2017年度夏学期　場の量子論 I（浜口）期末試験・解答例

2017 S-semester, Quantum Field Theory I (Hamaguchi), exam: Example Answers

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[1-1] From the lecture note,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a(\mathbf{p})e^{-ip\cdot x} + a^\dagger(\mathbf{p})e^{ip\cdot x}\right)$$

[1-2]

$$\begin{aligned}
[\phi(x), \dot\phi(y)] &= \int \frac{d^3p}{(2\pi)^3}\frac{i}{2}\left(e^{-ip\cdot(x-y)} + e^{ip\cdot(x-y)}\right)_{p^0=E_p} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{i}{2}\left(e^{-iE_p(x^0-y^0)}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{iE_p(x^0-y^0)}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right)
\end{aligned}$$

Note that, due to the factor $e^{\pm iE_p(x^0-y^0)}$ which depends on $\mathbf{p}$ though $E_p = \sqrt{\mathbf{p}^2+m^2}$, one can *not* further use $\int d^3p\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = (2\pi)^3\delta^{(3)}(\mathbf{x}-\mathbf{y})$ here. $\left([\phi(x), \dot\phi(y)] \neq \cos[E_p(x^0-y^0)]i\delta^{(3)}(\mathbf{x}-\mathbf{y}).\right)$

(**Derivation 1**) From the commutation relations of a and $a^\dagger$,

$$\begin{aligned}
[\phi(x), \dot\phi(y)] &= \left[\int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(a(\mathbf{p})e^{-ip\cdot x} + a^\dagger(\mathbf{p})e^{ip\cdot x}\right), \int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\left(-iq^0a(\mathbf{q})e^{-iq\cdot y} + iq^0a^\dagger(\mathbf{q})e^{iq\cdot y}\right)\right]_{p^0=E_p,q^0=E_q} \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\left(iq^0\underbrace{[a(\mathbf{p}), a^\dagger(\mathbf{q})]}_{(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})}e^{-ip\cdot x}e^{iq\cdot y} - iq^0\underbrace{[a^\dagger(\mathbf{p}), a(\mathbf{q})]}_{-(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})}e^{ip\cdot x}e^{-iq\cdot y}\right)_{p^0=E_p,q^0=E_q} \\
&= \text{(equation above)}
\end{aligned}$$

(**Derivation 2**) From the result obtained in the lecture (§1.4.8),

$$\begin{aligned}
[\phi(x), \phi(y)] &= \int \frac{d^3p}{(2\pi)^3 2E_p}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)_{p^0=E_p} \\
\therefore\ [\phi(x), \dot\phi(y)] = \frac{\partial}{\partial y^0}[\phi(x), \phi(y)] &= \int \frac{d^3p}{(2\pi)^3}\frac{i}{2}\left(e^{-ip\cdot(x-y)} + e^{ip\cdot(x-y)}\right)_{p^0=E_p}
\end{aligned}$$

[1-3] From [1-2],

$$[\phi(x), \dot\phi(y)]_{x^0=y^0} = \int \frac{d^3q}{(2\pi)^3}\frac{i}{2}\left(e^{+i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} + e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}\right) = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

[2-1]

$$\begin{aligned}
\varphi_I(t,\mathbf{x}) &= e^{iH_0t}\varphi(0,\mathbf{x})e^{-iH_0t}, \\
\chi_I(t,\mathbf{x}) &= e^{iH_0t}\chi(0,\mathbf{x})e^{-iH_0t},
\end{aligned}$$

where

$$H_0 = \int d^3x\left(\frac{1}{2}\dot\varphi(0,\mathbf{x})^2 + \frac{1}{2}\nabla\varphi(0,\mathbf{x})^2 + \frac{1}{2}M^2\varphi(0,\mathbf{x})^2 + \frac{1}{2}\dot\chi(0,\mathbf{x})^2 + \frac{1}{2}\nabla\chi(0,\mathbf{x})^2 + \frac{1}{2}m^2\chi(0,\mathbf{x})^2\right)$$

They follow EOM of free field, i.e., the Klein-Gordon eq. $(\Box + M^2)\varphi = 0$ and $(\Box + m^2)\chi = 0$. The Heisenberg fields are given by

$$\varphi(t, \mathbf{x}) = e^{iHt}\varphi(0, \mathbf{x})e^{-iHt},$$
$$\chi(t, \mathbf{x}) = e^{iHt}\chi(0, \mathbf{x})e^{-iHt},$$

with

$$H = H_0 + \int d^3x \left(\frac{f}{2}\varphi(0, \mathbf{x})\chi(0, \mathbf{x})^2 + \frac{1}{24}\lambda\chi(0, \mathbf{x})^4 \right)$$

and does not follow the free field EOM.

(χ^4 項を H_0 に含めたものもよしとする。／ The χ^4 term can be included in H_0.)

[2-2] (試験後追加コメント：すみません。ここで $\lambda \to 0$ として λ を無視するか、あるいは $H_I(t)$ に $\lambda\chi^4$ 項を加えておくべきでした。／ Note added after the exam: I am sorry, I should have taken $\lambda \to 0$ here, or included $\lambda\chi^4$ term in $H_I(t)$.)

From the lecture note,

$$\varphi(x) = u^\dagger(t)\varphi_I(x)u(t), \qquad t = x^0,$$

where

$$u(t) = U(t, 0) = \mathrm{T}\left[\exp\left(-i\int_0^t H_I(t')dt'\right)\right] \quad \text{for } t > 0,$$

$$u(t) = U(0, t)^\dagger = \sout{\mathrm{T}\left[\exp\left(i\int_t^0 H_I(t')dt'\right)\right]} \quad \text{for } t < 0$$

$$= \widetilde{\mathrm{T}}\left[\exp\left(i\int_t^0 H_I(t')dt'\right)\right] \qquad \widetilde{\mathrm{T}}: \text{ anti-time ordering} \quad \text{(corrected: 2018 July 26.)}$$

$$= \widetilde{\mathrm{T}}\left[\exp\left(-i\int_0^t H_I(t')dt'\right)\right]$$

Expanding it up to $\mathcal{O}(f)$,

$$\varphi(x) = \left(1 + i\int_0^t H_I(t')dt'\right)\varphi_I(x)\left(1 - i\int_0^t H_I(t')dt'\right) + \mathcal{O}(f^2)$$
$$= \varphi_I(x) + i\int_0^t [H_I(t'), \varphi_I(x)]dt' + \mathcal{O}(f^2)$$

Note that

(i) $\varphi_I(t', \mathbf{x})$ in $H_I(t')$ and $\varphi_I(x) = \varphi_I(t, \mathbf{x})$ are *not* at equal time ($t' \neq t$) and therefore the equal-time commutation relation cannot be used here. ($[H_I(t'), \varphi_I(x)] \neq 0$.)

(ii) $\varphi(x) = e^{iHt}e^{-iH_0t}\varphi_I(x)e^{iH_0t}e^{-iHt}$, but $\underline{e^{iHt}e^{-iH_0t} \neq e^{i(H-H_0)t}}$ and therefore $\underline{\varphi(x) \neq e^{i(H-H_0)t}\varphi_I(x)e^{-i(H-H_0)t}}$. In general $e^{A+B} \neq e^A e^B$ for $[A, B] \neq 0$.

(iii) $\underline{\int_0^t H_I(t')dt' \neq H_I(t)t}$ and therefore $\underline{\mathrm{T}(e^{i\int_0^t H_I(t')dt'})\varphi_I(x)\mathrm{T}(e^{-i\int_0^t H_I(t')dt'}) \neq e^{iH_It}\varphi_I(x)e^{-iH_It}}$ and $\underline{\int_0^t [H_I(t'), \varphi_I(x)]dt' \neq [H_I(t), \varphi_I(x)]\, t}$.

[2-3] From lecture note,

$$\langle 0|\varphi(x_1)\varphi(x_2)\chi(x_3)\chi(x_4)|0\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{{}_I\langle 0|\mathrm{T}\left[\varphi_I(x_1)\varphi_I(x_2)\chi_I(x_3)\chi_I(x_4)\exp\left(-i\int_{-t}^{t} H_I(t')dt'\right)\right]|0\rangle_I}{{}_I\langle 0|\mathrm{T}\left[\exp\left(-i\int_{-t}^{t} H_I(t')dt'\right)\right]|0\rangle_I}$$

(本当はここの H_I も χ^4 項を含むはずですね。すみません。／The H_I here should have included the χ^4 term as well. I am sorry.)

[2-4] (a) $\mathcal{O}(f)$ term is like ${}_I\langle 0|\varphi_I\varphi_I\chi_I\chi_I \cdot \frac{f}{2}\varphi_I\chi_I\chi_I|0\rangle_I$ and contains odd number of φ_I. From Wick's theorem, this vanishes.

[2-4] (b)

$$(-if)^2\int d^4y\int d^4z D_F(x_1-y,M)D_F(x_2-z,M)D_F(y-z,m) \\ \times\Big(D_F(x_3-y,m)D_F(x_4-z,m)+D_F(x_4-y,m)D_F(x_3-z,m)\Big)$$

[2-5]

$$\langle\chi(p_3)\chi(p_4);\mathrm{out}|\varphi(p_1)\varphi(p_2);\mathrm{in}\rangle = (-if)^2\left(\frac{i}{(p_1-p_3)^2-m^2}+\frac{i}{(p_1-p_4)^2-m^2}\right)(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)$$

$$i\mathcal{M}\Big(\varphi(p_1)\varphi(p_2)\to\chi(p_3)\chi(p_4)\Big) = (-if)^2\left(\frac{i}{(p_1-p_3)^2-m^2}+\frac{i}{(p_1-p_4)^2-m^2}\right)$$

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[3] Represented by 3-components field. Its representation matrices for the rotation around the z-axis and the boost in the z-directions are given by

$$D(\Lambda)=\begin{pmatrix} e^{i\theta_3} & & \\ & 1 & \\ & & e^{-i\theta_3}\end{pmatrix},\quad D(\Lambda)=\begin{pmatrix} -e^{\eta_3} & & \\ & 1 & \\ & & e^{\eta_3}\end{pmatrix}.$$

(**Derivation**) For $(A,B)=(0,1)$, $(2A+1)(2B+1)=1\cdot 3=3$, and hence 3 components. In this case, from the discussion in the lecture note, $D(A_3)=0_{3\times 3}$ and $D(B_3)=\mathrm{diag}(1,0,-1)$. ($D(B_3)\Phi_b=b\Phi_b$, $b=-1,0,1$). Thus, $D(J_3)=D(A_3)+D(B_3)=\mathrm{diag}(1,0,-1)$ and $D(K_3)=-iD(A_3)+iD(B_3)=\mathrm{diag}(i,0,-i)$. From this and $D(\Lambda)=\exp(i\theta_i D(J_i)+i\eta_i D(K_i))$,

- rotation around the z-axis: $D(\Lambda)=\exp(i\theta_3 D(J_3))=\exp(\mathrm{diag}(i\theta_3,0,-i\theta_3))=(e^{i\theta_3},1,e^{-i\theta_3})$,
- boost in the z-direction: $D(\Lambda)=\exp(i\eta_3 D(K_3))=\exp(\mathrm{diag}(-\eta_3,0,-\eta_3))=(e^{-\eta_3},1,e^{\eta_3})$.

You may use a different basis: $D(B_i)_{jk}=-i\epsilon_{ijk}=\left\{\begin{pmatrix}0 & & \\ & & -i\\ & i & \end{pmatrix},\begin{pmatrix} & & i\\ & 0 & \\ -i & & \end{pmatrix},\begin{pmatrix} & -i & \\ i & & \\ & & 0\end{pmatrix}\right\}$, which does not give the relation used in the lecture $D(B_3)\Phi_b=b\Phi_b$ but satisfies the same SU(2) algebra $[D(B_i),D(B_j)]=i\epsilon_{ijk}D(B_k)$. In this case, $D(J_3)=D(B_3)$ and $D(K_3)=iD(B_3)$ lead to

- rotation around the z-axis: $D(\Lambda)=\exp(i\theta_3 D(J_3))=\exp\begin{pmatrix} & \theta_3 & \\ -\theta_3 & & \\ & & 0\end{pmatrix}=\begin{pmatrix}\cos\theta_3 & \sin\theta_3 & \\ -\sin\theta_3 & \cos\theta_3 & \\ & & 1\end{pmatrix}$,

- boost in the z-direction: $D(\Lambda) = \exp(i\eta_3 D(K_3)) = \exp\begin{pmatrix} & i\eta_3 & \\ -i\eta_3 & & \\ & & 0 \end{pmatrix} = \begin{pmatrix} \cosh\eta_3 & i\sinh\eta_3 & \\ -i\sinh\eta_3 & \cosh\eta_3 & \\ & & 1 \end{pmatrix}$.

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[4] (i) For $\theta_3 \neq 0$, $\theta_1 = \theta_2 = \eta_k = 0$.

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\begin{aligned}
LHS &= D_R^\dagger(\Lambda)\sigma^\mu D_R(\Lambda) \\
&= \begin{pmatrix} e^{-i\theta_3/2} & \\ & e^{+i\theta_3/2} \end{pmatrix} \sigma^\mu \begin{pmatrix} e^{i\theta_3/2} & \\ & e^{i\theta_3/2} \end{pmatrix} = \left\{ I, \begin{pmatrix} 0 & e^{-i\theta_3} \\ e^{i\theta_3} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -ie^{-i\theta_3} \\ ie^{i\theta_3} & 0 \end{pmatrix}, \sigma_3 \right\} \\
RHS &= \Lambda^\mu{}_\nu \sigma^\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta_3 & \sin\theta_3 & \\ & -\sin\theta_3 & \cos\theta_3 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} I \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \left\{ I, \begin{pmatrix} 0 & e^{-i\theta_3} \\ e^{i\theta_3} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -ie^{-i\theta_3} \\ ie^{i\theta_3} & 0 \end{pmatrix}, \sigma_3 \right\}
\end{aligned}
$$

(ii) For $\eta_3 \neq 0$, $\eta_1 = \eta_2 = \theta_k = 0$,

$$
\begin{aligned}
LHS &= D_R^\dagger(\Lambda)\sigma^\mu D_R(\Lambda) \\
&= \begin{pmatrix} e^{\eta_3/2} & \\ & e^{-\eta_3/2} \end{pmatrix} \sigma^\mu \begin{pmatrix} e^{\eta_3/2} & \\ & e^{-\eta_3/2} \end{pmatrix} = \left\{ \begin{pmatrix} e^{\eta_3} & \\ & e^{-\eta_3} \end{pmatrix}, \sigma_1, \sigma_2, \begin{pmatrix} e^{\eta_3} & \\ & -e^{-\eta_3} \end{pmatrix} \right\} \\
RHS &= \Lambda^\mu{}_\nu \sigma^\nu = \begin{pmatrix} \cosh\eta_3 & & & \sinh\eta_3 \\ & 1 & & \\ & & 1 & \\ \sinh\eta_3 & & & \cosh\eta_3 \end{pmatrix} \begin{pmatrix} I \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \left\{ \begin{pmatrix} e^{\eta_3} & \\ & e^{-\eta_3} \end{pmatrix}, \sigma_1, \sigma_2, \begin{pmatrix} e^{\eta_3} & \\ & -e^{-\eta_3} \end{pmatrix} \right\}
\end{aligned}
$$