{t_2}^{t_1}H_I(t)dt\right)^{n-1}\right] \\ &= -iH_I(t_1)U(t_1,t_2) \qquad \blacksquare \end{aligned}$$

step (vi) From (4),

$$\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle = \lim_{t\to\infty(1-i\epsilon)}\frac{{}_I\langle 0|U(t,t_1)\phi_I(x_1)U(t_1,t_2)\phi_I(x_2)\cdots\phi_I(x_n)U(t_n,-t)|0\rangle_I}{{}_I\langle 0|U(t,-t)|0\rangle_I}$$

With (6), everything is written in terms of $\phi_I(x)$. Furthermore, the numerator is time-ordered ($t > t_1 > t_2 > \cdots t_n > -t$), and hence it can be written as

$$\begin{aligned}
\langle 0|\mathrm{T}\big(\phi(x_1)\cdots\phi(x_n)\big)|0\rangle &= \lim_{t\to\infty(1-i\epsilon)} \frac{{}_I\langle 0|\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_n)U(t,t_1)U(t_1,t_2)\cdots U(t_n,-t)\right]|0\rangle_I}{{}_I\langle 0|U(t,-t)|0\rangle_I} \\
&= \lim_{t\to\infty(1-i\epsilon)} \frac{{}_I\langle 0|\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_n)U(t,-t)\right]|0\rangle_I}{{}_I\langle 0|U(t,-t)|0\rangle_I} \qquad \text{—— (7)}
\end{aligned}$$

where we have used $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$.

step (vii) Substituting (6) to (7), we finally obtain

$$\boxed{\langle 0|\mathrm{T}\big(\phi(x_1)\cdots\phi(x_n)\big)|0\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{{}_I\langle 0|\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_n)\exp\left(-i\int_{-t}^{t} H_I(t')dt'\right)\right]|0\rangle_I}{{}_I\langle 0|\mathrm{T}\left[\exp\left(-i\int_{-t}^{t} H_I(t')dt'\right)\right]|0\rangle_I}}$$

$$\text{—— (8)}$$

Everything is written in terms of $\phi_I(x)$ and $|0\rangle_I$. By expanding $\exp(-i\int H_I)$, we can do the perturbation expansion as $\mathcal{O}(1) + \mathcal{O}(\lambda) + \mathcal{O}(\lambda^2)\cdots$.

§ 5.9 Wick's theorem

- All the terms in the numerator and the denominator of Eq. (8) have the following form:

$${}_I\langle 0|\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_n)\right]|0\rangle_I\,.$$

- Define $\varphi(x)$ as follows.

$$\phi_I(x) = \underbrace{\int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} a(\vec{p})e^{-ip\cdot x}}_{\equiv\, \varphi(x)} + \underbrace{\int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} a^\dagger(\vec{p})e^{ip\cdot x}}_{\equiv\, \varphi^\dagger(x)}\,.$$

and introduce

normal ordering N

$$\begin{aligned}
\mathrm{N}\left[\phi_I(x_1)\phi_I(x_2)\right] &= \mathrm{N}\left[\big(\varphi(x_1)+\varphi^\dagger(x_1)\big)\big(\varphi(x_2)+\varphi^\dagger(x_2)\big)\right] \\
&= \mathrm{N}\left[\varphi(x_1)\varphi(x_2) + \varphi^\dagger(x_1)\varphi(x_2) + \underline{\varphi(x_1)\varphi^\dagger(x_2)} + \varphi^\dagger(x_1)\varphi^\dagger(x_2)\right] \\
&= \varphi(x_1)\varphi(x_2) + \varphi^\dagger(x_1)\varphi(x_2) + \underline{\varphi^\dagger(x_2)\varphi(x_1)} + \varphi^\dagger(x_1)\varphi^\dagger(x_2)
\end{aligned}$$

(move $\varphi^\dagger$ to the left, and φ to the right.)

Then

$$\begin{cases}\varphi \sim a \\ \varphi^\dagger \sim a^\dagger\end{cases} \implies \begin{cases}\varphi\,|0\rangle_I = 0 \\ {}_I\langle 0|\,\varphi^\dagger = 0\end{cases} \implies {}_I\langle 0|\mathrm{N}\left[\phi_I(x_1)\cdots\phi_I(x_n)\right]|0\rangle_I = 0\,.$$

- We want to see the relation between

$$\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_n)\right] \overset{?}{\Longleftrightarrow} \mathrm{N}\left[\phi_I(x_1)\cdots\phi_I(x_n)\right]\,.$$

In the following, for simplicity we write

$$\phi_I(x_i) = \phi_i, \qquad \varphi(x_i) = \varphi_i, \qquad \therefore \quad \phi_i = \varphi_i + \varphi_i^\dagger.$$

Let's start from $n=2$.

- $\underline{n = 2}$

For $x_1^0 > x_2^0$,

$$\begin{aligned}\mathrm{T}(\phi_1\phi_2) = \phi_1\phi_2 &= (\varphi_1 + \varphi_1^\dagger)(\varphi_2 + \varphi_2^\dagger) \\ &= \varphi_1\varphi_2 + \varphi_1\varphi_2^\dagger + \varphi_1^\dagger\varphi_2 + \varphi_1^\dagger\varphi_2^\dagger \\ &= \mathrm{N}(\phi_1\phi_2) + [\varphi_1, \varphi_2^\dagger] \\ \text{where } [\varphi_1, \varphi_2^\dagger] &= \int \frac{d^3p_1}{(2\pi)^3\sqrt{2E_{p_1}}} e^{-ip_1\cdot x_1} \int \frac{d^3p_2}{(2\pi)^3\sqrt{2E_{p_2}}} e^{ip_2\cdot x_2} [a(\vec{p_1}), a^\dagger(\vec{p_2})] \\ &= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip\cdot(x_1-x_2)}\Big|_{p^0=E_p}\,.\end{aligned}$$

For $x_2^0 > x_1^0$, we have a similar formula with $x_1 \leftrightarrow x_2$. Therefore,

$$\boxed{\begin{aligned}&\mathrm{T}(\phi_1\phi_2) = \mathrm{N}(\phi_1\phi_2) + \wick{\c\phi_1\c\phi_2} \\ &\quad \wick{\c\phi_1\c\phi_2} = \int \frac{d^3p}{(2\pi)^3 2E_p} \times \begin{cases} e^{-ip\cdot(x_1-x_2)} & (x_1^0 > x_2^0) \\ e^{-ip\cdot(x_2-x_1)} & (x_2^0 > x_1^0)\end{cases}\end{aligned}}$$

not an operator,
but c-number.

$p^0 = \sqrt{\vec{p}^2 + m^2}$

The symbol $\wick{\c\phi_1\c\phi_2}$ is called "Wick contraction," and it can also be written as

$$\boxed{\wick{\c\phi_1\c\phi_2} = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x_1-x_2)} \qquad (\epsilon > 0,\ \epsilon \to 0).}$$

Feynman propagator

$p^0 \neq \sqrt{\vec{p}^2 + m^2}$ in general.
It's just an integration variable.

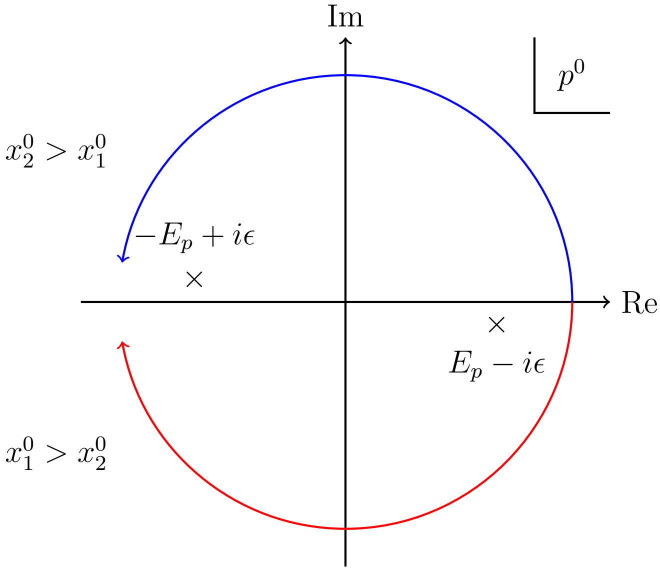
Problem

(b-33) Show

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x_1 - x_2)} = \int \frac{d^3p}{(2\pi)^3 2E_p} \times \begin{cases} e^{-ip\cdot(x_1-x_2)} & (x_1^0 > x_2^0) \\ e^{-ip\cdot(x_2-x_1)} & (x_2^0 > x_1^0) \end{cases}$$

$$(\epsilon > 0, \epsilon \to 0) \qquad\qquad (p^0 = E_p)$$

(Hint:)



(See the pdf file)

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x_1-x_2)} = \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi} \frac{i}{(p^0)^2 - \underbrace{(\vec{p}^2 + m^2)}_{E_p^2} + i\epsilon} e^{-ip\cdot(x_1-x_2)}$$

and, for $\epsilon \to 0$,[3]

$$\frac{i}{(p^0)^2 - E_p^2 + i\epsilon} \sim i \cdot \frac{1}{p^0 - (E_p - i\epsilon)} \cdot \frac{1}{p^0 + (E_p - i\epsilon)}$$

For $\underline{x_1^0 > x_2^0}$, $e^{-ip^0(x_1^0 - x_2^0)} \to 0$ for $p^0 \to -i\infty$,
For $\underline{x_2^0 > x_1^0}$, $e^{-ip^0(x_1^0 - x_2^0)} \to 0$ for $p^0 \to +i\infty$.
Take the contours as above

[3](Here, $E_p^2 - i\epsilon \sim (E_p - i\epsilon/(2E_p))^2$ and we renamed $\epsilon/(2E_p)$ as ϵ in the right hand side. The overall coefficient of ϵ doesn't matter as far as $\epsilon \to 0$.)

- $\underline{n=3}$
  For $x_3^0 > x_1^0, x_2^0$,

$$
\begin{aligned}
\mathrm{T}(\phi_1\phi_2\phi_3) = \phi_3 \mathrm{T}(\phi_1\phi_2) &= \phi_3 \mathrm{N}(\phi_1\phi_2) + \phi_3 \wick{\c{\phi}_1 \c{\phi}_2} \\
&= \varphi_3 \mathrm{N}(\phi_1\phi_2) + \varphi_3^\dagger \mathrm{N}(\phi_1\phi_2) + \phi_3 \wick{\c{\phi}_1 \c{\phi}_2} \\
&\quad \begin{cases} \varphi_3 \mathrm{N}(\phi_1\phi_2) = N(\phi_1\phi_2\varphi_3) + \wick{\c{\phi}_1 \c{\phi}_3}\phi_2 + \phi_1 \wick{\c{\phi}_2 \c{\phi}_3} \\ \varphi_3^\dagger \mathrm{N}(\phi_1\phi_2) = N(\phi_1\phi_2\varphi_3^\dagger) \end{cases} \\
&= \mathrm{N}(\phi_1\phi_2\phi_3) + \wick{\c{\phi}_1 \c{\phi}_2}\phi_3 + \wick{\c{\phi}_1 \phi_2 \c{\phi}_3} + \phi_1 \wick{\c{\phi}_2 \c{\phi}_3}\,.
\end{aligned}
$$

  Similar for $x_1^0 > x_2^0, x_3^0$ and $x_2^0 > x_1^0, x_3^0$.

- $\underline{n=4}$

$$
\mathrm{T}(\phi_1\phi_2\phi_3\phi_4) = \mathrm{N}(\phi_1\phi_2\phi_3\phi_4) + \underbrace{\wick{\c{\phi}_1 \c{\phi}_2} \mathrm{N}(\phi_3\phi_4) + \wick{\c{\phi}_1 \c{\phi}_3} \mathrm{N}(\phi_2\phi_4) + \cdots}_{\text{6 terms}} + \underbrace{\wick{\c1{\phi}_1 \c1{\phi}_2 \c2{\phi}_3 \c2{\phi}_4} + \cdots}_{\text{3 terms}}
$$

- $\underline{\text{In general}}$

Wick's theorem

$$
\begin{aligned}
\mathrm{T}(\phi_1 \cdots \phi_n) &= \mathrm{N}(\phi_1 \cdots \phi_n) \\
&+ \sum_{\text{pairs}} \wick{\c{\phi}_i \c{\phi}_j}\ \mathrm{N}(\phi_1 \cdots \underset{\hat{\imath}}{\cdot} \cdots \underset{\hat{\jmath}}{\cdot} \cdots \phi_n) \\
&+ \sum_{\text{2 pairs}} \wick{\c1{\phi}_i \c1{\phi}_j \c2{\phi}_k \c2{\phi}_\ell}\ \mathrm{N}(\phi_1 \cdots \underset{\hat{\imath}\,\hat{\jmath}\,\hat{k}\,\hat{\ell}}{\cdots} \cdots \phi_n) \\
&+ \cdots \\
&+ \begin{cases} \displaystyle\sum_{\frac{n}{2}\text{ pairs}} \wick{\c{\phi}_i \c{\phi}_j} \cdots \wick{\c{\phi}_p \c{\phi}_q} & (n = \text{ even}) \\ \displaystyle\sum_{\frac{n-1}{2}\text{ pairs}} \wick{\c{\phi}_i \c{\phi}_j} \cdots \wick{\c{\phi}_p \c{\phi}_q}\phi_r & (n = \text{ odd})\,. \end{cases}
\end{aligned}
$$

Problem

(b-34) Prove it by induction.

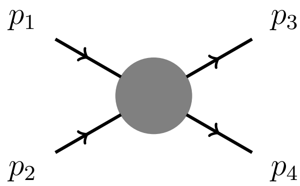
- Therefore,

$$
\boxed{{}_I\langle 0|\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_n)\right]|0\rangle_I = \begin{cases} \displaystyle\sum_{n/2\text{ pairs}} \wick{\c{\phi}_i \c{\phi}_j} \cdots \wick{\c{\phi}_p \c{\phi}_q} & (n = \text{ even}) \\ 0 & (n = \text{ odd})\,. \end{cases}}
$$

§ 5.10 Summary, Feynman rules, examples

- Let's calculate the cross section for $2 \to 2$ scattering in the ϕ^4 theory,

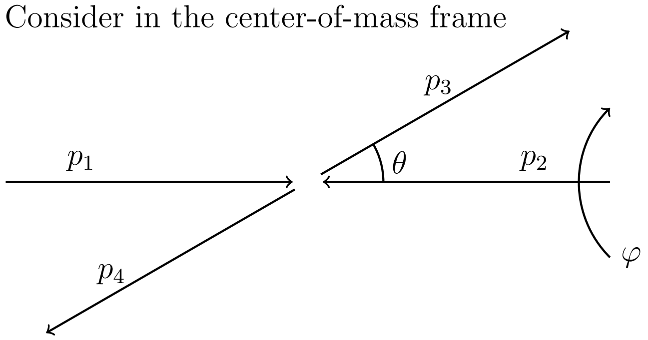
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{24}\phi^4\,.$$



- From § 5.2,

$$\sigma(p_1, p_2 \to \phi\phi) = \frac{1}{2E_1 \cdot 2E_2|v_1 - v_2|} \underbrace{\int d\Phi_2}_{\text{final state}} |\mathcal{M}(p_1, p_2 \to p_3, p_4)|^2 \underbrace{\times\frac{1}{2}}_{\substack{\text{identical}\\ \text{final}\\ \text{particles}}}$$

Consider in the center-of-mass frame



$$\vec{p}_1 = -\vec{p}_2$$
$$\vec{p}_3 = -\vec{p}_4$$
$$|\vec{p}_1| = |\vec{p}_2| = |\vec{p}_3| = |\vec{p}_4|$$
$$E_1 = E_2 = E_3 = E_4$$
$$= \sqrt{|\vec{p}_1|^2 + m^2}$$

Then

$$\sigma(p_1, p_2 \to \phi\phi) = \frac{1}{128\pi}\frac{1}{E_1^2}\int \frac{d\Omega}{4\pi}|\mathcal{M}|^2 \qquad \text{———————(1)},$$
$$\int d\Omega = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta.$$

Problem

(b-35) Show (1).

- From § 5.2,

$$\langle p_3, p_4; \text{out}|p_1, p_2; \text{in}\rangle = (2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4) \cdot i\mathcal{M}(p_1, p_2 \to p_3, p_4) \qquad \text{———————(2)}.$$

► On the other hand, <u>from the LSZ formula in § 5.5</u>,

$$
\begin{aligned}
\langle p_3, p_4; \mathrm{out}|p_1, p_2; \mathrm{in}\rangle &= \underbrace{\prod_{i=1,2}\left[i\int d^4x_i e^{-ip_i\cdot x_i}\left(\Box_i + m^2\right)\right] \times \prod_{i=3,4}^{n}\left[i\int d^4x_i e^{+ip_i\cdot x_i}\left(\Box_i + m^2\right)\right]}_{\text{We call it ``LSZ factor''}} \\
&\quad \times \langle 0|\mathrm{T}\Big(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\Big)|0\rangle
\end{aligned}
$$

where, <u>from § 5.8</u>,

$$
\begin{aligned}
\langle 0|\mathrm{T}\Big(\phi(x_1)\cdots\phi(x_4)\Big)|0\rangle &= \frac{{}_I\langle 0|\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_4)\exp\left(-i\int\frac{\lambda}{24}\phi_I(x)^4\right)\right]|0\rangle_I}{{}_I\langle 0|\mathrm{T}\left[\exp\left(-i\int\frac{\lambda}{24}\phi_I(x)^4\right)\right]|0\rangle_I} \qquad \text{———(3).} \\
&\equiv \frac{(\mathrm{3N})}{(\mathrm{3D})}.
\end{aligned}
$$

► Namely,

$$
\underline{\langle p_3, p_4; \mathrm{out}|p_1, p_2; \mathrm{in}\rangle = \frac{(\text{LSZ factor}) \times (\mathrm{3N})}{(\mathrm{3D})}}.
$$

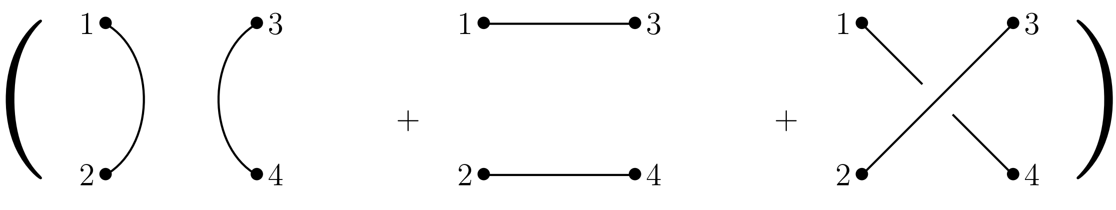
We can expand it with respect to λ.

► $\mathcal{O}(\lambda^0)$ term of (3D) $= {}_I\langle 0|0\rangle_I = 1$.

► $\mathcal{O}(\lambda^0)$ term of (3N)

$= {}_I\langle 0|\mathrm{T}(\phi_1\phi_2\phi_3\phi_4)|0\rangle_I$ (from <u>Wick's theorem in § 5.9</u>)

$$
= \contraction{}{\phi_1}{}{\phi_2}\contraction{\phi_1\phi_2}{\phi_3}{}{\phi_4}\phi_1\phi_2\phi_3\phi_4 \;+\; \contraction{}{\phi_1}{\phi_2}{\phi_3}\contraction[2ex]{\phi_1}{\phi_2}{\phi_3}{\phi_4}\phi_1\phi_2\phi_3\phi_4 \;+\; \contraction[2ex]{}{\phi_1}{\phi_2\phi_3}{\phi_4}\contraction{\phi_1}{\phi_2}{}{\phi_3}\phi_1\phi_2\phi_3\phi_4
$$



where

$$
\contraction{}{\phi_1}{}{\phi_2}\phi_1\phi_2 = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}e^{-ip\cdot(x_1-x_2)} \equiv D_F(x_1 - x_2).
$$

► Now, <u>(LSZ factor) $\times D_F(x_1 - x_2) =$?</u>

▶ In general,

$$
\begin{aligned}
&(\text{LSZ factor})_i \times D_F(x_i - y) \\
&= i\int d^4x_i e^{\mp ip_i\cdot x_i} \underbrace{\left(\Box_i + m^2\right) D_F(x_i - y)}_{\left(\begin{aligned}&= \int \frac{d^4p}{(2\pi)^4}\frac{i(-p_i^2+m^2)}{p_i^2 - m^2 + i\epsilon}e^{-ip_i\cdot(x_1-x_2)} \\ &= -i\delta^{(4)}(x_i - y)\end{aligned}\right)} \\
&= e^{\mp ip_i\cdot y} \qquad \text{———————— (4).}
\end{aligned}
$$

$\boxed{\textbf{POINT}}$ LSZ factor cancels the $D_F(x_i - y)$ factor of the external line.

——— on July 9, up to here. ———

——— July 23, from here. ———

Where were we?

$\mathcal{L} \to$ quantization § **5.3**

$\downarrow$ § **5.6**—§ **5.9**

• $\langle 0|T[\phi\cdots\phi]|0\rangle$

$\downarrow$ • LSZ, § **5.5**

• S-matrix, amplitude $\mathcal{M}$

$\downarrow$ § **5.2**

• observables (σ and Γ)

$\boxed{\text{§ 5.10, here.}}$

Summary and Feynman rule

§ 5.10 $\quad \langle p_3, p_4; \text{out}|p_1, p_2; \text{in}\rangle = \dfrac{(\text{LSZ factor}) \times (\text{3N})}{(\text{3D})}.$

$$(\text{3D}) = 1 + \mathcal{O}(\lambda).$$

$$(\text{3N}) = \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \mathcal{O}(\lambda).$$

$$\phi_1\phi_2 = D_F(x_1 - x_2).$$

$$(\text{LSZ factor})_i \times D_F(x_i - y) = e^{\mp ip_i\cdot y}. \text{———————— (4).}$$

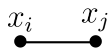
← last week

→ today

► In the case of $\phi_1\phi_2 = D_F(x_1 - x_2)$, both x_1 and x_2 are at the external lines, so

$$i\int d^4x_i e^{-ip_2\cdot x_2}\left(\Box_2 + m^2\right) \underbrace{i\int d^4x_i e^{-ip_1\cdot x_1}\left(\Box_1 + m^2\right) D_F(x_1 - x_2)}_{e^{-ip_1\cdot x_2}}$$

$$= i\int d^4x_i e^{-ip_2\cdot x_2}\underbrace{\left(-p_1^2 + m^2\right)}_{=0} e^{-ip_1\cdot x_2} \qquad = 0.$$

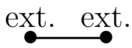
$\boxed{\textbf{POINT}}$ If two external points are directly connected,  $= 0.$

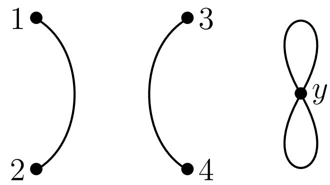
► Thus, $\mathcal{O}(\lambda^0)$ term in (LSZ)×(3N) = 0.

► Next, $\mathcal{O}(\lambda)$ term in (3N)

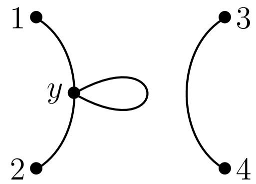
$$= {}_I\langle 0|\mathrm{T}\left(\phi_1\phi_2\phi_3\phi_4(-i)\int d^4y \frac{\lambda}{24}\phi_y\phi_y\phi_y\phi_y\right)|0\rangle_I$$

(4 pairs = 105 combinations)

= terms including  (= 0)

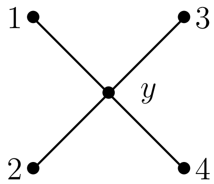


$$\phi_1\phi_2\phi_3\phi_4 \cdot \phi_y\phi_y\phi_y\phi_y \quad \text{etc,} \quad 9 \text{ terms}$$



$$\phi_1\phi_2\phi_3\phi_4 \cdot \phi_y\phi_y\phi_y\phi_y \quad \text{etc,} \quad 72 \text{ terms}$$

$$+ \quad \phi_1\phi_2\phi_3\phi_4\frac{-i\lambda}{24}\int d^4y\phi_y\phi_y\phi_y\phi_y \qquad 4! = 24 \text{ terms} \qquad \text{(in total 105 terms)}$$



$\left(24 \text{ of } \frac{\lambda}{4!} = \frac{\lambda}{24} \text{ was introduced for this.}\right)$

$$= (-i\lambda)\int d^4y D_F(x_1 - y)D_F(x_2 - y)D_F(x_3 - y)D_F(x_4 - y).$$

- Thus, from (4),

$$\begin{aligned}
&\mathcal{O}(\lambda) \text{ term in (LSZ)} \times \text{(3N)} \\
&= (-i\lambda) \int d^4y e^{-ip_1\cdot y} e^{-ip_2\cdot y} e^{+ip_3\cdot y} e^{+ip_4\cdot y} \\
&= (-i\lambda)(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \qquad \text{———————————}(5).
\end{aligned}$$

- Thus,

$$\begin{aligned}
\langle p_3, p_4; \text{out} | p_1, p_2; \text{in} \rangle &= \frac{\text{(LSZ)} \times \text{(3N)}}{\text{(3D)}} \\
&= \frac{\overbrace{\mathcal{O}(\lambda^0)}^{=0} + \overbrace{\mathcal{O}(\lambda)}^{=(5)} + \mathcal{O}(\lambda^2) + \cdots}{1 + \mathcal{O}(\lambda) + \cdots}, \\
&= (-i\lambda)(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) + \mathcal{O}(\lambda^2)
\end{aligned}$$

- Thus, from (2), we eventually obtain the amplitude at the leading order,

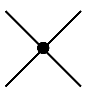
$$\underline{i\mathcal{M}(p_1, p_2 \to p_3, p_4) = -i\lambda} \qquad + \mathcal{O}(\lambda^2)$$

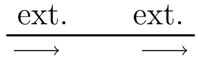
- and substituting it to (1), the cross section

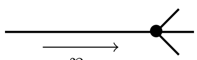
$$\begin{aligned}
\sigma(p_1, p_2 \to \phi\phi) &= \frac{1}{128\pi} \frac{1}{E_1^2} \underbrace{\int \frac{d\Omega}{4\pi}}_{=1} \underbrace{|\mathcal{M}|^2}_{=\lambda^2} \\
&= \underline{\frac{\lambda^2}{128\pi} \cdot \frac{1}{E_1^2}}. \\
&= 2.5 \times 10^{-3} \lambda^2 \text{GeV}^{-2} \left(\frac{\text{GeV}}{E_1} \right)^2 \\
&= 1.0 \times 10^{-30} \lambda^2 \text{cm}^2 \left(\frac{\text{GeV}}{E_1} \right)^2.
\end{aligned}$$

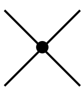
-

Feynman rules for $\mathcal{M}$

$i\mathcal{M} =$ diagrams $=$  $+\cdots$

(1) diagram with ext. $\xrightarrow{p_i}$ ext. $\xrightarrow{p_j}$  $= 0$.

(2) external line  $\xrightarrow{p_i}$ $= 1$.

(3) vertex  $= -i\lambda$.

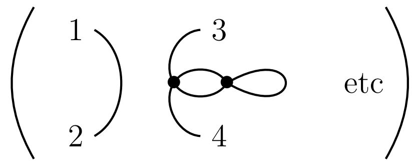
(cont'd)

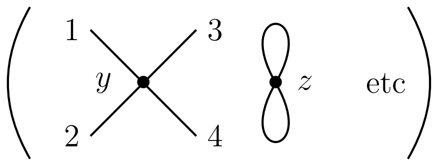
► Higher order terms:

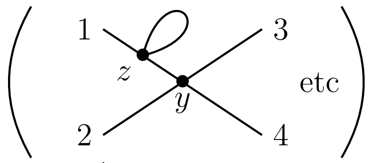
$\mathcal{O}(\lambda^2)$ term in (3N)

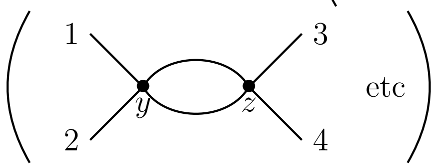
$$= {}_I\langle 0|\mathrm{T}\left(\phi_1\phi_2\phi_3\phi_4\frac{(-i)^2}{2}\int d^4y\frac{\lambda}{24}\phi_y^4\int d^4z\frac{\lambda}{24}\phi_z^4\right)|0\rangle_I$$

(6 pairs $\to$ 10395 combinations!)

$=$ terms with $\overset{\text{ext.}}{\underset{p_i}{\longrightarrow}}$ —— $\overset{\text{ext.}}{\underset{p_j}{\longrightarrow}}$  $\to 0$

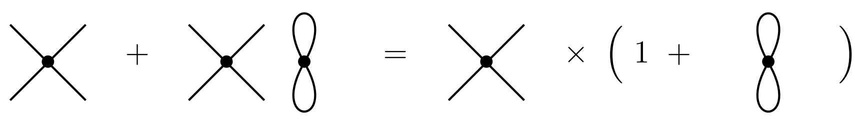
$+$ terms with bubbles 

$+$ terms with loops at external lines 

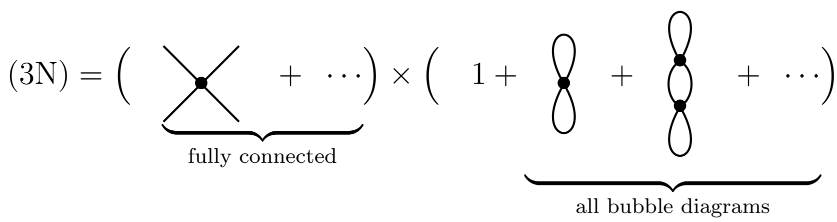
$+$ other loops  .

► In general, terms with loop diagrams are often divergent, and requires "renormalization". Here, we just give qualitative discussion.

► Terms with bubbles are, together with the leading order term,

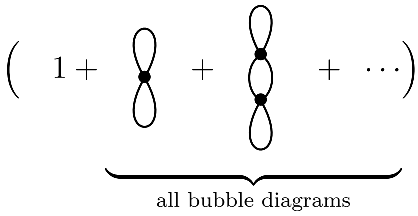


In general,



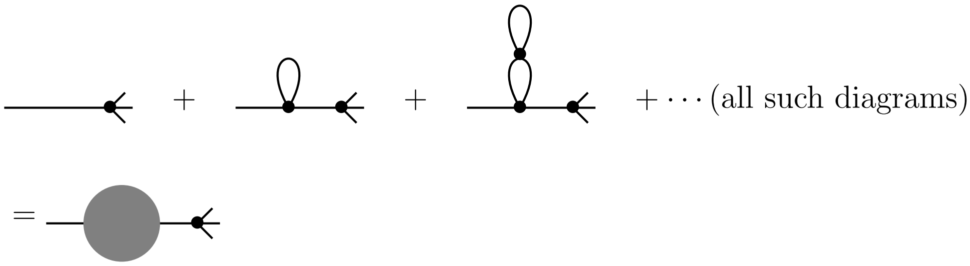
On the other hand,

$$(\mathrm{3D}) = {}_I\langle 0|\mathrm{T}\left(\exp\left[-i\int \frac{\lambda}{24}\phi_I^4\right]\right)|0\rangle_I =$$



Therefore, <u>all bubble diagrams cancel out between the numerator and the denominator.</u>

► <u>Loops in the external line</u>



$$= \frac{iZ}{p^2 - m^2}.$$

The factor Z is absorbed by the "field renormalization." (In general, the mass "m" here is also different from the parameter "m" in the Lagrangian.) We do not discuss it here.

►

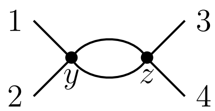
Feynman rules (cont'd)

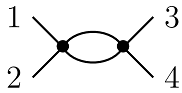
(4) Ignore the bubble diagrams.

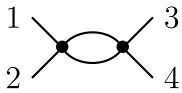
(5) We can also ignore the loops in the external lines if we take into account the renormalization.

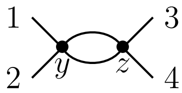
► The other loops.

Example:



The  term in $\langle \text{out}|\text{in}\rangle$

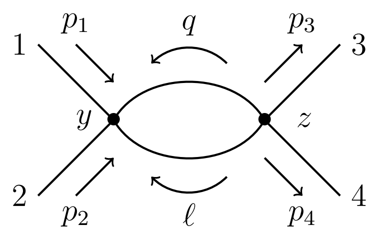
$= (\text{LSZ}) \times$ the  term in (3N)

$= (\text{LSZ}) \times$ the  term in ${}_I\langle 0|\mathrm{T}\left(\phi_1\phi_2\phi_3\phi_4 \frac{(-i)^2}{2}\int d^4y \frac{\lambda}{24}\phi_y^4 \int d^4z \frac{\lambda}{24}\phi_z^4\right)|0\rangle_I$

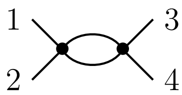
$$
\begin{aligned}
&= (\text{LSZ}) \times \frac{1}{2}\left(\frac{-i\lambda}{24}\right)^2 \int d^4y \int d^4z \;\; \phi_1\phi_2 \cdot \phi_y\phi_y\phi_y\phi_y \cdot \phi_z\phi_z\phi_z\phi_z \cdot \phi_3\phi_4 \\
&\qquad \times 12 \;(1,2 \leftrightarrow y^4 \text{ combinations}) \\
&\qquad \times 12 \;(3,4 \leftrightarrow z^4 \text{ combinations}) \\
&\qquad \times 2 \;(\text{remaining } y^2 \leftrightarrow z^2 \text{ combinations}) \\
&\qquad \times 2 \;(\text{replacing } y \leftrightarrow z) \\
&= (\text{LSZ}) \times \frac{1}{2}(-i\lambda)^2 \int d^4y \int d^4z \; D_F(x_1-y)D_F(x_2-y)D_F(x_3-z)D_F(x_4-z)D_F(y-z)^2 \\
&= \frac{1}{2}(-i\lambda)^2 \int d^4y \int d^4z \; e^{-ip_1\cdot y}e^{-ip_2\cdot y}e^{+ip_3\cdot z}e^{+ip_4\cdot z} D_F(y-z)^2 \qquad (\because\ (4)) \\
&= \frac{1}{2}(-i\lambda)^2 \int d^4y \int d^4z \; e^{-ip_1\cdot y}e^{-ip_2\cdot y}e^{+ip_3\cdot z}e^{+ip_4\cdot z} \\
&\qquad \times \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}e^{-iq\cdot(y-z)} \int \frac{d^4\ell}{(2\pi)^4}\frac{i}{\ell^2-m^2+i\epsilon}e^{-i\ell\cdot(y-z)}
\end{aligned}
$$

Here

$$
\begin{cases}
\displaystyle\int d^4y e^{-i(p_1+p_2+q+\ell)\cdot y} = (2\pi)^4\delta^{(4)}(p_1+p_2+q+\ell) \\
\displaystyle\int d^4z e^{+i(p_3+p_4+q+\ell)\cdot y} = (2\pi)^4\delta^{(4)}(p_3+p_4+q+\ell)
\end{cases}
$$

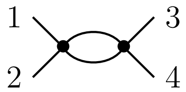


which represents the $\boxed{\text{momentum conservation}}$ at each vertex.

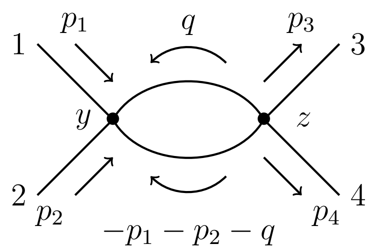
Thus, the  term in $\langle \text{out}|\text{in}\rangle$

$$
\begin{aligned}
&= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon} \int \frac{d^4\ell}{(2\pi)^4}\frac{i}{\ell^2-m^2+i\epsilon} \\
&\qquad \times (2\pi)^4\delta^{(4)}(p_1+p_2+q+\ell)(2\pi)^4\delta^{(4)}(p_3+p_4+q+\ell) \\
&= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon} \cdot \frac{i}{(-p_1-p_2-q)^2-m^2+i\epsilon} \underbrace{(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)}_{\text{the factor in (2)}}
\end{aligned}
$$

and finally, from (2),

The 

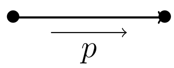
term in $i\mathcal{M} = \underline{\frac{1}{2}(-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \cdot \frac{i}{(-p_1 - p_2 - q)^2 - m^2 + i\epsilon}}$.



►

Feynman rules (cont'd)

(6) Momentum conservation at each vertex.

(7) Internal line 

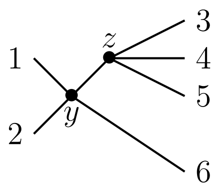
$= \dfrac{i}{p^2 - m^2 + i\epsilon}$.

(8) Loop momentum should be integrated by $\displaystyle\int \frac{d^4p}{(2\pi)^4}$.

(9) Multiply the "symmetry factor" (such as 1/2 in the above example).

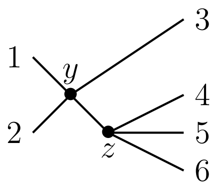
► Note that, if there is no loop, there is no symmetry factor and all coefficients cancel:

Example: $2 \to 4$ scattering.



$$
\begin{aligned}
&= (\text{LSZ}\times)\frac{1}{2}\left(\frac{-i\lambda}{24}\right)^2 \int d^4y \int d^4z\ \phi_1\phi_2 \cdot \phi_y\phi_y\phi_y\phi_y \cdot \phi_z\phi_z\phi_z\phi_z \cdot \phi_3\phi_4\phi_5\phi_6 \\
&\qquad \times 4!\ (y \text{ contraction}) \\
&\qquad \times 4!\ (z \text{ contraction}) \\
&\qquad \times 2\ (y \leftrightarrow z) \\
&= (\text{LSZ}\times)(-i\lambda)^2 \int d^4y \int d^4z D_F(x_1-y)D_F(x_2-y)D_F(x_6-y) \\
&\qquad\qquad\qquad\qquad\qquad \times D_F(x_3-z)D_F(x_4-z)D_F(x_5-z)D_F(y-z) \\
&= \cdots \\
&= \underbrace{(-i\lambda)^2\frac{i}{(p_1+p_2-p_6)^2-m^2-i\epsilon}}_{\text{the corresponding term in } i\mathcal{M}} \times (2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4-p_5-p_6).
\end{aligned}
$$

- Different diagrams give different terms: for instance,

The 

term in $i\mathcal{M} \;=\; (-i\lambda)^2\dfrac{i}{(p_1+p_2-p_3)^2-m^2-i\epsilon}$.

- (That's all for this semester. Thank you for your attendance!)

References

[1] M. Srednicki, Quantum Field Theory.

[2] M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory.

[3] S. Weinberg, The Quantum Theory of Fields volume I.

[4] M. D. Schwartz, Quantum Field Theory and the Standard Model.

[5] 「ゲージ場の量子論 I」九後汰一郎、培風館.

[6] 「場の量子論」坂井典佑、裳華房.