2017 S-semester

Quantum Field Theory

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Contents

§ 0 Introduction

about this lecture

(i) **Language**

(The rule of the department: For the graduate course, as far as there is one international student who prefers English, the lecture should be given in English. But for the undergraduate course, the lectures are usually given in Japanese. Now, this is a common lecture for both students, and there is no clear rule. . . .

Which one do you prefer?

		Speaking	
		E	J
Writing	E		(*)
	J		-

. . . (*) We choose this option.)

(ii) **Web page**

Google: Koichi Hamaguchi → Lectures → Quantum Field Theory I

- All the announcements will also be given in this web page.
- The lecture note will also be updated (every week, hopefully).

(iii) **Schedule**

April 10, 17, 24,
May 1, 8, 15, 29, (no class on May 22)
June 5, 12, 19, 26,
July 3, 10, **Exam on July 24.**
(maybe an extra class on July 31.)
(the extra class is an bonus lecture after the exam and irrelevant to the grades)

(I don't check the attendance. You don't have to attend the classes if you can learn by yourself, submit the homework problems, and attend the exam.)

(iv) **Grades**

based on the scores of homework problems (twice?) and the exam on July 24.
In the exam, you can bring notes, textbooks, laptop, etc.

(v) **Textbooks**

This course is not based on a specific textbook, but I often refer to the following textbooks during preparing the lecture note.

- M. Srednicki, Quantum Field Theory.
- M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory.
- S. Weinberg, The Quantum Theory of Fields volume I.

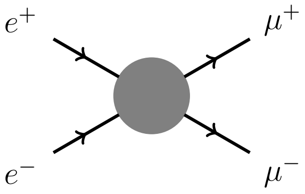
§ 0.1 Course objectives

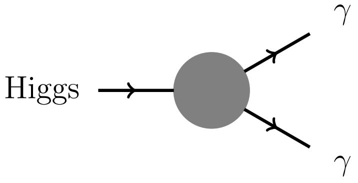
To learn the basics of Quantum Field Theory (QFT).

One of the goals is to understand how to calculate the transition probabilities (such as the cross section and the decay rate) in QFT. (See § 0.5.)

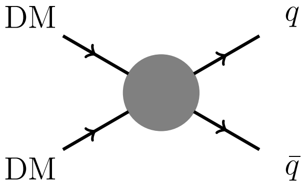
Examples

- at colliers





- in the early universe



§ 0.2 Quantum mechanics and quantum field theory

Quantum Field Theory (QFT) is just Quantum Mechanics (QM) applied to fields.

- QM: $q_i(t)$ $\quad i = 1, 2, \cdots$ discrete
- QFT: $\phi(\vec{x}, t)$ $\quad \vec{x} \cdots$ continous (infinite number of degrees of freedom)
  (Note: uncountably infinite, 非可算無限)

	QM	QFT
operators (Heisenberg picture)	$q_i(t), p_i(t)$ or $q_i(t), \dot{q}_i(t)$ $i = 1, 2, \cdots$ discrete $[q_i, p_j] = i\hbar\delta_{ij}$	$\phi(\vec{x}, t), \pi(\vec{x}, t)$ or $\phi(\vec{x}, t), \dot{\phi}(\vec{x}, t)$ $\vec{x} \cdots$ continous $[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y})$
states	(e.g., Harmonic Oscillator) $\|0\rangle$: ground state $a^\dagger \|0\rangle, a^\dagger a^\dagger \|0\rangle, \cdots$ $a^\dagger$ written in terms of q and p	$\|0\rangle$: ground state $a^\dagger_{\vec{p}} \|0\rangle, a^\dagger_{\vec{p}} a^\dagger_{\vec{p}'} \|0\rangle, \cdots$ $a^\dagger_{\vec{p}}$ written in terms of ϕ and π
observables (expectation value)	$\langle \cdot \| \vec{p} \| \cdot \rangle, \langle \cdot \| H \| \cdot \rangle, \cdots$	$\langle \cdot \| \vec{p} \| \cdot \rangle, \langle \cdot \| H \| \cdot \rangle, \cdots$
transition probability	$P(i \to f) = \|\langle f \| i \rangle\|^2$	$P(i \to f) = \|\langle f \| i \rangle\|^2$

In this lecture, we focus on the relativistic QFT.
(QFT can also be applied to non-relativistic system: condensed matter, bound state,. . .)

Relativistic QFT is based on QM and SR (special relativity).

- QM: $\hbar \neq 0$ (important at small scale)
- SR: $c < \infty$ (important at large velocity)
- QFT: $\hbar \neq 0$ and $c < \infty$
  (physics at small scale & large velocity: Particle Physics, Early Universe,. . .)

§ 0.3 Notation and convention

- We will use the natural units

$$\hbar = c = 1\,,$$

where

$$\begin{aligned} &\hbar \simeq 1.055 \times 10^{-34}\ \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{sec}^{-1}\,, \\ &c = 2.998 \times 10^{8}\ \mathrm{m} \cdot \mathrm{sec}^{-1}\,. \end{aligned}$$

For instance, we write

  - $E^2 = p^2 + m^2$ instead of $E^2 = p^2c^2 + m^2c^4$, and
  - $[x, p] = i$ instead of $[x, p] = i\hbar$.

- We will use the following metric .

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

(The sign convention depends on the textbook. $g_{\mu\nu}(\text{here}) = g_{\mu\nu}^{\text{Peskin}} = -g_{\mu\nu}^{\text{Srednicki}}$)

$$\begin{aligned} x^\mu &= (x^0, x^1, x^2, x^3) = (t, \vec{x}) \\ x_\mu &= (x_0, x_1, x_2, x_3) = g_{\mu\nu}x^\nu = (t, -\vec{x}) \\ p^\mu &= (p^0, p^1, p^2, p^3) = (E, \vec{p}) \\ p_\mu &= g_{\mu\nu}p^\nu = (E, -\vec{p}) \\ p \cdot x &= p_\mu x^\nu = p^\mu x_\nu \\ &= p^0x^0 - p^1x^1 - p^2x^2 - p^3x^3 \\ &= p^0x^0 - \vec{p} \cdot \vec{x} \\ &= Et - \vec{p} \cdot \vec{x} \end{aligned}$$

If p^μ is the 4-momentum of a particle with mass m,

$$\begin{aligned} p^2 = p^\mu p_\mu &= (p^0)^2 - |\vec{p}|^2 = E^2 - |\vec{p}|^2 \\ &= m^2 \,. \end{aligned}$$

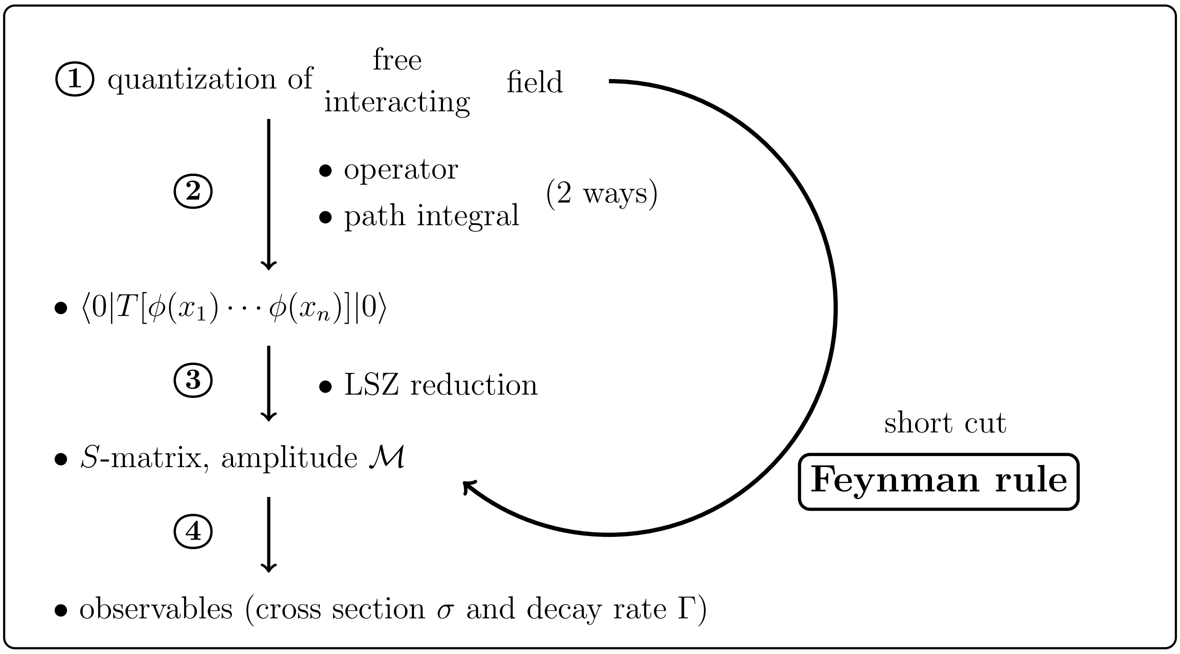
§ 0.4 Various fields

		spin	equation of motion for free fields	
scalar field	$\phi(x)$	0	$(\Box + m^2)\phi = 0$	Klein-Gordon eq.
fermionic field	$\psi_\alpha(x)$	1/2	$(i\gamma^\mu\partial_\mu - m)\psi = 0$	Dirac eq.
gauge field	$A_\mu(x)$	1	$\partial^\mu F_{\mu\nu} = 0$ $(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$	(part of) Maxwell eq.

We start from the scalar field.

> The Standard Model of Particle Physics is also written in terms of QFT:
> - quarks (u, d, s, c, b, t) and leptons (e, μ, τ, ν_i) ... fermionic fields
> - γ (photon), $W^\pm$, Z (weak bosons), g (gluon) ... gauge fields
> - H (Higgs) ... scalar field

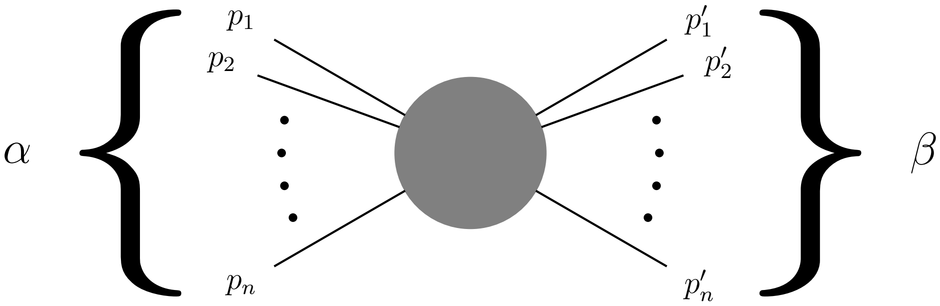
§ 0.5 Outline: what we will learn



- ▶ First, we will learn ①②③... with a scalar field. (→ next, fermionic field, ...)
- ▶ A long way to go,... Today, let's discuss ④ in advance.

§ 0.6 S-matrix, amplitude $\mathcal{M} \Longrightarrow$ observables (σ and Γ)

Let's consider the probability of the following process, $P(\alpha \to \beta)$.



- If the initial and final states are normalized as $\langle \alpha | \beta \rangle = \delta_{\alpha\beta}$, then

$$P(\alpha \to \beta) = |\langle \beta, \text{out} | \alpha, \text{in} \rangle|^2$$

(The meaning of "in" and "out" will be explained later.)

- We are interested in states with fixed momenta.

$$|\alpha\rangle = |\sigma_1, \vec{p}_1, \sigma_2, \vec{p}_2, \cdots \sigma_n, \vec{p}_n\rangle \ ,$$

(σ_i ; spins and other quantum numbers of the particle i)

Let's consider one-particle state $|\sigma, \vec{p}\rangle$.
Since the momentum $\vec{p}$ is continuous, we cannot normalize the states as $\langle \alpha | \beta \rangle = \delta_{\alpha\beta}$.
Instead, we normalize it as

$$\langle \sigma, \vec{p} | \sigma', \vec{q} \rangle = (2\pi)^3 2E_p \delta^{(3)}(\vec{p} - \vec{q}) \delta_{\sigma\sigma'} \ . \quad \text{———————} (1)$$

Comments

(i) What's the mass dimension of $|\sigma, \vec{p}\rangle$ then? (It's -1.)

(ii) Why $E_p \delta^{(3)}(\vec{p} - \vec{q})$, not just $\delta^{(3)}(\vec{p} - \vec{q})$?
$\to E_p \delta^{(3)}(\vec{p} - \vec{q})$ is Lorentz invariant.
For instance, for a boost Lorentz transformation along the z direction,

$$\begin{pmatrix} E' \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E \\ p_z \end{pmatrix}, \qquad \beta = v/c, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}},$$

one can show that (check it yourself)

$$E' \delta(p'_z - q'_z) = E \delta(p_z - q_z).$$

▶ **S-matrix:** The transition amplitude

$$\boxed{\langle \sigma_1', \vec{p'}_1, \cdots \sigma_m', \vec{p'}_m \,;\text{out}|\sigma_1, \vec{p}_1, \cdots \sigma_n, \vec{p}_n \,;\text{in}\rangle = \langle \sigma_1', \vec{p'}_1, \cdots \sigma_m', \vec{p'}_m|S|\sigma_1, \vec{p}_1, \cdots \sigma_n, \vec{p}_n\rangle}$$

is called S-matrix.

Comments

(i) The definition of in and out-states will be given later.

(ii) S-matrix is Lorentz invariant.

(iii) Why "matrix"?

$$\langle \beta \,;\, \text{out}|\alpha \,;\, \text{in}\rangle = \langle \beta|S|\alpha\rangle = S_{\beta\alpha}$$

is a "matrix" with an infinite dimension.

(iv) In the following, we omit the label σ_i and σ_i'.

▶ **invariant matrix element, or scattering amplitude, $\mathcal{M}$:**
As long as the total energy and momentum are conserved,

$$\begin{aligned}\langle \vec{p'}_1 \cdots \vec{p'}_m|S|\vec{p}_1 \cdots \vec{p}_n\rangle \;&\propto\; \delta(\underbrace{\sum_f E_f'}_{\text{final}} - \underbrace{\sum_i E_i'}_{\text{final}}) \times \delta^{(3)}(\sum_f \vec{p'}_f - \sum_i \vec{p'}_i)\\ &= \delta^{(4)}(\sum_f p_f' - \sum_i p_i)\end{aligned}$$

We define the invariant matrix element, or scattering amplitude, $\mathcal{M}$ as

$$\boxed{\langle \vec{p'}_1 \cdots \vec{p'}_m|S|\vec{p}_1 \cdots \vec{p}_n\rangle = (2\pi)^4\delta^{(4)}(\sum_f p_f' - \sum_i p_i) \cdot i\mathcal{M}(\vec{p}_1 \cdots \vec{p}_n \to \vec{p'}_1 \cdots \vec{p'}_m) \qquad \text{———(2)}}$$

Comments

(i) Since the S-matrix is Lorentz invariant, the amplitude $\mathcal{M}$ is also Lorentz invariant.

(ii) The amplitude $\mathcal{M}$ can be calculated by the Feynman rule. In this lecture, we learn how the Feynman rule is derived from the Lagrangian.

(iii) In this subsection, we derive the formula for

$$\mathcal{M} \to \text{transition probability.}$$

——— on April 10, up to here. ———

Questions after the lecture:

Q: Why do you promote the discrete label i in QM to a continuous label $\vec{x}$ in QFT? (After all, what's the motivation of QFT?)

A: A short answer is, because it works! After all the Standard Model (including the QED) has been extremely successful in explaining a large number of phenomena in particle physics and other fields. . . .

In fact, in this lecture, I skip the contents of what is called "relativistic quantum mechanics", and directly start from the QFT. A typical introduction in the relativistic quantum mechanics is as follows. In QM, the Schrödinger equation of a free field is $i(\partial/\partial t)\psi = -(1/2m)(\partial/\partial x)^2\psi$, which corresponds to $E = p^2/2m$. Promoting this relation to a relativistic relation, $E^2 = p^2 + m^2$, one obtains $-(\partial/\partial t)^2\phi = -[(\partial/\partial x)^2 + m^2]\phi = 0$, or $(\Box + m^2)\phi = 0$. This is nothing but the Klein-Gordon equation. Now, this is not yet the QFT, as far as ϕ is regarded as a wave function as in QM. There is still a logical gap from here to the QFT. (In general, there should be a logical jump when learning a new theory. For instance, one can not "derive" the QM from the classical mechanics!) Here, I do not try to fill this gap (e.g., with the arguments of negative energy etc. . .), but just directly start from quantizing the fields.

Q: What are the prerequisites for this lecture, in particular about the Special Relativity?

A: Not much. If you understand for instance the notation of $x^\mu p_\mu = g_{\mu\nu}x^\mu p^\nu$, and the fact that it is Lorentz invariant, then (I hope) the lecture is more or less understandable. If you are not sure what the statement "$x^\mu p_\mu$ is Lorentz invariant" means, then you should probably review the basics of the special relativity, Lorentz transformation, Lorentz invariance, etc.

Q: What is the difference between the lower and upper indices, x^μ and x_μ?

A: You should review the basics of the special relativity and get used to these notations. . . .

Q: Your normalization in Eq.(1) is proportional to $E_p\delta^{(3)}(\vec{p}-\vec{q})$. Why don't you use a (seemingly manifestly Lorentz invariant) normalization $\delta^{(4)}(\vec{p}-\vec{q})$?

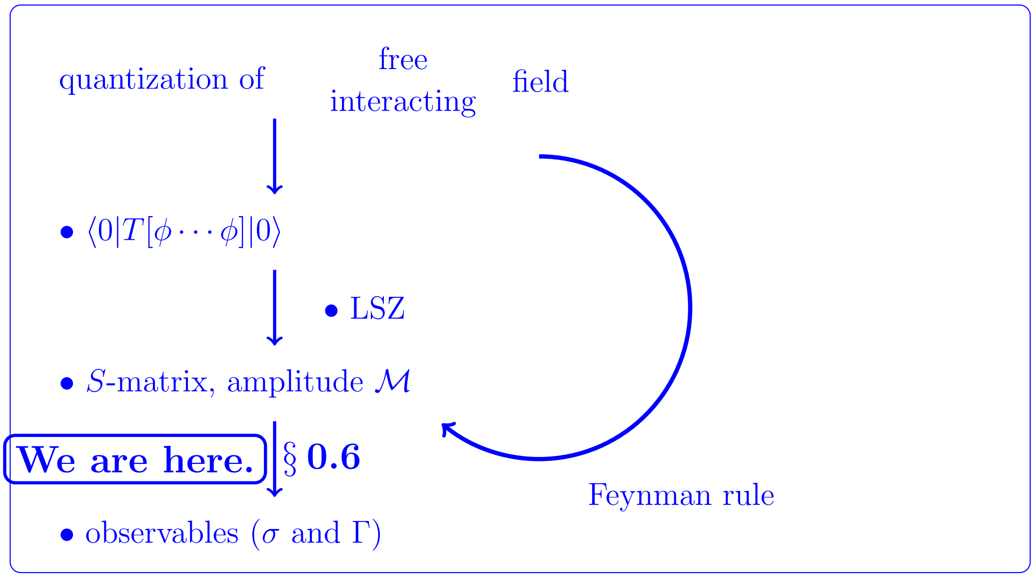
A: A good question. For one particle state with a definite mass m, $\delta^{(3)}(\vec{p}-\vec{q})$ implies $E_p = E_q$, and therefore $\delta^{(4)}(p-q) = \delta(E_p - E_q) \times \delta^{(3)}(\vec{p}-\vec{q})$ becomes proportional to $\delta(0)$, which is not very useful when normalizing fields.

Q: You said that $E_p\delta^{(3)}(\vec{p}-\vec{q})$ is Lorentz invariant, after Eq.(1). Then you also used (implicitly), in the comment after Eq.(2), the fact that $\delta^{(4)}(\vec{p}-\vec{q})$ is Lorentz invariant. Are these two statements compatible?

A: This is also a good question. Yes, both are correct. Check them yourself.

—————— on April 17, from here. ——————

Outline



quantization of free / interacting field

- $\langle 0|T[\phi\cdots\phi]|0\rangle$

- LSZ

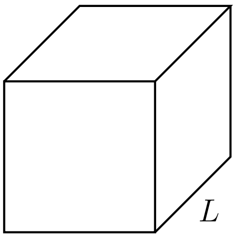
- S-matrix, amplitude $\mathcal{M}$

We are here. § **0.6**

- observables (σ and Γ)

Feynman rule

- For simplicity, we normalize the system with a box.



$$V = L^3, \qquad \vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

Then

$$\begin{aligned} \delta^{(3)}(\vec{p}-\vec{q}) &= \frac{1}{(2\pi)^3}\int d^3x e^{i(\vec{p}-\vec{q})\cdot\vec{x}} \\ &= \frac{1}{(2\pi)^3}\cdot V \cdot \underbrace{\delta_{\vec{p},\vec{q}}}_{\text{discrete}} . \end{aligned} \tag{3}$$

Define

$$|\vec{p}\rangle_{\text{Box}} = \frac{1}{\sqrt{2E_pV}}\,|\vec{p}\rangle . \tag{4}$$

Then,

$$\begin{aligned}
{}_{\text{Box}}\langle\vec{q}|\vec{p}\rangle_{\text{Box}} &= \frac{1}{\sqrt{2E_qV}}\frac{1}{\sqrt{2E_pV}}\langle\vec{q}|\vec{p}\rangle \\
&= \frac{1}{\sqrt{2E_qV}}\frac{1}{\sqrt{2E_pV}}(2\pi)^3 2E_p\delta^{(3)}(\vec{p}-\vec{q}) \qquad [\because (1)] \\
&= \frac{(2\pi)^3}{V}\delta^{(3)}(\vec{p}-\vec{q}) \qquad [\because E_p = E_q \text{ for } \vec{p}=\vec{q}] \\
&= \delta_{\vec{p},\vec{q}}. \qquad [\because (3)]
\end{aligned}$$

Therefore, $|\vec{p}\rangle_{\text{Box}}$ is the correct normalization to give the transition probability. For instance, if there is no interaction at all, for one particle state,

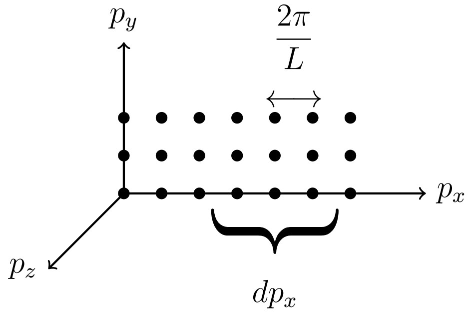
$$\text{Probability } P(\vec{p}\to\vec{p'}) = \left|{}_{\text{Box}}\langle\vec{p'}|\vec{p}\rangle_{\text{Box}}\right|^2 = \delta_{\vec{p'},\vec{p}} = \begin{cases} 1 & (\vec{p'}=\vec{p}) \\ 0 & (\vec{p'}\neq\vec{p}) \end{cases}$$

Thus,

$$\begin{aligned}
&\text{Probability } P(\vec{p}_1\cdots\vec{p}_n \to \vec{p'}_1\cdots\vec{p'}_m) \\
&= \left|{}_{\text{Box}}\langle\vec{p'}_1\cdots\vec{p'}_m|S|\vec{p}_1\cdots\vec{p}_n\rangle_{\text{Box}}\right|^2 \\
&= \left|\frac{1}{\sqrt{2E'_1V}}\cdots\frac{1}{\sqrt{2E'_mV}}\langle\vec{p'}_1\cdots\vec{p'}_m|S|\vec{p}_1\cdots\vec{p}_n\rangle\frac{1}{\sqrt{2E_1V}}\cdots\frac{1}{\sqrt{2E_mV}}\right|^2 \quad \because (4) \\
&= \left(\prod_{f=1}^{m}\frac{1}{2E'_f}\right)\left(\prod_{i=1}^{n}\frac{1}{2E_i}\right)\left(\frac{1}{V}\right)^{n+m}\left|\langle\vec{p'}_1\cdots\vec{p'}_m|S|\vec{p}_1\cdots\vec{p}_n\rangle\right|^2 \text{———————} (5)
\end{aligned}$$

► But this becomes zero for $V\to\infty$.
What is the (differential) probability that the final state is within $[\vec{p'}_f, \vec{p'}_f + d\vec{p'}_f]$?

$$dP = P(\vec{p}_1\cdots\vec{p}_n \to \vec{p'}_1\cdots\vec{p'}_m)\times\underbrace{dN}_{\text{number of states within } [\vec{p'}_f,\vec{p'}_f+d\vec{p'}_f]}$$



$$dN = \frac{dp_x}{(2\pi/L)}\cdot\frac{dp_y}{(2\pi/L)}\cdot\frac{dp_z}{(2\pi/L)} = \frac{d^3p}{(2\pi)^3}\cdot V$$

For the m particle final states, $\vec{p'}_1 \cdots \vec{p'}_m$,

$$dN = \prod_{f=1}^{m} \left(\frac{d^3 p'_f}{(2\pi)^3} \cdot V \right) \text{————————}(6)$$

From (5) and (6),

$$\begin{aligned} dP &= P(\vec{p}_1 \cdots \vec{p}_n \to \vec{p'}_1 \cdots \vec{p'}_m) \times dN \\ &= \left(\prod_{f=1}^{m} \frac{d^3 p'_f}{(2\pi)^3} \frac{1}{2E'_f} \right) \left(\prod_{i=1}^{n} \frac{1}{2E_i} \right) \left(\frac{1}{V} \right)^n \left| \langle \vec{p'}_1 \cdots \vec{p'}_m | S | \vec{p}_1 \cdots \vec{p}_n \rangle \right|^2 \text{————————}(7) \end{aligned}$$

- ► On the other hand, from the definition of $\mathcal{M}$ (2),

$$\begin{aligned} &\left| \langle \vec{p'}_1 \cdots \vec{p'}_m | S | \vec{p}_1 \cdots \vec{p}_n \rangle \right|^2 \\ &= (2\pi)^4 \delta^{(4)}(\sum p'_f - \sum p'_i) \cdot (2\pi)^4 \delta^{(4)}(\sum p'_f - \sum p'_i) \cdot |\mathcal{M}|^2 \\ &= (2\pi)^4 \delta^{(4)}(\sum p'_f - \sum p'_i) \cdot (2\pi)^4 \delta^{(4)}(0) \cdot |\mathcal{M}|^2 \\ &\qquad \left(\delta^{(4)}(0) = \int \frac{d^4 x}{(2\pi)^4} e^{i0\cdot x} = \frac{V \cdot T}{(2\pi)^4} \quad T : \text{time } (\to \infty) \right) \\ &= (2\pi)^4 \delta^{(4)}(\sum p'_f - \sum p'_i) \cdot V \cdot T \cdot |\mathcal{M}|^2 \end{aligned}$$

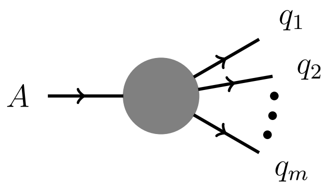
- ► Substituting it in Eq.(7) and dividing by T, we obtain the differential transition rate

$$\frac{dP}{T} = V^{1-n} \left(\prod_{i=1}^{n} \frac{1}{2E_i} \right) \underbrace{\left(\prod_{f=1}^{m} \frac{d^3 p'_f}{(2\pi)^3} \frac{1}{2E'_f} \right) (2\pi)^4 \delta^{(4)}(\sum p'_f - \sum p'_i)}_{\equiv d\Phi_m} \cdot |\mathcal{M}(\vec{p}_1 \cdots \vec{p}_n \to \vec{p'}_1 \cdots \vec{p'}_m)|^2$$

$$\text{————————}(8)$$

- ► Now let's discuss the cases $n = 1$ and $n = 2$.

$\boxed{n = 1}$



particle decay

From Eq.(8), the probability that the particle A decays into the range of final states $[\vec{p'}_f, \vec{p'}_f + d\vec{p'}_f]$ per unit time is

$$\frac{dP(p_A \to q_1 \cdots q_m)}{T} = \frac{1}{2E_A} d\Phi_m |\mathcal{M}(p_A \to q_1 \cdots q_m)|^2$$

Integrating over the final momenta, we have

Decay Rate

$$\begin{aligned}
&\Gamma(A \to 1, 2, \cdots) \\
&= \frac{1}{2m_A} \int d\Phi_m |\mathcal{M}(p_A \to q_1 \cdots q_m)|^2 \\
&= \underbrace{\frac{1}{2m_A}}_{\text{at rest frame}} \prod_{f=1}^{m} \int \frac{d^3 q_f}{(2\pi)^3 2E_f} (2\pi)^4 \delta^{(4)}(p_A - \sum_f q_f) |\mathcal{M}(p_A \to q_1 \cdots q_m)|^2 \\
&\qquad (\times \text{ symmetry factor})
\end{aligned}$$

Comments

(i) The mass dimension of Γ is $(\text{energy})^{+1} \sim (\text{time})^{-1}$.

(CHECK) $\langle \vec{q} | \vec{p} \rangle = (2\pi)^3 \underbrace{2E_p}_{E} \underbrace{\delta^{(3)}(\vec{p} - \vec{q})}_{E^{-3}} \qquad \longrightarrow \qquad |\vec{p}\rangle \sim E^{-1}.$

$$\underbrace{\langle \vec{q}_1 \cdots \vec{q}_m | S | \vec{p}_A \rangle}_{E^{-m-1}} = \underbrace{(2\pi)^4 \delta^{(4)}(\sum q - p_A)}_{E^{-4}} \times \underbrace{i\mathcal{M}(p_A \to q_1 \cdots)}_{\to E^{3-m}}$$

$$\Gamma = \underbrace{\underbrace{\frac{1}{2m_A}}_{E^{-1}} \underbrace{\prod_{f=1}^{m} \int \frac{d^3 q_f}{(2\pi)^3 2E_f}}_{E^{2m}} \underbrace{(2\pi)^4 \delta^{(4)}(p_A - \sum_f q_f)}_{E^{-4}} \underbrace{|\mathcal{M}(p_A \to q_1 \cdots q_m)|^2}_{E^{6-2m}}}_{E^{+1}}$$

(ii) If there is more than one decay modes, their sum

$$\Gamma(A \to \text{all}) = \Gamma(A \to 1, 2 \cdots) + \Gamma(A \to 1', 2' \cdots) + \cdots$$

is called the total decay rate, and its inverse

$$\tau_A = \frac{1}{\Gamma(A \to \text{all})}$$

gives the lifetime of A.

(Example: muon.

$$\begin{aligned}
\Gamma(\mu \to \text{all}) \simeq \Gamma(\mu \to e\bar{\nu}_e \nu_\mu) &\simeq 3 \times 10^{-19} \text{GeV}. \\
\tau_\mu \simeq \frac{1}{3 \times 10^{-19} \text{GeV}} &\simeq 2.2 \times 10^{-6} \text{sec}.
\end{aligned}$$

)

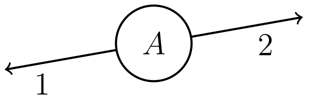
(iii) If not in the rest frame,

$$\Gamma = \frac{1}{2E_A} \underbrace{\int d\Phi_m |\mathcal{M}|^2}_{\text{Lorentz inv.}}$$

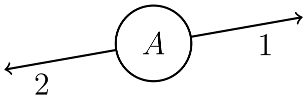
In the frame which is boosted by a velocity β, the energy is $E_A = \gamma m_A$.
($\gamma = 1/\sqrt{1-\beta^2}$.)
→ Γ becomes smaller by a factor of $1/\gamma$.
→ The lifetime becomes longer by a factor of γ.
(This is consistent with the Special Relativity!)

(iv) If there are identical particles in the final state, one should divide by a symmetry factor.

(Example) If particles 1 and 2 are identical,



and



are indistinguishable.

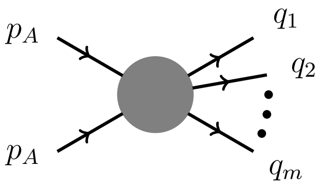
Thus, we should

① reduce the integration range ($\theta = [0, \pi] \to [0, \pi/2]$),

or

② divide by a symmetry factor (= 2) after integration.

$\boxed{n = 2}$

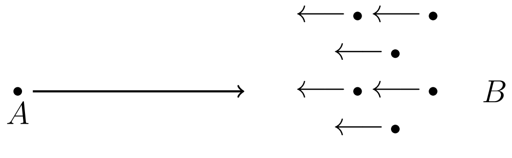


particle scattering

From Eq.(8), the probability that the final particles are in the range of $[\vec{p'}_f, \vec{p'}_f + d\vec{p'}_f]$ per unit time is

$$\frac{dP(p_A, p_B \to q_1 \cdots q_m)}{T} = \frac{1}{V} \frac{1}{2E_A \cdot 2E_B} d\Phi_m |\mathcal{M}(p_A \to q_1 \cdots q_m)|^2 \qquad \text{———————}(9)$$

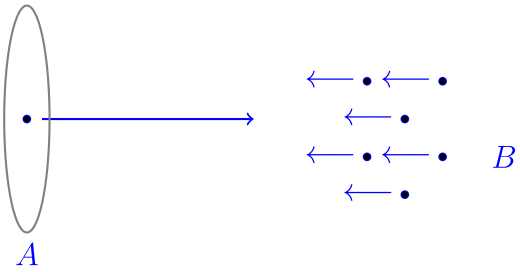
► In this case, we consider a quantity called "scattering cross section" (or just cross section).



Suppose that a particle A collides with a bunch of particles B (with number density n_B) with a relative velocity v_{rel}. The number that the scattering $A, B \to 1, 2 \cdots$ occurs per unit time is given by

$$\frac{P(p_A, p_B \to 1, 2 \cdots)}{T} = n_B \cdot v_{\text{rel}} \cdot \underbrace{\sigma(p_A, p_B \to 1, 2 \cdots)}_{\text{cross section}} \qquad \text{———————}(10)$$

Why "cross section"?



If we think a disk with an area σ, the number of B particles which goes through this disk within time T is given by

$$N_B = \sigma \cdot v_{\text{rel}} \cdot T \cdot n_B.$$

This is consistent with (10). (For small T, $N_B < 1$ and it gives the probability.)

- ▶ In the situation of Eq.(9), there is only one B particle, so $n_B = 1/V$. Thus, the differential cross section that the final state goes within $[\vec{p'}_f, \vec{p'}_f + d\vec{p'}_f]$ is

$$\begin{aligned} d\sigma(p_A, p_B \to 1, 2 \cdots) &= \frac{1}{v_{\text{rel}}} V \frac{dP(p_A, p_B \to 1, 2 \cdots)}{T} \qquad [\because (10)] \\ &= \frac{1}{v_{\text{rel}}} \frac{1}{2E_A \cdot 2E_B} d\Phi_m |\mathcal{M}(p_A \to q_1 \cdots q_m)|^2 \qquad [\because (9)] \end{aligned}$$

——————— on April 17, up to here. ———————

Questions after the lecture: (only some of them)

Q: Is the integration only over the final state momenta $\vec{q}_f$ (or $\vec{p'_f}$) but not over the initial ones p_i?

A: Right. The initial momenta are specified (by the collider experiment, for instance).

Q: Does the amplitude $\mathcal{M}$ depend on the final state momenta $\vec{q}_f$?

A: Yes, it does.

Q: If there are identical particles in the final state, one should divide by a symmetry factor. What about the initial state?

A: Even if there are identical particles in the initial state, it is not necessary to divide by a symmetry factor. In the case of final state, when integrating over the momentum phase space, you should avoid the double counting. (See the figure in the comment.) For the initial state, there is no integration, and hence there is no double counting.

Q: I understand the volume factor V in $\delta^{(3)}(0)$. What is the factor T in $\delta^{(4)}(0)$? And what do you mean by $T \to \infty$?

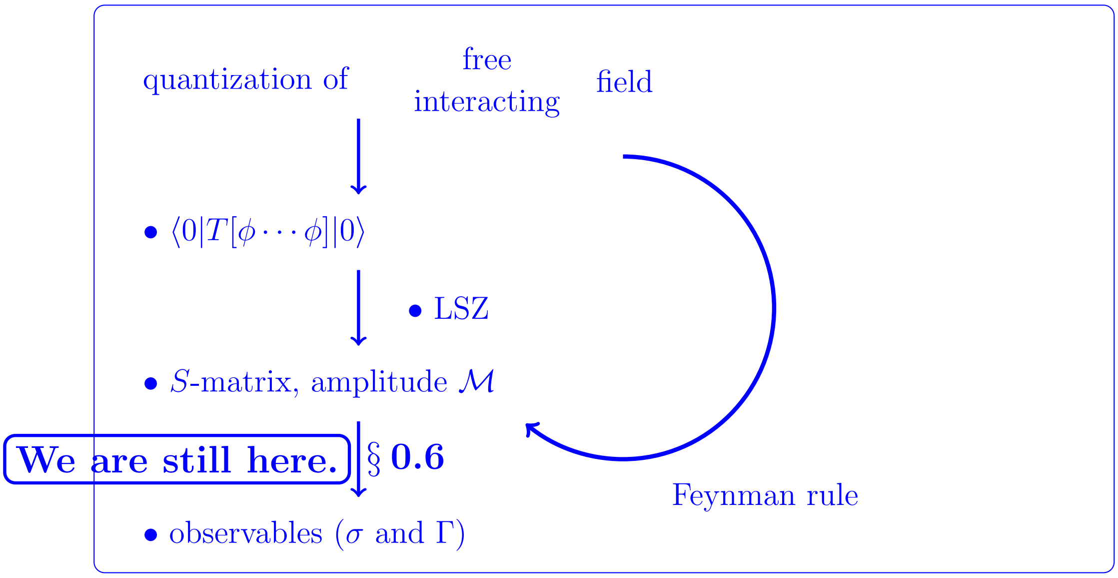
A: It's basically the same as the volume factor. Suppose that the interaction is turned on for only a time T. Then, the delta function corresponding to the energy conservation, $\delta(\sum E_f - \sum E_i)$, gives for $\sum E_f = \sum E_i$, $\delta(0) = \int_{-T/2}^{T/2} dt/(2\pi) e^{i0t} = T/(2\pi)$.

To all: Here, for the derivation of the formulae for the decay rate and the cross section, I referred to Section 3.4 of Weinberg's textbook [3] ("Rates and Cross-Sections"), with a modified normalization. There are derivations without a box normalization; see for instance Section 11 of Srednicki's textbook [1] ("Cross sections and decay rates") and Section 4.5 of Peskin's textbook [2] ("Cross Sections and the S-Matrix).

In general, if you look at more than one textbook for a certain topic, it can help your understanding a lot.

on April 24, from here.

Outline



Integrating over the final momenta,

Cross Section

$$\begin{aligned}
&\sigma(p_A, p_B \to 1, 2 \cdots) \\
&= \frac{1}{2E_A \cdot 2E_B \cdot v_{\rm rel}} \int d\Phi_m |\mathcal{M}(p_A \to q_1 \cdots q_m)|^2 \\
&= \frac{1}{2E_A \cdot 2E_B \cdot v_{\rm rel}} \prod_{f=1}^{m} \int \frac{d^3 q_f}{(2\pi)^3 2E_f} (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum_f q_f) |\mathcal{M}(p_A, p_B \to q_1 \cdots)|^2 \\
&\qquad\qquad (\times \text{ symmetry factor})
\end{aligned}$$

Comments

(i) The mass dimension of σ is $(\text{energy})^{-2} \sim (\text{length})^2 \sim (\text{area})$.

(ii) If there are identical particles, divide by the symmetry factor (same as Γ).

(iii) The relative velocity $v_{\rm rel}$ is given by

$$v_{\rm rel} = \left| \frac{\vec{p}_A}{E_A} - \frac{\vec{p}_B}{E_B} \right|$$

(For a head-on collision with speeds of light, $v_{\rm rel} = 2$.)

(iv) $E_A E_B v_{\rm rel} = |E_B \vec{p}_A - E_A \vec{p}_B|$ is not Lorentz inv., and therefore σ in the above formula is not Lorentz inv. either.
(Lorentz inv. cross section can be defined by replacing as $E_A E_B v_{\rm rel} \to \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}$.)

Problems

(a) Show that the mass dim. of σ is -2.

(b) Show that $\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} = E_A E_B v_{\rm rel}$ for $\vec{p}_A \parallel \vec{p}_B$.

(Sometimes I give problems. They are mostly just for exercises. Some of those "problems" may be included in the "homework problems" which will be posted later, and which you have to submit.)

§ 1 Scalar (spin 0) Field

We consider a real scalar field $\phi(x)$.

- real: $\phi(x)^\dagger = \phi(x)$ (Hermitian operator).
- scalar: Lorentz transformation of the field is given by

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)\,.$$

Now let's briefly review the Lorentz transformation (before starting QFT).

§ 1.1 Lorentz transformation

§ 1.1.1 Lorentz transformation of coordinates

— is a linear, homogeneous change of coordinates from x^μ to x'^μ,

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu\,,$$

where Λ is a 4×4 matrix satisfying

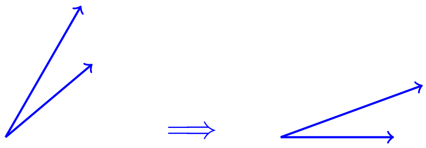
$$g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma} \quad (\text{or} \quad \Lambda^T g \Lambda = g \quad \text{in matrix notation})\,.$$

Comments

(i) It preserves inner products of four vectors:

$$\begin{aligned} & x \cdot y = g_{\mu\nu} x^\mu y^\nu \\ \to \ & x' \cdot y' = g_{\mu\nu} x'^\mu y'^\nu = g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho y^\sigma = g_{\rho\sigma} x^\rho y^\sigma = x \cdot y\,. \end{aligned}$$

- This is similar to orthogonal transformation $\vec{v} \to \vec{v'} = R\vec{v}$ where R is an orthogonal matrix satisfying $R^T R = \mathbf{1}$. (e.g., $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ in 2-dim.)
  Inner products are preserved: $\vec{u} \cdot \vec{v} \to \vec{u'} \cdot \vec{v'} = (R\vec{u}) \cdot (R\vec{v}) = \vec{u}^T R^T R \vec{v} = \vec{u} \cdot \vec{v}$.



(ii) The set of all Lorentz transformations (LTs) forms a group (Lorentz group).

- Product of two LTs Λ_1 and Λ_2 is defines as $(\Lambda_2\Lambda_1)^\mu{}_\nu = (\Lambda_2)^\mu{}_\rho (\Lambda_1)^\rho{}_\nu$.
- closure: if $\Lambda_1^T g \Lambda_1 = g$ and $\Lambda_2^T g \Lambda_2 = g$, then $(\Lambda_2\Lambda_1)^T g (\Lambda_2\Lambda_1) = g$.

- associativity: $(\Lambda_1\Lambda_2)\Lambda_3 = \Lambda_1(\Lambda_2\Lambda_3)$.
- identity: $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$.
- inverse: $(\Lambda^{-1})^\mu{}_\nu = g^{\mu\rho}g_{\nu\sigma}\Lambda^\sigma{}_\rho = \Lambda_\nu{}^\mu$.

Problems

(a) Write the explicit form Λ for a rotation along the z-axis. Show that it satisfies $\Lambda^T g\Lambda = g$.

(b) Write the explicit form of Λ for a boost along the z-axis. Show that it satisfies $\Lambda^T g\Lambda = g$.

(c) Show that the above Λ^{-1} satisfies $\Lambda^{-1}\Lambda = \mathbf{1}$, i.e., $(\Lambda^{-1})^\mu{}_\nu \Lambda^\nu{}_\rho = \delta^\nu{}_\rho$.

(d) For an infinitesimal LT, we can write

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu\,.$$

Show that $\delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu}$ and hence there are six independent $\delta\omega$.
We will discuss more on this later in Sec.§ 2.1.

§ 1.1.2 Lorentz transformation of quantum fields

— is represented by unitary operators acting on fields:

$$\Phi(x) \to \Phi'(x) = U(\Lambda)\Phi(x)U(\Lambda)^{-1} \qquad \Phi(x):\ \text{generic field}$$

Scalar fields are the fields which transform as

$$\phi(x) \to \phi'(x) = U(\Lambda)\phi(x)U(\Lambda)^{-1} = \phi(\Lambda^{-1}x)$$

Comments

(i) Note that it transforms the field $\phi(x)$ at all the spacetime x.

(ii) Substituting $x = y' = \Lambda y$, it means $\underline{\phi'(y') = \phi(y)}$ (for all y).

(iii) Why $\Phi' = U\Phi U^{-1}$ for fields Φ? Suppose that a state $|\cdot\rangle$ transforms as $|\cdot\rangle \to |\cdot\rangle' = U|\cdot\rangle$. Then, with operators O_i,

$$\begin{aligned} O_1O_2\cdots O_n|\cdot\rangle &\to U(O_1O_2\cdots O_n|\cdot\rangle) \\ &= (UO_1U^{-1})(UO_2U^{-1})\cdots(UO_nU^{-1})U|\cdot\rangle \\ &= O_1'O_2'\cdots O_n'|\cdot\rangle' \end{aligned}$$

§ 1.2 Lagrangian and Canonical Quantization of Real Scalar Field

In quantum mechanics, we consider a Lagrangian

$$L = L(\dot{q}, q) \qquad \dot{} = \frac{\partial}{\partial t}\,.$$

In QFT, we also start from a Lagrangian

$$L = \int d^3x \; \underbrace{\mathcal{L}[\dot{\phi}(\vec{x},t), \phi(\vec{x},t)]}_{\text{Lagrangian density}}$$

In this lecture, we consider the following Lagrangian (called ϕ^4 theory):

$$\begin{aligned} L &= \int d^3x \; \mathcal{L}[\dot{\phi}(\vec{x},t), \phi(\vec{x},t)] \\ &= \int d^3x \; \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{24}\phi^4\right) \\ &= \int d^3x \; \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{24}\phi^4\right) \end{aligned}$$

where λ is real and positive constant, and

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

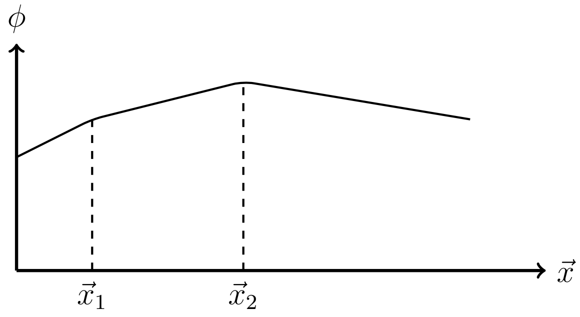
$$\partial_\mu\phi\partial^\mu\phi = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \left(\frac{\partial}{\partial x^0}\phi\right)^2 - \sum_{i=1}^{3}\left(\frac{\partial}{\partial x^i}\phi\right)^2 = \dot{\phi}^2 - \vec{\nabla}\phi\cdot\vec{\nabla}\phi$$

The $\lambda\phi^4$ term represents the interaction. For $\lambda = 0$, it becomes the Lagrangian of free scalar field:

$$\begin{aligned} L_{\text{free}} &= \int d^3x \; \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right) \\ &= \int d^3x \; \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi - \frac{1}{2}m^2\phi^2\right) \end{aligned}$$

If we regard $\vec{x}$ as just a label,

$$\phi(\vec{x},t) = \{\phi_{\vec{x}_1}(t), \phi_{\vec{x}_2}(t), \cdots\}$$



QM of infinite number of degrees of freedom

$$\begin{aligned} L &= \sum_{\vec{x}}\left(\frac{1}{2}\dot{\phi}_{\vec{x}}(t)^2 + \cdots\right) \\ &\sim \sum_i \frac{1}{2}\dot{q}_i(t)^2 + \cdots \end{aligned}$$

	QM	QFT
conjugate momentum	$p_i = \frac{\partial L}{\partial \dot{q}_i}$	$\pi(\vec{x},t) = \frac{\delta L}{\delta \dot{\phi}(\vec{x},t)} = \dot{\phi}(\vec{x},t)$ (functional derivative)
Hamiltonian	$H = \sum_i p_i \dot{q}_i - L$	$H = \int d^3x \left(\pi(\vec{x},t)\dot{\phi}(\vec{x},t) - \mathcal{L}\right)$ $= \int d^3x \left(\pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4\right)$ $= \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4\right)$
Canonical Quantization: Equal Time Commutation Relation		
	$[q_i(t), p_j(t)] = i\delta_{ij}$ $[q_i(t), q_j(t)] = 0$ $[p_i(t), p_j(t)] = 0$	$\boxed{\begin{array}{l} [\phi(\vec{x},t), \pi(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y}) \\ [\phi(\vec{x},t), \phi(\vec{y},t)] = 0 \\ [\pi(\vec{x},t), \pi(\vec{y},t)] = 0 \end{array}}$ ↑ ↗ equal time

Comments

(i) The action

$$S = \int dt L = \int dt d^3x \mathcal{L} = \int d^4x \mathcal{L}$$

is Lorentz invariant.

Check:

$$\text{Under} \quad \phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x),$$

$$\int d^4x \mathcal{L}[\phi(x), \partial_\mu\phi(x)] \to \int d^4x \mathcal{L}\left[\phi(\Lambda^{-1}x), \frac{\partial\phi}{\partial x^\mu}(\Lambda^{-1}x)\right]$$

By a change of variable, $x = \Lambda y$ or $y^\nu = (\Lambda^{-1})^\nu{}_\mu x^\mu$,

$$\partial_\mu\phi(x) \to \frac{\partial\phi}{\partial x^\mu}(\Lambda^{-1}x) = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial\phi}{\partial y^\nu}(y) = (\Lambda^{-1})^\nu{}_\mu[\partial_\nu\phi](y)$$

and hence

$$\partial_\mu\phi\partial^\mu\phi(x) \to g^{\mu\rho}\frac{\partial\phi}{\partial x^\mu}(\Lambda^{-1}x)\frac{\partial\phi}{\partial x^\rho}(\Lambda^{-1}x) \qquad = \underbrace{g^{\mu\rho}(\Lambda^{-1})^\nu{}_\mu(\Lambda^{-1})^\sigma{}_\rho}_{=g^{\nu\sigma}}[\partial_\nu\phi](y)[\partial_\rho\phi](y)$$

$$= [\partial_\nu\phi\partial^\nu\phi](y)$$

Thus,

$$\begin{aligned}
S &= \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi(x) + \frac{1}{2}m^2\phi(x)^2 + \cdots\right) \\
&\to \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi(y) + \frac{1}{2}m^2\phi(y)^2 + \cdots\right) \\
&= \int d^4y \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi(y) + \frac{1}{2}m^2\phi(y)^2 + \cdots\right) \quad \left(d^4x = \left|\det\frac{\partial x}{\partial y}\right| d^4y = |\det\Lambda| d^4y = d^4y\right) \\
&= S\,.
\end{aligned}$$

(ii) Why $\mathcal{L}_{\text{interaction}} = -\frac{\lambda}{24}\phi^4$?

- For $\mathcal{L}_{\text{interaction}} \sim \phi^3$ or $-\phi^3$ or $+\phi^4$, the corresponding Hamiltonian term becomes

$$H_{\text{interaction}} \sim \int d^3x(-\phi^3 \quad \text{or} \quad \phi^3 \quad \text{or} \quad -\phi^4)$$

  and hence the energy becomes unbounded below. (Take, for instance $\phi(\vec{x},t) = const \to \pm\infty$.)
- Thus, $\mathcal{L}_{\text{interaction}} \sim -\phi^4$ is the simplest possibility.
- $24 = 4!$ is for later convenience (for Feynman rule).

——————— on April 24, up to here. ———————

Questions after the lecture: (only some of them)

Q: What does ϕ represent? (a particle?) And do we start from it? Just as a toy model?

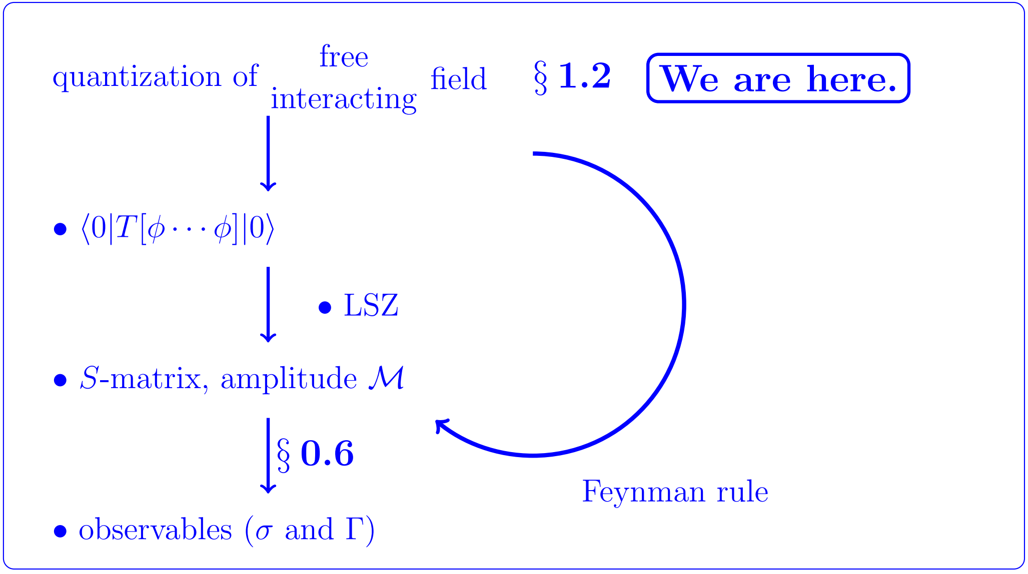
A: (sorry, I should have said in the lecture.) Yes, it represents creation and annihilation of a scalar particle. It will become clearer later. It is a good example of QFT as a simple toy model, but scalar particles do exist in nature. An example of a scalar particle is the Higgs boson (and it is the only known elementary scalar particle). The ϕ^4 interaction of the Higgs boson is assumed in the Standard Model, but it is not yet experimentally tested. There may also be interactions like ϕ^6, ϕ^8, or other forms....

Q: I am confused with the LT of field, $\phi'(x) = \phi(\Lambda^{-1}x)$...

A: (I was also confused when I learned it!) As I said during the lecture, the LT of fields should be distinguished from the LT of coordinates. The former is one of many field transformations. A simple transformation of field is the Z_2 transformation, $\phi(x) \to -\phi(x)$. The action of the ϕ^4 theory is invariant under this Z_2 transformation of the field. Similarly, the action is invariant under the LT of the field, $\phi(x) \to \phi(\Lambda^{-1}x)$. The latter seems more complicated because it is accompanied by a change of the argument, but they are both just transformations of field.

——— on May 1, from here. ———

Outline



quantization of $\begin{matrix}\text{free}\\ \text{interacting}\end{matrix}$ field **§ 1.2** $\boxed{\textbf{We are here.}}$

- $\langle 0|T[\phi\cdots\phi]|0\rangle$

$\downarrow$ • LSZ

- S-matrix, amplitude $\mathcal{M}$

$\downarrow$ **§ 0.6**

- observables (σ and Γ)

Feynman rule

(iii) Schrödinger representation and Heisenberg representation:

In QFT, usually the Heisenberg representation is used.

	state	operator
S-rep.	$\lvert\Psi(t)\rangle_S$ time-dependent	$\mathcal{O}_S$ time-independent
H-rep.	$\lvert\Psi\rangle_H$ time-independent	$\mathcal{O}_H(t)$ time-dependent

S-rep.

$$i\frac{d}{dt}\,|\Psi(t)\rangle_S = H(p,q)\,|\Psi(t)\rangle_S$$

$$|\Psi(t)\rangle_S = e^{-iH(t-t_0)}\,|\Psi(t_0)\rangle_S$$

Expectation value of an operator: ${}_S\langle\Psi(t)|\mathcal{O}_S|\Psi(t)\rangle_S$

H-rep.

$$\begin{cases}|\Psi\rangle_H & \equiv |\Psi(t_0)\rangle_S = e^{iH(t-t_0)}\,|\Psi(t)\rangle_S\\ \mathcal{O}_H(t) & \equiv e^{iH(t-t_0)}\mathcal{O}_S e^{-iH(t-t_0)}\end{cases}$$

Expectation value: ${}_H\langle\Psi|\mathcal{O}_H(t)|\Psi\rangle_H = \cdots = {}_S\langle\Psi(t)|\mathcal{O}_S|\Psi(t)\rangle_S$

$$\begin{aligned}i\frac{d}{dt}\mathcal{O}_H(t) &= -He^{iH(t-t_0)}\mathcal{O}_S e^{-iH(t-t_0)} + e^{iH(t-t_0)}\mathcal{O}_S H e^{-iH(t-t_0)}\\ &= -H\mathcal{O}_H(t) + \mathcal{O}_H(t)H\\ &= [\mathcal{O}_H(t), H]. \quad \text{Heisenberg eq.}\end{aligned}$$

§ 1.3 Equation of Motion (EOM)

- ▶ There are two ways to derive the EOM.

  (i) From the action principle $\delta S = \delta \int dt L = 0$,

$$\partial_\mu \frac{\delta L}{\delta(\partial_\mu \phi)} - \frac{\delta L}{\delta \phi} = 0 \quad \text{Euler-Langrange eq.}$$

  (ii) From Heisenberg eq.,

$$\begin{cases} i\dot{\phi} = [\phi, H] \\ i\dot{\pi} = [\pi, H] \end{cases}$$

- ▶ From (i), for the Lagrangian $L = \displaystyle\int d^3x \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{24}\phi^4 \right)$, we obtain

$$\partial_\mu \frac{\delta L}{\delta(\partial_\mu \phi)} - \frac{\delta L}{\delta \phi} = \partial_\mu (\partial^\mu \phi) + m^2 \phi + \frac{\lambda}{6}\phi^3 = 0$$

or

$$\text{EOM} \quad \boxed{\left(\Box + m^2\right) \phi(x) = -\frac{\lambda}{6}\phi(x)^3} \quad \text{------------------} (\star)$$

- ▶ Later we will derive the EOM $(\star)$ with (ii) again (see § 1.5.3).
- ▶ Here, there is an important difference between $\lambda = 0$ and $\lambda \neq 0$.

For free field ($\lambda = 0$), the EOM is linear in $\phi(x)$, and can be solved exactly (§ 1.4).

→ $\phi \sim a + a^\dagger$

→ The relations between a, $a^\dagger$, and H are obtained.

(creation and annihilation oprators)

For $\lambda \neq 0$, the EOM $(\star)$ is non-linear, and it cannot be simply solved.

→ what is $\phi(x)$ in this case? (more on this later)

§ 1.4 Free scalar field

§ 1.4.1 Solution of the EOM

- Starting from $\boxed{(\Box + m^2)\phi(x) = 0 \text{ (Klein-Gordon eq.)}}$, one can show that $\phi(x)$ can be expressed as

$$\boxed{\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x}\right)} \quad \text{———————————}(1)$$

where $\phi(x)$, $a(\vec{p})$, $a^\dagger(\vec{p})$ are operators, $p{\cdot}x = p_\mu x^\mu = p^0t - \vec{p}{\cdot}\vec{x}$, and $p^0 = E_p = \sqrt{\vec{p}^2 + m^2}$.

- Eq. (1) is the solution of the EOM, because

$$\begin{aligned}(\Box + m^2)e^{\pm ip\cdot x} &= (\partial_\mu\partial^\mu + m^2)e^{\pm ip\cdot x} \\ &= \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right) e^{\pm ip\cdot x} \\ &= (-E_p^2 + \underbrace{\vec{p}^2 + m^2}_{E_p^2})e^{\pm ip\cdot x} = 0.\end{aligned}$$

- **proof:** Now let's show that Eq.(1) is the general solution of the EOM.

(i) Fourier transform (FT) $\phi(x)$ with respect to $\vec{x}$:

$$\underbrace{\phi(\vec{x},t)}_{\text{operator}} = \int d^3p\, \underbrace{C(\vec{p},t)}_{\text{operator}}\, e^{i\vec{p}\cdot\vec{x}} \quad \text{———————————}(2)$$

(ii) From the condition $\phi = \phi^\dagger$ (real field),

$$\begin{aligned}\int d^3pC(\vec{p},t)e^{i\vec{p}\cdot\vec{x}} &= \int d^3pC^\dagger(\vec{p},t)e^{-i\vec{p}\cdot\vec{x}} \\ &= \int d^3p'C^\dagger(-\vec{p}',t)e^{i\vec{p}'\cdot\vec{x}} \quad (\vec{p}' = -\vec{p}) \\ &= \int d^3pC^\dagger(-\vec{p},t)e^{i\vec{p}\cdot\vec{x}}\end{aligned}$$

Using inverse FT,

$$C(\vec{p},t) = C^\dagger(-\vec{p},t) \quad \text{———————————}(3)$$

(iii) From $\left(\Box + m^2\right)\phi = \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi = 0$ and (2),

$$\int d^3p \left(\ddot{C}(\vec{p},t) + C(\vec{p},t)\underbrace{(\vec{p}^2 + m^2)}_{E_p^2}\right) e^{i\vec{p}\cdot\vec{x}} = 0$$

Using inverse FT,

$$\begin{aligned} &\ddot{C}(\vec{p},t) + E_p^2 C(\vec{p},t) = 0 \\ \therefore\quad &C(\vec{p},t) = C(\vec{p})e^{-iE_pt} + C'(\vec{p})e^{+iE_pt}. \end{aligned}$$

From (3), $C'(\vec{p}) = C^\dagger(-\vec{p})$, and hence

$$\underline{C(\vec{p},t) = C(\vec{p})e^{-iE_pt} + C^\dagger(-\vec{p})e^{+iE_pt}}.$$

(iv) Substituting it to (2) (and changing $\vec{p} \to -\vec{p}$ in the 2nd term),

$$\begin{aligned} \phi(\vec{x},t) &= \int d^3p \left(C(\vec{p})e^{-iE_pt}e^{i\vec{p}\cdot\vec{x}} + C^\dagger(\vec{p})e^{+iE_pt}e^{-i\vec{p}\cdot\vec{x}} \right) \\ &= \int d^3p \left(C(\vec{p})e^{-ip\cdot x} + C^\dagger(\vec{p})e^{+ip\cdot x} \right) \end{aligned}$$

Finally by normalizing as $a(\vec{p}) \equiv (2\pi)^3\sqrt{2E_p} \cdot C(\vec{p})$, we obtain (1). ■

- ► Note that the normalization depends on the convention (textbook).

$$a(\text{here}) = a(\text{Peskin}) = \frac{1}{\sqrt{2E_p}} a(\text{Srednicki}) = (2\pi)^{3/2} a(\text{Weinberg}).$$

- ► From (1), we can express the operators $a(\vec{p})$ and $a^\dagger(\vec{p})$ in terms of $\phi(x)$:

$$\begin{cases} a(\vec{p}) = \dfrac{1}{\sqrt{2E_p}} \displaystyle\int d^3x e^{+ip\cdot x} \left[i\dot{\phi}(x) + E_p\phi(x) \right] \\ a^\dagger(\vec{p}) = \dfrac{1}{\sqrt{2E_p}} \displaystyle\int d^3x e^{-ip\cdot x} \left[-i\dot{\phi}(x) + E_p\phi(x) \right] \end{cases} \qquad \text{———————}(4)$$

Problems

(a) Substitute (1) to the right-hand side (RHS) of (4) and show that it gives a & $a^\dagger$.

(b) The RHS of (4) seems to depend on $x^0 = t$, but the LHS does not. Show that $\frac{\partial}{\partial t}$[RHS of (4)]= 0, using the EOM. (Hint: integration by parts (部分積分))

(c) Substitute (4) to the RHS of (1) and show that it gives LHS.

Pay attention to which variables are just the integration variable. For instance, let's solve (a):

$$\text{from (1),}\quad \phi(x) = \int \underbrace{\frac{d^3q}{(2\pi)^3\sqrt{2E_q}}}_{=[dq]} \left(a(\vec{q})e^{-iq\cdot x} + a^\dagger(\vec{q})e^{+iq\cdot x} \right)$$

$$i\dot{\phi}(x) = \int [dq] \left(E_q a(\vec{q})e^{-iq\cdot x} - E_q a^\dagger(\vec{q})e^{+iq\cdot x} \right)$$

$$i\dot{\phi}(x) + E_p\phi(x) = \int [dq] \left((E_q + E_p)a(\vec{q})e^{-iq\cdot x} + (-E_q + E_p)a^\dagger(\vec{q})e^{-ip\cdot x} \right)$$

Thus,

$$\text{RHS of (4)} = \frac{1}{\sqrt{2E_p}}\int d^3x e^{+ip\cdot x}\int [dq]\left((E_q+E_p)a(\vec{q})e^{-iq\cdot x} + (-E_q+E_p)a^\dagger(\vec{q})e^{+iq\cdot x}\right)$$

$$\int d^3x e^{iE_p x^0} e^{-i\vec{p}\cdot\vec{x}} \cdot e^{-iE_q x^0} e^{i\vec{q}\cdot\vec{x}}$$

$$= (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\cdot e^{i(E_p-E_q)x^0} \qquad (2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\cdot e^{i(E_p+E_q)x^0}$$

$$= \frac{1}{\sqrt{2E_p}}\int \frac{d^3q}{\sqrt{2E_q}}\left((E_q+E_p)a(\vec{q})\delta^{(3)}(\vec{p}-\vec{q}) + \underbrace{(-E_q+E_p)}_{\to 0}a^\dagger(\vec{q})\delta^{(3)}(\vec{p}-\vec{q})e^{i(E_p+E_q)x^0}\right)$$

$$= a(\vec{p}) = \text{LHS of (4)} \quad \blacksquare$$

§ 1.4.2 Commutation relations

- From the commutation relation in § 1.2, we have the following commutation relations (recall $\pi(x) = \dot{\phi}$):

$$\boxed{\begin{aligned}&[\phi(\vec{x},t),\dot{\phi}(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y})\\&[\phi(\vec{x},t),\phi(\vec{y},t)] = 0\\&[\dot{\phi}(\vec{x},t),\dot{\phi}(\vec{y},t)] = 0\end{aligned}} \Longleftrightarrow \boxed{\begin{aligned}&[a(\vec{p}),a^\dagger(\vec{q})] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\\&[a(\vec{p}),a(\vec{q})] = 0\\&[a^\dagger(\vec{p}),a^\dagger(\vec{q})] = 0\end{aligned}} \text{ ——— (5)}$$

Problems

(a) Show that RHS of (5) $\Longrightarrow$ LHS of (5), using (1).

(b) Show that LHS of (5) $\Longrightarrow$ RHS of (5), using (4).

§ 1.4.3 $a^\dagger$ and a are the creation and annihilation operators.

- In this section we will see that

$a(\vec{p})$ ——— annihilate a particle with energy E_p, momentum $\vec{p}$.

$a^\dagger(\vec{p})$ ——— create a particle with energy E_p, momentum $\vec{p}$.

- First, we can show that $\boxed{\begin{aligned}&[H,a^\dagger(\vec{p})] = E_p a^\dagger(\vec{p})\\&[H,a(\vec{p})] = -E_p a^\dagger(\vec{p})\end{aligned}}$ ——— (6) (We will show it later.)

We can also show $\boxed{\begin{aligned}&[\hat{\vec{P}},a^\dagger(\vec{p})] = \vec{p}a^\dagger(\vec{p})\\&[\hat{\vec{P}},a(\vec{p})] = -\vec{p}a^\dagger(\vec{p})\end{aligned}}$

where $\hat{\vec{P}}$ is the "momentum" operator. ($\hat{\vec{P}} = -\int d^3x\pi\vec{\nabla}\phi$. We skip the details here.)

- Consider a state with energy E_X and momentum $\vec{p}_X$;

$$|X\rangle : \quad \begin{cases} H|X\rangle = E_X|X\rangle \\ \hat{\vec{P}}|X\rangle = \vec{p}_X|X\rangle \end{cases}$$

Then, for the state $a^\dagger(\vec{p})\,|X\rangle$,

$$\begin{aligned}
H\left(a^\dagger(\vec{p})\,|X\rangle\right) &= \left([H, a^\dagger(\vec{p})] + a^\dagger(\vec{p})H\right)|X\rangle \\
&= \left(E_p a^\dagger(\vec{p}) + a^\dagger(\vec{p})E_X\right)|X\rangle \\
&= (E_p + E_X)\left(a^\dagger(\vec{p})\,|X\rangle\right), \\
\hat{\vec{P}}\left(a^\dagger(\vec{p})\,|X\rangle\right) &= \left([\hat{\vec{P}}, a^\dagger(\vec{p})] + a^\dagger(\vec{p})\hat{\vec{P}}\right)|X\rangle \\
&= \left(\vec{p}a^\dagger(\vec{p}) + a^\dagger(\vec{p})\vec{p}_X\right)|X\rangle \\
&= (\vec{p} + \vec{p}_X)\left(a^\dagger(\vec{p})\,|X\rangle\right).
\end{aligned}$$

Thus, the state $a^\dagger(\vec{p})\,|X\rangle$ has energy $E_p + E_X$ and momentum $\vec{p} + \vec{p}_X$, namely, $a^\dagger(\vec{p})$ adds energy E_p and momentum $\vec{p}$. (creation operator)

- Similarly, we can show

$$\begin{aligned}
H\left(a(\vec{p})\,|X\rangle\right) &= (E_X - E_p)\left(a(\vec{p})\,|X\rangle\right), \\
\hat{\vec{P}}\left(a(\vec{p})\,|X\rangle\right) &= (\vec{p}_X - \vec{p})\left(a(\vec{p})\,|X\rangle\right).
\end{aligned}$$

and therefore $a(\vec{p})$ is an annihilation operator.

- Now let's show (6). There are two ways.

  (i) Express H in terms of a and $a^\dagger$.
  (ii) Use (4) and Heisenberg eq.

Here we do (i). [**Problem:** Do (ii): Show (6) by using (4) and Heisenberg eq.]

——————— on May 1, up to here. ———————

Questions and comments after the lecture: (only some of them)

comment: There is a typo at the end of §1.2. In Schrödinger rep., the argument of the RHS of $|\psi(t)\rangle_S$ should be t_0, not t.

A: Thanks! corrected.

Q: When you write $\int \frac{d^3p}{\sqrt{2E_p}}$, does E_p in the denominator depend on the integration variable $\vec{p}$?

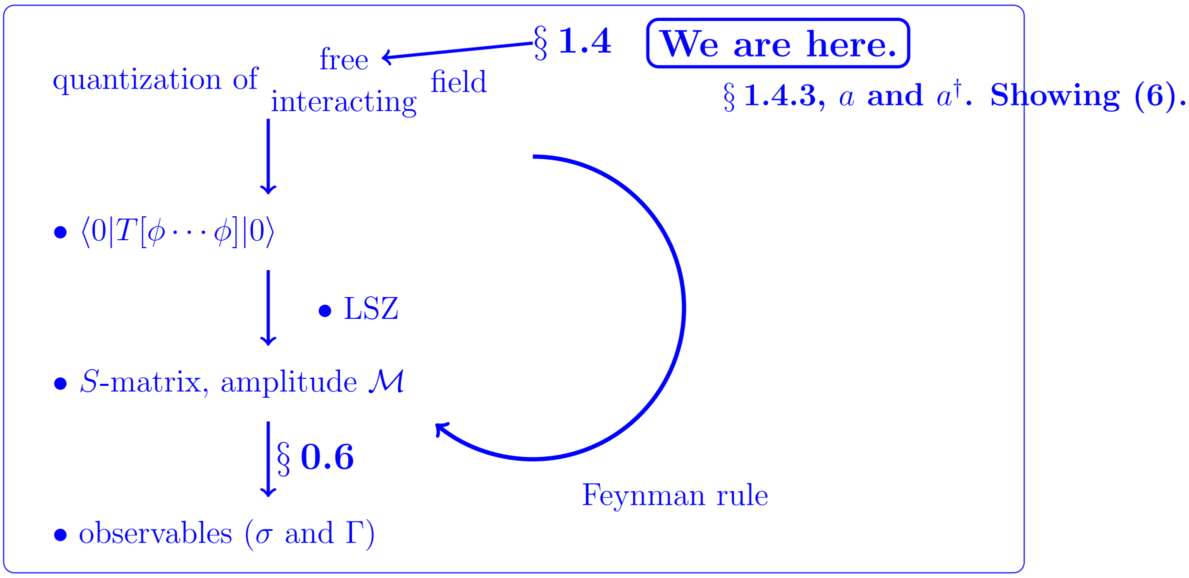
A: Yes.

Q: Can we construct a "number operator" from the creation and annihilation operator? Does it give a finite number even though the momentum is continuous?

A: Good question! One can indeed define a number operator $N = \int \frac{d^3p}{(2\pi)^3} a^\dagger(\vec{p})a(\vec{p})$, similar to the case of harmonic oscillator, $N = \sum_i a_i^\dagger a_i$. Now, next week we will see that a one-particle state is proportional to $a^\dagger(\vec{p})\left|0\right\rangle$. You can check explicitly (by using the commutation relation) that $N(a^\dagger(\vec{p})\left|0\right\rangle) = (a^\dagger(\vec{p})\left|0\right\rangle)$, which means the number is one. Similarly, you can check $N(a^\dagger(\vec{p})a^\dagger(\vec{p}')\left|0\right\rangle) = 2(a^\dagger(\vec{p})a^\dagger(\vec{p}')\left|0\right\rangle)$, etc.

——————— on May 8, from here. ———————

Outline



First,

$$\begin{aligned}
\phi(x) &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(a(\vec{p})e^{-iE_pt+i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{iE_pt-i\vec{p}\cdot\vec{x}}\right) \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(a(\vec{p})e^{-iE_pt} + a^\dagger(-\vec{p})e^{iE_pt}\right)e^{i\vec{p}\cdot\vec{x}} \quad (\vec{p}\to-\vec{p}\ \text{ in the 2nd term})
\end{aligned}$$

Let's define,

$$A(\vec{p},t) \equiv \frac{1}{(2\pi)^3\sqrt{2E_p}}a(\vec{p})e^{-iE_pt}$$

and omit t for simplicity: $A(\vec{p}) = A(\vec{p},t)$. Then

$$\begin{aligned}
\phi(x) &= \int d^3p\left(A(\vec{p}) + A^\dagger(-\vec{p})\right)e^{i\vec{p}\cdot\vec{x}} \\
\vec{\nabla}\phi(x) &= \int d^3p\left(A(\vec{p}) + A^\dagger(-\vec{p})\right)(i\vec{p})e^{i\vec{p}\cdot\vec{x}} \\
\dot{\phi}(x) &= \int d^3p(-iE_p)\left(A(\vec{p}) - A^\dagger(-\vec{p})\right)e^{i\vec{p}\cdot\vec{x}} \quad (\because \dot{A}(\vec{p},t) = (-iE_p)A(\vec{p},t))
\end{aligned}$$

Therefore,

$$
\begin{aligned}
H &= \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right) \\
&= \int d^3x \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right) \\
&= \int d^3x \int d^3p \int d^3q \;\; e^{i\vec{p}\cdot\vec{x}} e^{i\vec{q}\cdot\vec{x}} \quad \rightarrow (2\pi)^3\delta^{(3)}(\vec{p}+\vec{q}) \\
&\times \left[\frac{1}{2}(-iE_p)(-iE_q)\left(A(\vec{p}) - A^\dagger(-\vec{p})\right)\left(A(\vec{q}) - A^\dagger(-\vec{q})\right) \right. \\
&\qquad + \frac{1}{2}(i\vec{p})(i\vec{q})\left(A(\vec{p}) + A^\dagger(-\vec{p})\right)\left(A(\vec{q}) + A^\dagger(-\vec{q})\right) \\
&\qquad \left. + \frac{1}{2}m^2\left(A(\vec{p}) + A^\dagger(-\vec{p})\right)\left(A(\vec{q}) + A^\dagger(-\vec{q})\right) \right] \\
&= \int d^3q (2\pi)^3 \\
&\times \left[\frac{1}{2}(-E_q^2)\left(A(-\vec{q}) - A^\dagger(\vec{q})\right)\left(A(\vec{q}) - A^\dagger(-\vec{q})\right) \right. \\
&\qquad \left. + \frac{1}{2}\underbrace{(\vec{q}^{\,2} + m^2)}_{E_q^2}\left(A(-\vec{q}) + A^\dagger(\vec{q})\right)\left(A(\vec{q}) + A^\dagger(-\vec{q})\right) \right] \\
&= \int d^3q (2\pi)^3 E_q^2 \left[A(-\vec{q})A^\dagger(-\vec{q}) + A^\dagger(\vec{q})A(\vec{q}) \right] \\
&= \int d^3q (2\pi)^3 E_q^2 \left[A(\vec{q})A^\dagger(\vec{q}) + A^\dagger(\vec{q})A(\vec{q}) \right] \quad (\vec{q} \rightarrow -\vec{q} \text{ in the 1st term}) \\
&= \int d^3q (2\pi)^3 E_q^2 \frac{1}{(2\pi)^6 2E_q} \left[a(\vec{q})a^\dagger(\vec{q}) + a^\dagger(\vec{q})a(\vec{q}) \right]
\end{aligned}
$$

Here, note that the t-dependence of $A(\vec{q},t)$ cancels in H, and hence H is time independent.

By using

$$
a(\vec{q})a^\dagger(\vec{q}) = a^\dagger(\vec{q})a(\vec{q}) + (2\pi)^3\delta^{(3)}(0),
$$

we obtain

$$
H = \int \frac{d^3q}{(2\pi)^3} E_q \left(a^\dagger(\vec{q})a(\vec{q}) + \frac{1}{2}(2\pi)^3\delta^{(3)}(0) \right)
$$

The constant term,

$$
\int d^3q E_q \frac{1}{2}\delta^{(3)}(0)
$$

is the zero-point energy. (This corresponds to the $\frac{1}{2}\hbar\omega$ term in the energy spectrum of the harmonic oscillator, $E = \hbar\omega(a^\dagger a + \frac{1}{2})$.)

The zero-point energy cannot be observed (except through the gravitational force), so we neglect it in the following.

> In fact, there is an ordering ambiguity to quantize the theory from a classical level. If we define the Hamiltonian by
>
> $$H =: \int d^3x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right] : \qquad : aa^\dagger :=: a^\dagger a : \quad \text{normal ordering}$$
>
> then there is no zero-point energy.

In any case, we have $\boxed{H = \int \frac{d^3q}{(2\pi)^3} E_q a^\dagger(\vec{q}) a(\vec{q})}$ (+ const.)

Therefore,

$$\begin{aligned}[H, a^\dagger(\vec{p})] &= \int \frac{d^3q}{(2\pi)^3} E_q a^\dagger(\vec{q}) \underbrace{[a(\vec{q}), a^\dagger(\vec{p})]}_{(2\pi)^3\delta^{(3)}(\vec{q}-\vec{p})} \\ &= E_p a^\dagger(\vec{p}) \\ \text{similarly} \qquad [H, a(\vec{p})] &= -E_p a^\dagger(\vec{p}) \qquad \blacksquare\end{aligned}$$

§ 1.4.4 Consistency check

Now that $\phi(x)$ and H are expressed in terms of a and $a^\dagger$, let's do some consistency check.

(i) Heisenberg eq. $\quad i\dot{\phi}(x) = [\phi(x), H]$.[1]

(ii) $\phi(x)$ is a Heisenberg operator: $\quad \phi(x) = \phi(t, \vec{x}) = e^{iH(t-t_0)}\phi(t_0, \vec{x})e^{-iH(t-t_0)}$.

Problems

(a) Show (i) by using (1) and (6).

(b) Show that $e^{iHt}a(\vec{p})^\dagger e^{-iHt} = a(\vec{p})^\dagger e^{iE_pt}$ and $e^{iHt}a(\vec{p})e^{-iHt} = a(\vec{p})^\dagger e^{-iE_pt}$ by using (6).

(c) Show (ii) by using the result of (b) and eq.(1).

---

[1] (A typo here was corrected (May 27).)

§ 1.4.5 vacuum state

The operator $a(\vec{p})$ decreases the energy:

$$\begin{array}{cccccccc} & |X\rangle & \to & a(\vec{p})\,|X\rangle & \to & a(\vec{q})a(\vec{p})\,|X\rangle & \cdots \\ \text{energy} & E_X & & E_X - E_p & & E_X - E_p - E_q & \end{array}$$

The ground state (lowest energy) state $|0\rangle$ is a state which satisfies

$$a(\vec{p})\,|0\rangle = 0$$

and Lorentz invariant

$$U(\Lambda)\,|0\rangle = |0\rangle\,.$$

§ 1.4.6 One-particle state

The one-particle state in § 0.6 is given by (for free theory)

$$|\vec{p}\rangle = \sqrt{2E_p}a^\dagger(\vec{p})\,|0\rangle\ .$$

normalization

$$\begin{aligned}\langle\vec{q}|\vec{p}\rangle &= \sqrt{2E_q}\sqrt{2E_p}\langle 0|a(\vec{q})a^\dagger(\vec{p})|0\rangle \\ &= \sqrt{2E_q}\sqrt{2E_p}\langle 0|\Big(\underbrace{[a(\vec{q}),a^\dagger(\vec{p})]}_{(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})}+a^\dagger(\vec{p})\underbrace{a(\vec{q})}_{\to 0}\Big)|0\rangle \\ &= (2\pi)^3 2E_p\delta^{(3)}(\vec{p}-\vec{q})\,,\end{aligned}$$

reproducing the normalization in § 0.6.

Lorentz transform:
From the Lorentz transformation $U(\Lambda)\phi(x)U(\Lambda)^{-1} = \phi(\Lambda^{-1}x)$, we expect

$$U(\Lambda)\,|\vec{p}\rangle = \left|\vec{p'}\right\rangle\ .$$

where $p' = \Lambda^{-1}p$. (This may be opposite to ordinary convention.) Let's show it.

$$\begin{aligned}\text{LHS} &= \sqrt{2E_p}U(\Lambda)a^\dagger(\vec{p})U(\Lambda)^{-1}U(\Lambda)\,|0\rangle \\ &= \sqrt{2E_p}U(\Lambda)a^\dagger(\vec{p})U(\Lambda)^{-1}\,|0\rangle \\ \text{RHS} &= \sqrt{2E_{p'}}a^\dagger(\vec{p'})\,|0\rangle\end{aligned}$$

So it is sufficient to show

$$U(\Lambda)a^\dagger(\vec{p})U(\Lambda)^{-1} = \sqrt{\frac{E_{p'}}{E_p}}a^\dagger(\vec{p'}) \qquad \text{------------}(\star).$$

§ 1.4.7 Lorentz transformation of a and $a^\dagger$

Let's show ($\star$).

(i) First of all, for any $f(\vec{p})$,

$$\boxed{\int d^3p \frac{1}{2E_p} f(\vec{p}) = \int d^4p\ \delta(p^2-m^2)\big|_{p^0>0} f(\vec{p})}$$

This is because

$$\begin{aligned}\delta(p^2-m^2) &= \delta((p^0)^2-\vec{p}^2-m^2)\\ &= \frac{\delta(p^0-\sqrt{\vec{p}^2+m^2})}{|2p^0|}+\frac{\delta(p^0+\sqrt{\vec{p}^2+m^2})}{|2p^0|}\\ &\left(\delta(f(x)) = \sum_{x_i;f(x_i)=0}\frac{\delta(x-x_i)}{f'(x_i)}\right)\end{aligned}$$

$$\therefore \int dp^0\delta(p^2-m^2)|_{p^0>0} = \int dp^0\frac{\delta(p^0-\sqrt{\vec{p}^2+m^2})}{|2p^0|} = \frac{1}{2E_p}$$

(ii) Therefore

$$\begin{aligned}\phi(x) &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(a(\vec{p})e^{-ip\cdot x}+a^\dagger(\vec{p})e^{ip\cdot x}\right)\\ &= \int d^4p\delta(p^2-m^2)|_{p^0>0}\frac{\sqrt{2E_p}}{(2\pi)^3}\left(a(\vec{p})e^{-ip\cdot x}+a^\dagger(\vec{p})e^{ip\cdot x}\right)\end{aligned}$$

and its <u>4-momentum</u> FT is given by

$$\begin{aligned}\widetilde{\phi}(k) &\equiv \int d^4x e^{ik\cdot x}\phi(x)\\ &= \int d^4p\delta(p^2-m^2)|_{p^0>0}\frac{\sqrt{2E_p}}{(2\pi)^3}\left(a(\vec{p})(2\pi)^4\delta^{(4)}(p-k)+a^\dagger(\vec{p})(2\pi)^4\delta^{(4)}(p+k)\right)\\ &= (2\pi)\delta(k^2-m^2)\sqrt{2E_k}\left(\theta(k^0)a(\vec{k})+\theta(-k^0)a^\dagger(-\vec{k})\right)\ .\end{aligned}$$

(iii) and its LT is

$$U(\Lambda)\widetilde{\phi}(k)U(\Lambda)^{-1} = (2\pi)\delta(k^2-m^2)\sqrt{2E_k}\left(\theta(k^0)U(\Lambda)a(\vec{k})U(\Lambda)^{-1} + \theta(-k^0)U(\Lambda)a^\dagger(-\vec{k})U(\Lambda)^{-1}\right) .$$

$$\begin{aligned}
\text{LHS} &= \int d^4x e^{ik\cdot x}U(\Lambda)\phi(x)U(\Lambda)^{-1} \\
&= \int d^4x e^{ik\cdot x}\phi(\Lambda^{-1}x) \\
&= \int d^4y e^{ik\cdot(\Lambda y)}\phi(y) \qquad (x=\Lambda y,\ \ d^4x = (\det\Lambda)d^4y = d^4y) \\
&= \int d^4y e^{i(\Lambda k')\cdot(\Lambda y)}\phi(y) \qquad (k' = \Lambda^{-1}k) \\
&= \int d^4y e^{ik'\cdot y}\phi(y) \\
&= \widetilde{\phi}(k') \\
&= (2\pi)\delta(k'^2-m^2)\sqrt{2E_{k'}}\left(\theta(k'^0)a(\vec{k'}) + \theta(-k'^0)a^\dagger(-\vec{k'})\right) .
\end{aligned}$$

Using $k'^2 = k^2$ and $\theta(k'^0) = \theta(k^0)$ and comparing it with RHS, we obtain

$$U(\Lambda)a(\vec{k})U(\Lambda)^{-1} = \sqrt{\frac{E_{k'}}{E_k}}a(\vec{k'}) \qquad \blacksquare$$

§ 1.4.8 $[\phi(x),\phi(y)]$ for $x^0 \neq y^0$

For $x^0 = y^0 = t$, we have $[\phi(x),\phi(y)] = 0$. What if $x^0 \neq y^0$?
From Eq.(1) and the commutation relations of a and $a^\dagger$ in § 1.4.2, we have

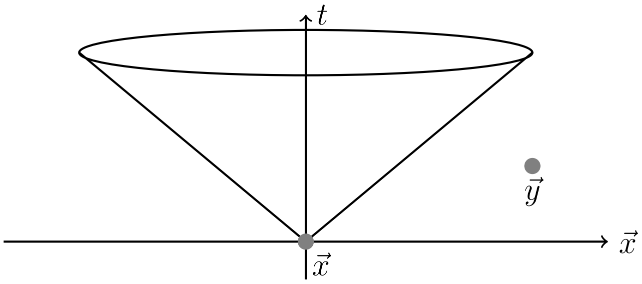
$$\boxed{\begin{aligned}
&[\phi(x),\phi(y)] = \int \frac{d^3p}{(2\pi)^3 2E_p}\left(e^{-ip\cdot(x-y)} - e^{+ip\cdot(x-y)}\right) \qquad \equiv i\Delta(x-y) \\
&\Delta(x) = (-i)\int \frac{d^3p}{(2\pi)^3 2E_p}\left(e^{-ip\cdot x} - e^{+ip\cdot x}\right)
\end{aligned}}$$

Properties of $\Delta(x)$:

(a) $(\Box + m^2)\Delta(x) = 0$.

(b) Lorentz invariant: $\Delta(\Lambda x) = \Delta(x)$.

(c) Local causality: $\Delta(x) = 0$ for $x^2 = (x^0)^2 - \vec{x}^2 < 0$ (space-like).



$\Delta(x-y) = 0$

Among them, (a) is clear from the definition of $\Delta(x)$. (b) can be shown by using the equation in (i) of the previous section § 1.4.7:

$$\begin{aligned}
\Delta(x) &= (-i)\int \frac{d^4p}{(2\pi)^4}\delta(p^2-m^2)\theta(p^0)\left(e^{-ip\cdot x}-e^{ip\cdot x}\right) \\
\Delta(x') &= (-i)\int \frac{d^4p}{(2\pi)^4}\delta(p^2-m^2)\theta(p^0)\left(e^{-ip\cdot x'}-e^{ip\cdot x'}\right) \qquad (x'=\Lambda x) \\
&= (-i)\int \frac{d^4q}{(2\pi)^4}\delta(q^2-m^2)\theta(q^0)\left(e^{-i(\Lambda q)\cdot(\Lambda x)}-e^{i(\Lambda q)\cdot(\Lambda x)}\right) \qquad p=\Lambda q,\ d^4p=d^4q \\
&= (-i)\int \frac{d^4q}{(2\pi)^4}\delta(q^2-m^2)\theta(q^0)\left(e^{-iq\cdot x}-e^{iq\cdot x}\right) \qquad p=\Lambda q,\ d^4p=d^4q \\
&= \Delta(x)\,.
\end{aligned}$$

Finally, (c) can be shown as follows. Fist,

$$\Delta(x^0=0,\vec{x}) = (-i)\int \frac{d^3p}{(2\pi)^3 2E_p}\left(e^{i\vec{p}\cdot\vec{x}}-e^{-i\vec{p}\cdot\vec{x}}\right) = 0\,.$$

On the other hand, for space-like x ($x^2=(x^0)^2-\vec{x}^2<0$), one can always Lorentz transform it to a frame with $x'^0=0$.

> For a Lorentz boost in the opposite direction to $\vec{x}$, x^0 is transformed as $x'^0 = \dfrac{1}{\sqrt{1-\beta^2}}(x^0-\vec{\beta}\cdot\vec{x})$. Taking $\vec{\beta}=\dfrac{x^0}{\vec{x}^2}\vec{x}$, we have $x'^0=0$. Note that this is impossible for a time-like x, where $x^2=(x^0)^2-\vec{x}^2>0$, because $\left|\dfrac{x^0}{\vec{x}^2}\vec{x}\right|>1$ in that case.

Therefore, we have $\Delta(x)=\Delta(x'^0=0,\vec{x'})=0$ for $x^2<0$.

——————— on May 8, up to here. ———————

——————— on May 15, from here. ———————

§ 1.5 Interacting Scalar Field

Lagrangian

$$L = \int d^3x \Big(\underbrace{\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2}_{\text{same as free theory}} \underbrace{-\frac{\lambda}{24}\phi^4}_{\text{Interaction}} \Big)$$

$(\lambda$: real and positive constant)

$$\phi(\vec{x},t) \longleftrightarrow \pi(\vec{x},t)$$

$$= \frac{\delta L}{\delta\dot{\phi}(\vec{x},t)} = \dot{\phi}(\vec{x},t) \qquad \text{(same as free theory)}$$

Equal-Time Commutation Relation

$$[\phi(\vec{x},t), \pi(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y})$$

$$[\phi(\vec{x},t), \phi(\vec{y},t)] = 0$$

$$[\pi(\vec{x},t), \pi(\vec{y},t)] = 0 \qquad \text{(same as free theory)}$$

§ 1.5.1 What is $\phi(x)$?

- In the case of free theory $(\lambda = 0)$,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(e^{-iE_pt}e^{i\vec{p}\cdot\vec{x}}a(\vec{p}) + e^{iE_pt}e^{-i\vec{p}\cdot\vec{x}}a^\dagger(\vec{p}) \right)$$

We could exactly solve the t-dependence by using Fourier transform and the Klein-Gordon eq. $(\Box + m^2)\phi = 0$.

- With the interaction, $\phi(\vec{x},t) = ??$

- The EOM is (see § 1.3)

$$\boxed{\left(\Box + m^2\right)\phi(x) = -\frac{\lambda}{6}\phi(x)^3.}$$

This is non-linear.

- Let's try Fourier transform $\underline{\text{at } t = 0}$.

$$\phi(t=0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \Big(C(\vec{p})e^{i\vec{p}\cdot\vec{x}} + \underbrace{C^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}}_{\text{from } \phi=\phi^\dagger} \Big)$$

Defining $a(\vec{p})$ by $C(\vec{p}) = a(\vec{p})/\sqrt{2E_p}$,

$$\phi(t=0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a(\vec{p})e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}} \right). \qquad \text{———— (1)}$$

Note that, at this stage, $a(\vec{p})$ and $a(\vec{p})^\dagger$ are just coefficients of the Fourier transformation.

- In the Heisenberg picture,

$$\begin{aligned}\phi(t,\vec{x}) &= e^{iHt}\phi(0,\vec{x})e^{-iHt}\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\Big(\underset{\sim\sim\sim\sim\sim\sim\sim\sim}{e^{iHt}a(\vec{p})e^{-iHt}}e^{i\vec{p}\cdot\vec{x}} + \underset{\sim\sim\sim\sim\sim\sim\sim\sim}{e^{iHt}a^\dagger(\vec{p})e^{-iHt}}e^{-i\vec{p}\cdot\vec{x}}\Big).\end{aligned}$$

- What happened in the case of free theory?

$$\begin{aligned}\text{(free theory)}\quad &[H,a(\vec{p})] = -E_p a(\vec{p})\\ \to\quad &e^{iHt}a(\vec{p})e^{-iHt} = a(\vec{p})e^{-iE_pt}\end{aligned}$$

This is the problem (b) of § 1.4.4. An example solution is as follows:

$$\begin{aligned}\text{define}\quad f(t) &= e^{iHt}a(\vec{p})e^{-iHt}\\ \text{then}\quad \dot{f}(t) &= e^{iHt}i[H,a(\vec{p})]e^{-iHt}\\ &= e^{iHt}(-iE_p)a(\vec{p})e^{-iHt}\\ &= (-iE_p)f(t)\\ \text{Thus,}\ f(t) &= e^{-iE_pt}f(0) = e^{-iE_pt}a(\vec{p}). \qquad \blacksquare\end{aligned}$$

- Similarly, $e^{iHt}a^\dagger(\vec{p})e^{-iHt} = a^\dagger(\vec{p})e^{iE_pt}$. Therefore,

$$\text{(free theory)}\quad \phi(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\Big(a(\vec{p})e^{-iE_pt}e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{iE_pt}e^{-i\vec{p}\cdot\vec{x}}\Big),$$

which is a linear combination of $a(\vec{p})$ and $a^\dagger(\vec{p})$.

- However, with the interaction term,

$$\begin{aligned}&H = H_0 + \underline{H_{\text{int}}}\\ &\qquad\qquad\ \sim \phi^4 \sim (a+a^\dagger)^4\\ \to\ &[H,a(\vec{p})] = -E_p a(\vec{p}) + \mathcal{O}(a^3, a^2a^\dagger, a(a^\dagger)^2, (a^\dagger)^3)\\ \to\ &e^{iHt}a(\vec{p})e^{-iHt}\quad \text{includes infinitely many } a \text{ and } a^\dagger.\\ \to\ &\phi(t,\vec{x})\quad \text{also includes infinitely many } a \text{ and } a^\dagger.\end{aligned}$$

Thus, $\phi(x)$ cannot be written as a linear combination of a and $a^\dagger$.
→ It cannot be considered as a field to create/annihilate just 1-particle state.
→ It includes (infinitely many) particle creation/annihilation.
→ We can't discuss scatterings just in terms of $\phi(x)$. →§ 1.5.2.

Comment

Here, we have used $[a, a^\dagger]$ etc, but a and $a^\dagger$ are just Fourier coefficients in Eq. (1). What are these a and $a^\dagger$?

$$\begin{cases} \bullet \ [a, a^\dagger] =? & \\ \bullet \ \dot{\phi}(t=0, \vec{x}) =? & \text{How is it written in terms of } a \text{ and } a^\dagger? \\ \bullet \ H =? & \\ \bullet \ [H, a] =?, \ [H, a^\dagger] =? & \end{cases}$$

In fact, $a(\vec{p})$ is not uniquely determined by

$$(1) \longleftrightarrow \phi(t=0, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a(\vec{p}) + a^\dagger(-\vec{p}) \right) e^{i\vec{p}\cdot\vec{x}}.$$

For <u>any operator $f(\vec{p})$</u>, replacing

$$a(\vec{p}) \to a(\vec{p}) + i \left(f(\vec{p}) + f^\dagger(-\vec{p}) \right)$$

does not change the above equation.

We will define $a(\vec{p})$ more precisely later. (In "interaction picture").

§ 1.5.2 In/out states and the LSZ reduction formula

- We want to define the in/out states in § 0.6.
- In the free theory, one particle state is (see § 1.4.6)

$$|p\rangle = \sqrt{2E_p} a^\dagger(\vec{p}) |0\rangle .$$

where (see § 1.4.1)

$$\begin{aligned} a^\dagger(\vec{p}) &= \frac{1}{\sqrt{2E_p}} \int d^3x \; e^{-ip\cdot x} \left(-i\dot{\phi}(x) + E_p \phi(x) \right) \\ &= \frac{-i}{\sqrt{2E_p}} \int d^3x \; e^{-ip\cdot x} \overleftrightarrow{\partial_0} \phi(x). \\ & \left(f \overleftrightarrow{\partial_0} g \equiv f \partial_0 g - (\partial_0 f) g, \quad \partial_0 = \frac{\partial}{\partial t} \right) \end{aligned}$$

- We consider the same operator <u>in the interacting theory</u>.

$$a^\dagger(\vec{p}, t) = \frac{-i}{\sqrt{2E_p}} \int d^3x \; e^{-ip\cdot x} \overleftrightarrow{\partial_0} \phi(x),$$

which is now <u>time-dependent</u>. ($\frac{\partial}{\partial t}(RHS)\neq 0$ for $\lambda \neq 0$.)

- And we define the in/out states by

$$
\begin{aligned}
|\vec{p_1}\cdots\vec{p_n};\ \mathrm{in}\rangle &= \sqrt{2E_{p_1}}a^\dagger(\vec{p_1},-\infty)\cdots\sqrt{2E_{p_n}}a^\dagger(\vec{p_n},-\infty)\,|0\rangle \\
|\vec{q_1}\cdots\vec{q_m};\ \mathrm{out}\rangle &= \sqrt{2E_{q_1}}a^\dagger(\vec{q_1},+\infty)\cdots\sqrt{2E_{q_n}}a^\dagger(\vec{q_m},+\infty)\,|0\rangle
\end{aligned}
$$

where

$$
\begin{aligned}
a^\dagger(\vec{p},-\infty) &= \lim_{x^0\to-\infty}\frac{-i}{\sqrt{2E_p}}\int d^3x\ e^{-ip\cdot x}\overleftrightarrow{\partial_0}\phi(x), \\
a^\dagger(\vec{p},+\infty) &= \lim_{x^0\to+\infty}\frac{-i}{\sqrt{2E_p}}\int d^3x\ e^{-ip\cdot x}\overleftrightarrow{\partial_0}\phi(x).
\end{aligned}
$$

Comments

(i) One can think of operators with wave-packets:

$$\widetilde{a}^\dagger(t) = \int d^3p f(\vec{p})a^\dagger(\vec{p},t)$$

$$\text{with}\quad f(\vec{p}) \sim \exp\left(-\frac{(\vec{p}-\vec{p_1})^2}{4\sigma^2}\right)$$

and then later take $\sigma \to 0$. See e.g., the textbooks by Srednicki [1] and/or Peskin [2].

(ii) Here, the vacuum state $|0\rangle$ is the ground state of the full Hamiltonian $H = H_0 + H_{\mathrm{int}}$. (This comment is added after the lecture. See the comment at the end of this subsection.)

- Then, one can show

LSZ reduction formula

$$
\begin{aligned}
&\langle\vec{p_1}\cdots\vec{p_n};\ \mathrm{in}|\vec{q_1}\cdots\vec{q_m};\ \mathrm{out}\rangle \\
&= \prod_{i=1}^{m}\left[i\int d^4x_i e^{+iq_i\cdot x_i}\left(\Box_{x_i}+m^2\right)\right] \\
&\times \prod_{i=1}^{n}\left[i\int d^4y_i e^{-ip_i\cdot y_i}\left(\Box_{y_i}+m^2\right)\right] \\
&\times \langle 0|\mathrm{T}\left(\phi(x_i)\cdots\phi(x_n)\ \phi(y_1)\cdots\phi(y_m)\right)|0\rangle
\end{aligned}
$$

where

T : time-ordering

$$\mathrm{T}\left(\phi(x)\phi(y)\right) = \begin{cases}\phi(x)\phi(y) & \text{for}\quad x^0 > y^0 \\ \phi(y)\phi(x) & \text{for}\quad y^0 > x^0\end{cases}$$

$$\mathrm{T}\left(\phi(x_1)\phi(x_2)\phi(x_3)\cdots\right) = \phi(x_{i_1})\phi(x_{i_2})\phi(x_{i_3})\cdots \quad \text{for}\quad x^0_{i_1} > x^0_{i_2} > x^0_{i_3} > \cdots$$

- The LSZ reduction formula shows the relation between $\underline{\text{the S-matrix } \langle \text{in}|\text{out}\rangle}$ (see § 0.6) and $\underline{\text{the time-ordered correlation function } \langle 0|\mathrm{T}\left(\phi(x)\cdots\right)|0\rangle}$.

- Let's show it. First, by the definition of in/out states,

$$\begin{aligned}&(\text{LHS of the LSZ formula})\\ &=\sqrt{2E_{p_1}}\cdots\sqrt{2E_{q_1}}\cdots\langle 0|a(\vec{q_1},+\infty)\cdots a(\vec{q_m},+\infty)a^\dagger(\vec{p_1},-\infty)\cdots a^\dagger(\vec{p_n},-\infty)|0\rangle\end{aligned}$$

This is already time-ordered. Thus, we can put in the time-ordering operator T without changing anything:

$$\begin{aligned}&(\text{LHS of the LSZ formula})\\ &=\sqrt{2E_{p_1}}\cdots\sqrt{2E_{q_1}}\cdots\langle 0|\mathrm{T}\left(a(\vec{q_1},+\infty)\cdots a(\vec{q_m},+\infty)a^\dagger(\vec{p_1},-\infty)\cdots a^\dagger(\vec{p_n},-\infty)\right)|0\rangle\end{aligned}$$

——————————(1)

- Next,

$$\begin{aligned}a^\dagger(\vec{p},+\infty)-a^\dagger(\vec{p},-\infty)&=\int_{-\infty}^{\infty}dt\;\partial_0\;a^\dagger(\vec{p},t)\\ &=\int_{-\infty}^{\infty}dt\;\partial_0\left[\frac{-i}{\sqrt{2E_p}}\int d^3x\;e^{-ip\cdot x}\overleftrightarrow{\partial_0}\,\phi(x)\right]\\ &=\frac{-i}{\sqrt{2E_p}}\int d^4x\;\underbrace{\partial_0\;\left(e^{-ip\cdot x}\overleftrightarrow{\partial_0}\,\phi(x)\right)}\end{aligned}$$

$$\begin{aligned}&=e^{-ip\cdot x}(\partial_0^2+E_p^2)\phi(x)\\ &=e^{-ip\cdot x}(\partial_0^2+\vec{p}^2+m^2)\phi(x)\\ &=e^{-ip\cdot x}(\partial_0^2-\overleftarrow{\nabla}^2+m^2)\phi(x)\\ &=e^{-ip\cdot x}(\partial_0^2-\overrightarrow{\nabla}^2+m^2)\phi(x)\end{aligned}$$

$$\left[\begin{aligned}&\because\int d^3xe^{i\vec{p}\cdot\vec{x}}\overleftarrow{\nabla}^2f(\vec{x})\\ &=\int d^3xe^{i\vec{p}\cdot\vec{x}}\overrightarrow{\nabla}^2\int d^3qe^{-i\vec{q}\cdot\vec{x}}\int\frac{d^3y}{(2\pi)^3}e^{i\vec{q}\cdot\vec{y}}f(\vec{y})\\ &=\underline{\int d^3xe^{i\vec{p}\cdot\vec{x}}}\int d^3q(-\vec{q}^2)\underline{e^{-i\vec{q}\cdot\vec{x}}}\int\frac{d^3y}{(2\pi)^3}e^{i\vec{q}\cdot\vec{y}}f(\vec{y})\\ &=\int d^3q(-\vec{q}^2)\underline{(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})}\int\frac{d^3y}{(2\pi)^3}e^{i\vec{q}\cdot\vec{y}}f(\vec{y})\\ &=(-\vec{p}^2)\int d^3ye^{i\vec{p}\cdot\vec{y}}f(\vec{y})\\ &=\int d^3y\left(\overrightarrow{\nabla}^2e^{i\vec{p}\cdot\vec{y}}\right)f(\vec{y})\end{aligned}\right]$$

and therefore

$$a^\dagger(\vec{p},+\infty) - a^\dagger(\vec{p},-\infty) = \frac{-i}{\sqrt{2E_p}} \int d^4x\, e^{-ip\cdot x}(\Box^2 + m^2)\phi(x)$$

or

$$a^\dagger(\vec{p},-\infty) = a^\dagger(\vec{p},+\infty) + \frac{i}{\sqrt{2E_p}} \int d^4x\, e^{-ip\cdot x}(\Box^2 + m^2)\phi(x)$$

Similarly, one can show

$$a(\vec{p},+\infty) = a(\vec{p},-\infty) + \frac{i}{\sqrt{2E_p}} \int d^4x\, e^{ip\cdot x}(\Box^2 + m^2)\phi(x)$$

- ▶ Substituting these equations to (1),

(LHS of the LSZ formula)

$$= \sqrt{2E_{p_1}}\cdots\sqrt{2E_{q_1}}\cdots\langle 0|\mathrm{T}\left(\underbrace{a(\vec{q_1},+\infty)}_{=\ a(\vec{q_1},-\infty)+\frac{i}{\sqrt{2E_{q_1}}}\int\cdots}\cdots a(\vec{q_m},+\infty)\underbrace{a^\dagger(\vec{p_1},-\infty)}_{=\ a^\dagger(\vec{p_1},+\infty)+\frac{i}{\sqrt{2E_{q_1}}}\int\cdots}\cdots a^\dagger(\vec{p_n},-\infty)\right)|0\rangle$$

time-ordering
$a(\vec{q_1},-\infty)\,|0\rangle = 0$

time-ordering
$\langle 0|\, a^\dagger(\vec{p_1},+\infty) = 0$

$$= \langle 0| \left(i\int d^4x_1\, e^{iq_1\cdot x_1}(\Box^2_{x_1}+m^2)\phi(x_1)\right)\cdots\left(i\int d^4x_m\, e^{iq_m\cdot x_m}(\Box^2_{x_m}+m^2)\phi(x_m)\right)$$
$$\times\left(i\int d^4y_1\, e^{-ip_1\cdot y_1}(\Box^2_{y_1}+m^2)\phi(y_1)\right)\cdots\left(i\int d^4y_n\, e^{ip_n\cdot y_n}(\Box^2_{y_n}+m^2)\phi(y_n)\right)|0\rangle$$

= (RHS of the LSZ formula) ■

——— on May 15, up to here. ———

Questions and comments after the lecture: (only some of them)

comment: There is a typo at § 1.4.4. The Heisenberg equation should be $i\dot{\phi}(x) = [\phi(x), H]$ instead of $i\dot{\phi}(x) = [H, \phi(x)]$.

A: Thanks! corrected.

Q: What happens to the LSZ formula, in particular the time-ordering, if you do a Lorentz transformation?

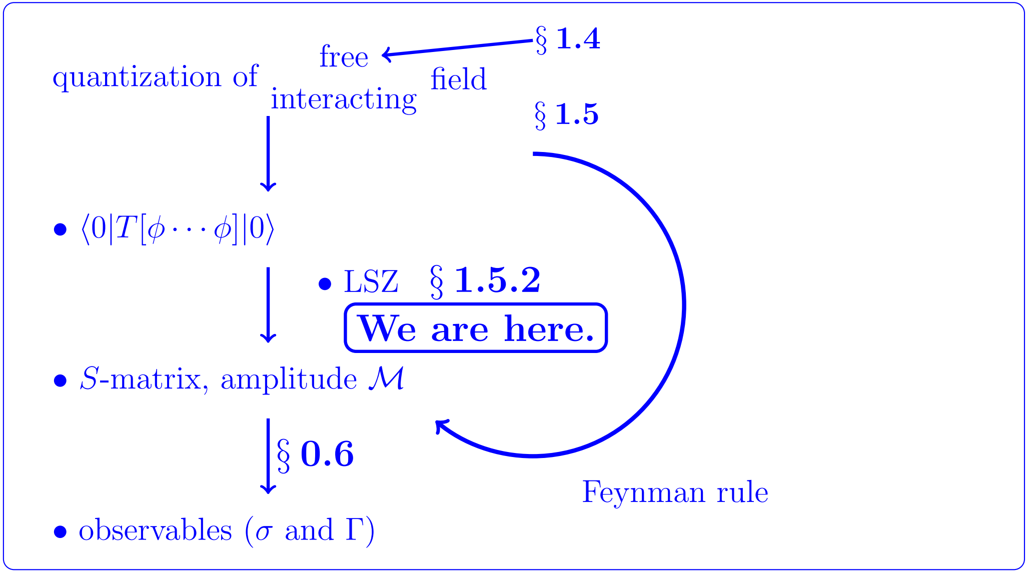
A: Good question. It is implicitly assumed that all the participating particles are causally connected, i.e., $(x_i - y_j)^2 > 0$ (time-like). In this case, the ordering $x_i^0 > y_j^0$ doesn't change, the proof can be done in the same way, and therefore the LSZ formula in the Lorentz-boosted frame has the same form as in the original frame.

Q: Why $a^\dagger(\vec{p}, \pm\infty) = \lim_{x^0 \to \pm\infty} \frac{-i}{\sqrt{2E_p}} \int d^3x \; e^{-ip\cdot x} \overleftrightarrow{\partial_0} \phi(x)$?

A: That's the definition of the operator $a^\dagger(\vec{p}, \pm\infty)$.

——————— on May 29, from here. ———————

Outline



Comments

(i) In the derivation of the LSZ formula, we have used

$$a(\vec{p}, \pm\infty) \,|0\rangle = 0$$

where $|0\rangle$ is the ground state (lowest energy state) of the full Hamiltonian $H = H_0 + H_{\text{int}}$. There is a subtlety here, but we will not discuss the details in this lecture.

Under certain assumptions (axioms) on the quantum field theory, such as “spectral conditions” (スペクトル条件), “asymptotic completion” (漸近的完全性), and “LSZ asymptotic condition”, one can show the above equation $a(\vec{p}, \pm\infty)\,|0\rangle = 0$.

For instance, the “asymptotic completeness” (漸近的完全性) says that, the Fock space spanned by the “in”-operators:

$$\mathcal{V}^{\text{in}} = \Big\{ |0\rangle \,, a^\dagger(\vec{p}, -\infty)\,|0\rangle \,, a^\dagger(\vec{p}, -\infty) a^\dagger(\vec{p}', -\infty)\,|0\rangle \,, \cdots \Big\}$$

and that by the "out"-operators:

$$\mathcal{V}^{\text{out}} = \left\{ |0\rangle \, , a^\dagger(\vec{p}, +\infty) \, |0\rangle \, , a^\dagger(\vec{p}, +\infty) a^\dagger(\vec{p}', +\infty) \, |0\rangle \, , \cdots \right\}$$

are the same as the Fock space spanned by the Heisenberg operator $\phi(x)$, $\mathcal{V}$:

$$\mathcal{V}^{\text{in}} = \mathcal{V}^{\text{out}} = \mathcal{V}.$$

(For more details, see e.g.,
Kugo-san's textbook [5] 「ゲージ場の量子論 I」九後汰一郎、培風館
and Sakai-san's textbook [6] 「場の量子論」坂井典佑、裳華房.
They are in Japanese. I have checked several QFT textbooks in English, such as Peskin [2], Schrednicki [1], Weinberg [3], etc, but I couldn't find the corresponding explanation.)

(ii) In general, the operator defined by

$$a^\dagger(\vec{p}, \pm\infty) = \lim_{x^0 \to \pm\infty} \frac{-i}{\sqrt{2E_p}} \int d^3x e^{-ip\cdot x} \overleftrightarrow{\partial_0} \phi(x)$$

does not give the correct normalization for the 1-particle state.
($\sqrt{2E_p} a^\dagger(\vec{p}, \pm\infty) \, |0\rangle \propto |\vec{p}\rangle$, but normalization is not correct in general.)

One should either define the operator by

$$a^\dagger(\vec{p}, \pm\infty) = \frac{1}{\sqrt{Z}} \lim_{x^0 \to \pm\infty} \frac{-i}{\sqrt{2E_p}} \int d^3x e^{-ip\cdot x} \overleftrightarrow{\partial_0} \phi(x)$$

(see e.g, Kugo-san's and Sakai-san's textbooks [5, 6]),
or rescale the field as

$$\phi(x) = \sqrt{Z} \phi_r(x) \qquad (\phi_r(x) : \text{rescaled, or renormalized field})$$

(see e.g., Srednicki's textbook [1])
where

$$Z = |\langle p|\phi(x)|0\rangle|^2$$

represents how much the state $\phi(x) \, |0\rangle$ contains the one-particle state $|p\rangle$.
(Note that $\langle p|\phi(x)|0\rangle = e^{ip\cdot x}$ and $Z = 1$ for the free theory.)

In this lecture, we do not discuss the renormalization, and take $Z = 1$ as the leading order perturbation.

§ 1.5.3 Heisenberg field and Interaction picture field

$$\begin{tikzcd} \langle 0|\mathrm{T}(\phi(x_1)\cdots\phi(x_n))|0\rangle \arrow[d, "\S\,1.5.2\text{, LSZ formula}"] & \arrow[l] \textbf{\underline{Now we want to calculate this.}} \\ \langle \mathrm{out}|\mathrm{in}\rangle \arrow[d, "\S\,0.6"] & \\ \sigma, \Gamma & \end{tikzcd}$$

Idea: perturbative expansion in the coupling λ. There are two ways:

- Here, we perform the perturbation in terms of the "interaction picture field".
- Another way is the "path integral" formalism.

The two ways give the same result for $\langle 0|\mathrm{T}(\phi(x_1)\cdots\phi(x_n))|0\rangle$.

► Let's start from $\underline{\phi(0,\vec{x})}$ and $\underline{\dot{\phi}(0,\vec{x})}$ at $t=0$. The Equal-time commutation relation at $t=0$ is

$$[\phi(0,\vec{x}),\dot{\phi}(0,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}). \qquad \text{———————①}$$

Define the Hamiltonian at $t=0$.

$$H = \underbrace{\int d^3x\left(\frac{1}{2}\dot{\phi}(0,\vec{x})^2 + \frac{1}{2}\left(\vec{\nabla}\phi(0,\vec{x})\right)^2 + \frac{1}{2}m^2\phi(0,\vec{x})^2\right)}_{H_0} + \underbrace{\int d^3x\left(\frac{\lambda}{24}\phi(0,\vec{x})^4\right)}_{H_{\text{int}}}$$

Note that H_0 and H_{int} are defined in terms of $\underline{\phi(0,\vec{x})}$ and $\underline{\dot{\phi}(0,\vec{x})}$ at $t=0$, and they are time-independent.

► For $t \neq 0$,

$$\begin{cases} \text{Heisenberg field} & \phi(t,\vec{x}) = e^{iHt}\phi(0,\vec{x})e^{-iHt} \quad \text{(evolved by } H\text{)} \\ \text{Interaction piecture field } \boxed{\textbf{new!}} & \phi_I(t,\vec{x}) = e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t} \quad \text{(evolved by } H_0\text{)} \end{cases}$$

———————②

► Properties of $\phi(x)$ and $\phi_I(x)$.

(i) From ②, Heisenberg equations are

$$\begin{cases} \dot{\phi} = i[H,\phi], & \ddot{\phi} = i[H,\dot{\phi}], \\ \dot{\phi}_I = i[H_0,\phi_I], & \ddot{\phi}_I = i[H_0,\dot{\phi}_I]. \end{cases}$$

(ii) One can also show

$$\begin{cases} \dot{\phi}(t,\vec{x}) = e^{iHt}\dot{\phi}(0,\vec{x})e^{-iHt} \\ \dot{\phi}_I(t,\vec{x}) = e^{iH_0t}\dot{\phi}_I(0,\vec{x})e^{-iH_0t} \end{cases} \qquad \text{———————} ③$$

$$\because \Big(\text{define } \; A(t) = e^{-iHt}\dot{\phi}(t,\vec{x})e^{iHt}$$

$$\text{then } \; \dot{A}(t) = e^{-iHt}\left(-i[H,\dot{\phi}] + \ddot{\phi}\right)e^{iHt} = 0$$

$$\text{so } \; A(t) = A(0) = \dot{\phi}(0,\vec{x}). \quad \blacksquare$$

$$\phi_I \text{ is similar.}\Big)$$

(ii)' and

$$\begin{aligned} \dot{\phi}_I(0,\vec{x}) &= i[H_0, \phi_I(0,\vec{x})] \\ &= i[H - H_{\text{int}}, \phi(0,\vec{x})] \\ &= i[H, \phi(0,\vec{x})] \\ &= \dot{\phi}(0,\vec{x}). \end{aligned}$$

Note that, $\underline{\phi_I(0,\vec{x}) = \phi(0,\vec{x})}$ and $\underline{\dot{\phi}_I(0,\vec{x}) = \dot{\phi}(0,\vec{x})}$ but $\underline{\ddot{\phi}_I(0,\vec{x}) \neq \ddot{\phi}(0,\vec{x})}$.

(iii) The equal-time commutation relations hold even at $t \neq 0$, both for ϕ and ϕ_I.

$$\begin{aligned} [\phi(t,\vec{x}), \dot{\phi}(t,\vec{y})] &= e^{iHt}[\phi(0,\vec{x}), \dot{\phi}(0,\vec{y})]e^{-iHt} \qquad (\because ② ③) \\ &= e^{iHt} i\delta^{(3)}(\vec{x}-\vec{y})e^{-iHt} \qquad (\because ①) \\ &= i\delta^{(3)}(\vec{x}-\vec{y}) \\ [\phi_I(t,\vec{x}), \dot{\phi}_I(t,\vec{y})] &= e^{iH_0t}[\phi(0,\vec{x}), \dot{\phi}(0,\vec{y})]e^{-iH_0t} = i\delta^{(3)}(\vec{x}-\vec{y}) \end{aligned}$$

(iv) H can be written in terms of $\phi(t,\vec{x})$, and
H_0 can be written in terms of $\phi_I(t,\vec{x})$.

$$\begin{aligned} H &= e^{iHt}He^{-iHt} \\ &= e^{iHt}\int d^3x \left(\frac{1}{2}\dot{\phi}(0,\vec{x})^2 + \cdots + \frac{\lambda}{24}\phi(0,\vec{x})^4\right)e^{-iHt} \\ &= \int d^3x \left(\frac{1}{2}\dot{\phi}(t,\vec{x})^2 + \cdots + \frac{\lambda}{24}\phi(t,\vec{x})^4\right) \qquad (\because ② ③) \end{aligned}$$

Each time in the RHS is time-dependent, but the sum is time-independent.
Similarly,

$$\begin{aligned} H_0 &= e^{iH_0t}H_0e^{-iH_0t} \\ &= e^{iH_0t}\int d^3x \left(\frac{1}{2}\dot{\phi}_I(0,\vec{x})^2 + \frac{1}{2}\left(\vec{\nabla}\phi_I(0,\vec{x})\right)^2 + \frac{1}{2}m^2\phi_I(0,\vec{x})^2\right)e^{-iH_0t} \\ &= \int d^3x \left(\frac{1}{2}\dot{\phi}_I(t,\vec{x})^2 + \frac{1}{2}\left(\vec{\nabla}\phi_I(t,\vec{x})\right)^2 + \frac{1}{2}m^2\phi_I(t,\vec{x})^2\right) \qquad (\because ② ③) \end{aligned}$$

The RHS is (the sum is) time-independent.
Namely,

	if written in terms of $\phi(t,\vec{x})$	if written in terms of $\phi_I(t,\vec{x})$
H_0	wrong	OK (t-independent)
H_{int}	wrong	wrong
$H = H_0 + H_{\text{int}}$	OK (t-independent)	wrong

(v) Equation of motion: From (i),

$$\begin{aligned}\ddot{\phi}(x) &= i[H, \dot{\phi}(x)] \\ &= i\int d^3y \left[\frac{1}{2}\dot{\phi}(y)^2 + \frac{1}{2}\left(\vec{\nabla}\phi(y)\right)^2 + \frac{1}{2}m^2\phi(y)^2 + \frac{\lambda}{24}\phi(y)^4,\ \dot{\phi}(x)\right]\end{aligned}$$

From (iv), we can take $x^0 = y^0$. Then

1st term: $[\dot{\phi}(y)^2, \dot{\phi}(x)]_{x^0=y^0} = 0.$

2nd term:
$$\begin{aligned} & i\int d^3y \frac{1}{2}\left[\left(\vec{\nabla}\phi(y)\right)^2, \dot{\phi}(x)\right]_{x^0=y^0} \\ &= i\int d^3y \frac{1}{2}\left(\vec{\nabla}\phi(y)\cdot\vec{\nabla}_y[\phi(y),\dot{\phi}(x)] + \vec{\nabla}_y[\phi(y),\dot{\phi}(x)]\cdot\vec{\nabla}\phi(y)\right)_{x^0=y^0} \\ &= i\int d^3y\, \vec{\nabla}\phi(y)\cdot\vec{\nabla}_y i\delta^{(3)}(\vec{x}-\vec{y})\Big|_{x^0=y^0} \\ &= i\int d^3y \left(-\vec{\nabla}^2\phi(y)\right) i\delta^{(3)}(\vec{x}-\vec{y})\Big|_{x^0=y^0} \qquad (\because \text{integration by parts}) \\ &= \vec{\nabla}^2\phi(x). \end{aligned}$$

3rd term:
$$\begin{aligned} & i\int d^3y \frac{1}{2}m^2[\phi(y)^2, \dot{\phi}(x)]_{x^0=y^0} \\ &= -m^2\phi(x). \qquad (\because [\phi^2,\dot{\phi}] = \phi[\phi,\dot{\phi}] + [\phi,\dot{\phi}]\phi) \end{aligned}$$

4th term: $= -\frac{\lambda}{6}\phi(x)^3.$

Thus,

$$\ddot{\phi} = (\vec{\nabla}^2 - m^2)\phi - \frac{\lambda}{6}\phi^3$$

$$\boxed{(\Box + m^2)\phi = -\frac{\lambda}{6}\phi^3}$$

Similarly,

$$\begin{aligned}\ddot{\phi}_I &= i[H_0, \dot{\phi}_I] \\ &= \cdots (\text{write } H_0 \text{ in terms of } \phi_I) \\ &= (\vec{\nabla}^2 - m^2)\phi_I.\end{aligned}$$

$$\therefore \boxed{(\Box + m^2)\phi_I = 0} \qquad \underline{\phi_I \text{ is a free field!}}$$

§ 1.5.4 a and $a^\dagger$ (again)

- ϕ_I satisfies $(\Box + m^2)\phi_I = 0$ and therefore it can be solved exactly (see § 1.4.1).

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x}\right)\Big|_{p^0=E_p}$$

where $a(\vec{p})$ and $a^\dagger(\vec{p})$ are the expansion coefficients of the interaction picture field ϕ_I. (We can also write them as a_I and $a_I^\dagger$.)

Thus, the original $\phi(0,\vec{x})$ and $\dot{\phi}(0,\vec{x})$ as well as H_0 and H_{int} can be expanded in terms of a and $a^\dagger$.

$$\left.\begin{aligned} \phi(0,\vec{x}) &= \phi_I(0,\vec{x}) = \cdots \\ \dot{\phi}(0,\vec{x}) &= \dot{\phi}_I(0,\vec{x}) = \cdots \\ H_0 &= \cdots \\ \xrightarrow[\text{substitute}]{} \quad H_{\text{int}} &= \cdots \end{aligned}\right\} \quad \text{all written in terms of } a, a^\dagger.$$

- From the commutation relation of ϕ_I, $\dot{\phi}_I$, and H_0, one can show the following relations (see § 1.4.2 and § 1.4.3)

$$\begin{aligned} &[a(\vec{p}), a^\dagger(\vec{q})] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \\ &[a, a] = 0 \\ &[a^\dagger, a^\dagger] = 0 \\ &[H_0, a^\dagger(\vec{p})] = E_p a^\dagger(\vec{p}) \\ &[H_0, a(\vec{p})] = -E_p a(\vec{p}) \end{aligned}$$

Note that the last two equations hold for H_0, not H.

- The state annihilated by $a(\vec{p})$:

$$|0\rangle_I : \qquad a(\vec{p})\,|0\rangle_I = 0, \qquad H_0\,|0\rangle_I = 0 \quad (H_0 : \text{ normal ordered})$$

is <u>NOT</u> the ground state of the full Hamiltonian:

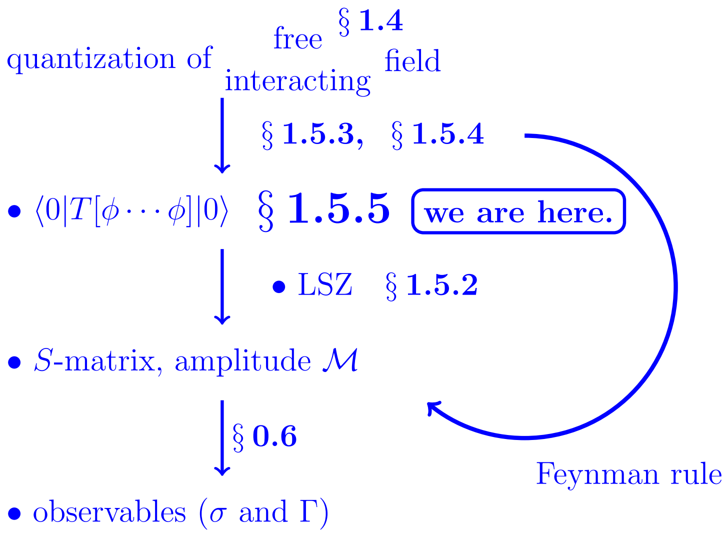
$$\begin{aligned} H\,|0\rangle_I = (H_0 + H_{\text{int}})\,|0\rangle_I &\neq 0 \\ \because H_{\text{int}} \sim \phi_I^4 &\sim (a + a^\dagger)^4 \end{aligned}$$

$$\boxed{|0\rangle_I \neq |0\rangle}$$

——————— on May 29, up to here. ———————

——————— on June 5, from here. ———————

Outline



quantization of $\begin{matrix}\text{free}\\ \text{interacting}\end{matrix}$ field § **1.4**

§ **1.5.3**, § **1.5.4**

- $\langle 0|T[\phi\cdots\phi]|0\rangle$ § **1.5.5** **we are here.**

- LSZ § **1.5.2**

- S-matrix, amplitude $\mathcal{M}$

§ **0.6**

- observables (σ and Γ)

Feynman rule

§ **1.5.5** $\langle 0|\mathrm{T}\left(\phi(x_1)\cdots\phi(x_n)\right)|0\rangle = ?$

We want to express it in terms of ϕ_I (a and $a^\dagger$).
Step (i) ∼ (vii).

(i) redefine the space-time points such that $x_1^0 > x_2^0 > \cdots x_n^0$,

$$\langle 0|\mathrm{T}\left(\phi(x_1)\cdots\phi(x_n)\right)|0\rangle = \langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle. \qquad \text{———} \; (1)$$

(ii) $\phi(x) = ?$

$$\begin{cases}\phi(x) = e^{iHt}\phi(0,\vec{x})e^{-iHt}\\ \phi_I(x) = e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t}\end{cases}$$

$$\to \; \phi(x) = e^{iHt}e^{-iH_0t}\phi_I(x)\underbrace{e^{iH_0t}e^{-iHt}}_{\equiv u(t)}$$

$$\phi(x) = u^\dagger(t)\phi_I(x)u(t). \qquad \text{———} \; (2)$$

(iii) $|0\rangle = ?$

$$\begin{aligned} {}_I\langle 0|\, u(t) &= {}_I\langle 0|\, e^{iH_0t}e^{-iHt} \\ &= {}_I\langle 0|\, e^{-iHt} \qquad (\because H_0\,|0\rangle_I = 0) \end{aligned}$$

Insert an identity operator:

$$\mathbf{1} = |0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|$$

where $|n\rangle$ represent the eigenstates of H with eigenvalues $E_n > E_0 = 0$. (The summation includes continuous parameter (integral).) Then

$$\begin{aligned}
{}_I\langle 0|\, u(t) &= {}_I\langle 0| \left[|0\rangle \langle 0| + \sum_{n=1} |n\rangle \langle n| \right] e^{-iHt} \\
&= {}_I\langle 0|0\rangle \langle 0|\, e^{-iHt} + \sum_{n=1} {}_I\langle 0|n\rangle \langle n|\, e^{-iHt} \\
&= {}_I\langle 0|0\rangle \langle 0| \qquad + \sum_{n=1} {}_I\langle 0|n\rangle \langle n|\, e^{-iE_n t}
\end{aligned}$$

The 2nd term oscillates for $t \to \infty$. Thus, for regularization, we take

$$t \to \infty(1 - i\epsilon) \qquad (\epsilon > 0, \epsilon \to 0)$$
$$\text{then, } e^{-iE_n t} \to e^{-iE_n \infty(1-i\epsilon)} \propto e^{-E_n \infty \cdot \epsilon} \to 0$$

Therefore

$$\lim_{t\to\infty(1-i\epsilon)} {}_I\langle 0|\, u(t) = {}_I\langle 0|0\rangle \langle 0|\,.$$

Similarly

$$\lim_{t\to\infty(1-i\epsilon)} u^\dagger(-t)\, |0\rangle_I = |0\rangle \langle 0|0\rangle_I.$$

Thus

$$\begin{aligned}
\langle 0|\mathcal{O}|0\rangle &= \frac{{}_I\langle 0|0\rangle\langle 0|\mathcal{O}|0\rangle\langle 0|0\rangle_I}{{}_I\langle 0|0\rangle\, \underbrace{\langle 0|0\rangle}_{1}\langle 0|0\rangle_I} \\
&= \lim_{t\to\infty(1-i\epsilon)} \frac{{}_I\langle 0|u(t)\mathcal{O}u^\dagger(-t)|0\rangle_I}{{}_I\langle 0|u(t)u^\dagger(-t)|0\rangle_I} \qquad \text{———————} \ (3)
\end{aligned}$$

(iv) Substituting (2) (3) to (1),

$$\begin{aligned}
&\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle \\
&= \lim_{t\to\infty(1-i\epsilon)} \frac{1}{{}_I\langle 0|u(t)u^\dagger(-t)|0\rangle} \\
&\times\ {}_I\langle 0|\, \underbrace{u(t)\cdot u^\dagger(t_1)}\, \phi_I(x_1)\, \underbrace{u(t_1)\cdot u^\dagger(t_2)}\, \phi_I(x_2)\, \underbrace{u(t_2)\cdots}\cdots \underbrace{\cdots u^\dagger(t_n)}\, \phi_I(x_n)\, \underbrace{u(t_n)\cdot u^\dagger(-t)}\, |0\rangle_I \\
&\qquad \text{———————} \ (4)
\end{aligned}$$

(v)

$$\begin{aligned}
U(t_1, t_2) &\equiv u(t_1)u^\dagger(t_2) \qquad (t_1 > t_2) \\
&= e^{iH_0 t_1} e^{-iH(t_1-t_2)} e^{-iH_0 t_2} \qquad =?
\end{aligned}$$

It satisfies

$$\begin{cases} U(t,t) = 0 \\ \dfrac{\partial}{\partial t_1} U(t_1,t_2) = -iH_I(t_1)U(t_1,t_2) \\ \dfrac{\partial}{\partial t_2} U(t_1,t_2) = iU(t_1,t_2)H_I(t_2) \end{cases} \qquad \text{———————} \ (5)$$

where

$$\begin{aligned} H_I(t) &\equiv e^{iH_0t}H_{\text{int}}e^{-iH_0t} \\ &= e^{iH_0t}\int d^3x \frac{\lambda}{24}\phi(0,\vec{x})^4 e^{-iH_0t} \\ &= \int d^3x \frac{\lambda}{24}\phi_I(x)^4 \end{aligned}$$

$$\left(\begin{array}{lrcl} \text{check (5):} & \dfrac{d}{dt}u(t) & = & \dfrac{d}{dt}e^{iH_0t}e^{-iHt} \\ & & = & e^{iH_0t}(iH_0 - iH)e^{-iHt} \\ & & = & e^{iH_0t}(-iH_{\text{int}})e^{-iH_0t}e^{iH_0t}e^{-iHt} \\ & & = & -iH_I(t)u(t) \\ & \dfrac{d}{dt}u^\dagger(t) & = & \cdots \\ & & = & iu^\dagger(t)H_I(t) \end{array} \right)$$

The solution of (5) is, if $H_I(t)$ at different t are commuting,

$$\times \quad U(t_1,t_2) = \exp\left(-i\int_{t_2}^{t_1} H_I(t)dt\right),$$

but this is wrong. The correct solution is

$$\begin{aligned} U(t_1,t_2) &= \mathrm{T}\left[\exp\left(-i\int_{t_2}^{t_1} H_I(t)dt\right)\right] \\ &= \mathrm{T}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(-i\int_{t_2}^{t_1} H_I(t)dt\right)^n\right] \qquad \text{———————} \ (6) \end{aligned}$$

Let's check it.

$$\begin{aligned} \frac{\partial}{\partial t_1}U(t_1,t_2) &= \sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{T}\left[\frac{\partial}{\partial t_1}\left(-i\int_{t_2}^{t_1} H_I(t)dt\right)^n\right] \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{T}\left[\sum_{k=1}^{n}\left(-i\int_{t_2}^{t_1} H_I(t)dt\right)^{k-1}\underbrace{(-iH_I(t_1))}_{}\left(-i\int_{t_2}^{t_1} H_I(t)dt\right)^{n-k}\right] \end{aligned}$$

Here, t_1 of $H_I(t_1)$ is larger than other t ($t_1 \geq t \geq t_2$), and therefore $H_I(t_1)$ can be moved in front of the time-ordering:

$$\begin{aligned}\frac{\partial}{\partial t_1}U(t_1,t_2) &= -iH_I(t_1)\sum_{n=1}^{\infty}\frac{1}{n!}\mathrm{T}\left[n\cdot\left(-i\int_{t_2}^{t_1}H_I(t)dt\right)^{n-1}\right]\\ &= -iH_I(t_1)U(t_1,t_2) \qquad \blacksquare\end{aligned}$$

(vi) From (4),

$$\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle = \lim_{t\to\infty(1-i\epsilon)}\frac{{}_I\langle 0|U(t,t_1)\phi_I(x_1)U(t_1,t_2)\phi_I(x_2)\cdots\phi_I(x_n)U(t_n,-t)|0\rangle_I}{{}_I\langle 0|U(t,-t)|0\rangle_I}$$

With (6), everything is written in terms of $\phi_I(x)$. Furthermore, the numerator is time-ordered ($t > t_1 > t_2 > \cdots t_n > -t$), and hence it can be written as

$$\begin{aligned}\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle &= \lim_{t\to\infty(1-i\epsilon)}\frac{{}_I\langle 0|\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_n)U(t,t_1)U(t_1,t_2)\cdots U(t_n,-t)\right]|0\rangle_I}{{}_I\langle 0|U(t,-t)|0\rangle_I}\\ &= \lim_{t\to\infty(1-i\epsilon)}\frac{{}_I\langle 0|\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_n)U(t,-t)\right]|0\rangle_I}{{}_I\langle 0|U(t,-t)|0\rangle_I} \qquad \text{———} \quad (7)\end{aligned}$$

where we have used $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$.

(vii) Substituting (6) to (7), we finally obtain

$$\boxed{\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle = \lim_{t\to\infty(1-i\epsilon)}\frac{{}_I\langle 0|\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_n)\exp\left(-i\int_{-t}^{t}H_I(t')dt'\right)\right]|0\rangle_I}{{}_I\langle 0|\mathrm{T}\left[\exp\left(-i\int_{-t}^{t}H_I(t')dt'\right)\right]|0\rangle_I}}$$

——— (8)

Everything is written in terms of $\phi_I(x)$ and $|0\rangle_I$. By expanding $\exp(-i\int H_I)$, we can do the perturbation expansion as $\mathcal{O}(1) + \mathcal{O}(\lambda) + \mathcal{O}(\lambda^2)\cdots$.

§ 1.5.6 Wick's theorem

- All the terms in the numerator and the denominator of Eq. (8) has as the following form:

$${}_I\langle 0|\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_n)\right]|0\rangle_I\,.$$

Define $\varphi(x)$ as follows.

$$\phi_I(x) = \underbrace{\int\frac{d^3p}{(2\pi)^3\sqrt{2E_p}}a(\vec{p})e^{-ip\cdot x}}_{\equiv\varphi(x)} + \underbrace{\int\frac{d^3p}{(2\pi)^3\sqrt{2E_p}}a^\dagger(\vec{p})e^{ip\cdot x}}_{\equiv\varphi^\dagger(x)}\,.$$

Then

$$\begin{aligned}\phi_I(x) &= \varphi(x) + \varphi^\dagger(x),\\ \varphi(x)\left|0\right\rangle_I &= 0,\\ {}_I\!\left\langle 0\right|\varphi^\dagger(x) &= 0.\end{aligned}$$

- Now we introduce "normal ordering".

Normal Ordering

move $\varphi^\dagger$ to the left, and φ to the right.

$$\begin{aligned}\mathrm{N}\left[\phi_I(x_1)\phi_I(x_2)\right] &= \mathrm{N}\left[\left(\varphi(x_1)+\varphi^\dagger(x_1)\right)\left(\varphi(x_2)+\varphi^\dagger(x_2)\right)\right]\\ &= \mathrm{N}\Big[\varphi(x_1)\varphi(x_2)+\varphi^\dagger(x_1)\varphi(x_2)+\underset{\sim}{\varphi(x_1)\varphi^\dagger(x_2)}+\varphi^\dagger(x_1)\varphi^\dagger(x_2)\Big]\\ &= \varphi(x_1)\varphi(x_2)+\varphi^\dagger(x_1)\varphi(x_2)+\underset{\sim}{\varphi^\dagger(x_2)\varphi(x_1)}+\varphi^\dagger(x_1)\varphi^\dagger(x_2)\end{aligned}$$

(It can also be written as $:\phi_I(x_1)\phi_I(x_2):$ by using ":".)

$$\underline{\text{POINT}}\qquad {}_I\langle 0|\mathrm{N}\left[\phi_I(x_1)\cdots\phi_I(x_n)\right]|0\rangle_I = 0\,.$$

- Now we want to see the relation between the time-ordering and the normal ordering.

$$\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_n)\right] \overset{?}{\Longleftrightarrow} \mathrm{N}\left[\phi_I(x_1)\cdots\phi_I(x_n)\right]\,.$$

In the following, we write

$$\phi_I(x_i) = \phi_i \qquad \varphi(x_i) = \varphi_i$$

for simplicity. Let's start from $n = 2$.

- $\underline{n = 2}$

$$\begin{aligned}\text{For } x_1^0 > x_2^0,\qquad \mathrm{T}(\phi_1\phi_2) &= \phi_1\phi_2\\ &= (\varphi_1+\varphi_1^\dagger)(\varphi_2+\varphi_2^\dagger)\\ &= \varphi_1\varphi_2+\underset{\sim}{\varphi_1\varphi_2^\dagger}+\varphi_1^\dagger\varphi_2+\varphi_1^\dagger\varphi_2^\dagger\\ &= \mathrm{N}(\phi_1\phi_2)+[\varphi_1,\varphi_2^\dagger]\\ [\varphi_1,\varphi_2^\dagger] &= \int\frac{d^3p_1}{(2\pi)^3\sqrt{2E_{p_1}}}e^{-ip_1\cdot x_1}\int\frac{d^3p_2}{(2\pi)^3\sqrt{2E_{p_2}}}e^{ip_2\cdot x_2}[a(\vec{p_1}),a^\dagger(\vec{p_2})]\\ &= \int\frac{d^3p}{(2\pi)^3 2E_p}e^{-ip\cdot(x_1-x_2)}\end{aligned}$$

For $x_2^0 > x_1^0$, we have a similar formula with $x_1 \leftrightarrow x_2$. Therefore,

$$\mathrm{T}(\phi_1\phi_2) = \mathrm{N}(\phi_1\phi_2) + \overbracket{\phi_1\phi_2}$$

$$\overbracket{\phi_1\phi_2} = \int \frac{d^3p}{(2\pi)^3 2E_p} \times \begin{cases} e^{-ip\cdot(x_1-x_2)} & (x_1^0 > x_2^0) \\ e^{-ip\cdot(x_2-x_1)} & (x_2^0 > x_1^0) \end{cases}$$

not an operator,
but c-number.

$p^0 = \sqrt{\vec{p}^2 + m^2}$

The symbol $\overbracket{\phi_1\phi_2}$ is called "Wick contraction," and it can also be written as

$$\overbracket{\phi_1\phi_2} = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x_1-x_2)} \qquad (\epsilon > 0,\ \epsilon \to 0).$$

Feynman propagator

p^0 is not necessarily $\sqrt{\vec{p}^2 + m^2}$.
It's just an integration variable.

Check

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x_1-x_2)} = \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi} \frac{i}{(p^0)^2 - \underbrace{(\vec{p}^2 + m^2)}_{E_p^2} + i\epsilon} e^{-ip\cdot(x_1-x_2)}$$

Here, for $\epsilon \to 0$,[2]

$$\frac{i}{(p^0)^2 - E_p^2 + i\epsilon} \sim i \cdot \frac{1}{p^0 - (E_p - i\epsilon)} \cdot \frac{1}{p^0 + (E_p - i\epsilon)}$$

which has poles at $p^0 = E_p - i\epsilon$ and $p^0 = -E_p + i\epsilon$. (See Fig. 1.)

For $\underline{x_1^0 > x_2^0}$, $e^{-ip^0(x_1^0 - x_2^0)} \to 0$ for $p^0 \to -i\infty$, so closing the contour at Im$p^0 < 0$ (red line)

$$\begin{aligned} &\int \frac{d^3p}{(2\pi)^3} \oint \frac{dp^0}{2\pi} i \cdot \frac{1}{p^0 - (E_p - i\epsilon)} \cdot \frac{1}{p^0 + (E_p - i\epsilon)} e^{-ip\cdot(x_1-x_2)} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{i}{2\pi} \cdot (-2\pi i) \cdot \frac{1}{E_p + E_p} e^{-ip\cdot(x_1-x_2)} \Big|_{p^0 = E_p} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x_1-x_2)} \Big|_{p^0 = E_p}. \end{aligned}$$

---

[2] Here, $E_p^2 - i\epsilon \sim (E_p - i\epsilon/(2E_p))^2$ and we renamed $\epsilon/(2E_p)$ as ϵ in the right hand side. The overall coefficient of ϵ doesn't matter as far as $\epsilon \to 0$.

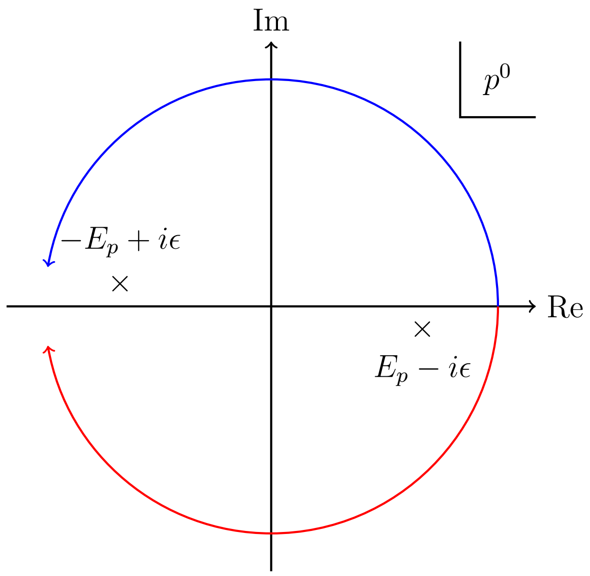


Figure 1:

For $\underline{x_2^0 > x_1^0,}$ $e^{-ip^0(x_1^0-x_2^0)} \to 0$ for $p^0 \to +i\infty$, so closing the contour at Im$p^0 > 0$ (blue line)

$$\begin{aligned}
&\int \frac{d^3p}{(2\pi)^3} \oint \frac{dp^0}{2\pi} i \cdot \frac{1}{p^0 - (E_p - i\epsilon)} \cdot \frac{1}{p^0 + (E_p - i\epsilon)} e^{-ip\cdot(x_1-x_2)} \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{i}{2\pi} \cdot \frac{1}{-E_p - E_p} \cdot (+2\pi i) e^{-ip\cdot(x_1-x_2)}\Big|_{p^0=-E_p} \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x_2-x_1)}\Big|_{p^0=+E_p},
\end{aligned}$$

where we have changed the integration variable $\vec{p} \to -\vec{p}$ in the las line. Therefore

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x_1-x_2)} = \int \frac{d^3p}{(2\pi)^3 2E_p} \times \begin{cases} e^{-ip\cdot(x_1-x_2)} & (x_1^0 > x_2^0) \\ e^{-ip\cdot(x_2-x_1)} & (x_2^0 > x_1^0) \end{cases} \quad \blacksquare$$

———————— on June 5, up to here. ————————

Questions and comments after the lecture: (only some of them)

Q: Can you explain how the time-ordering operator acts in Eq.(6) in § 1.5.5?

A: OK. Let's write Eq.(6) in § 1.5.5 again:

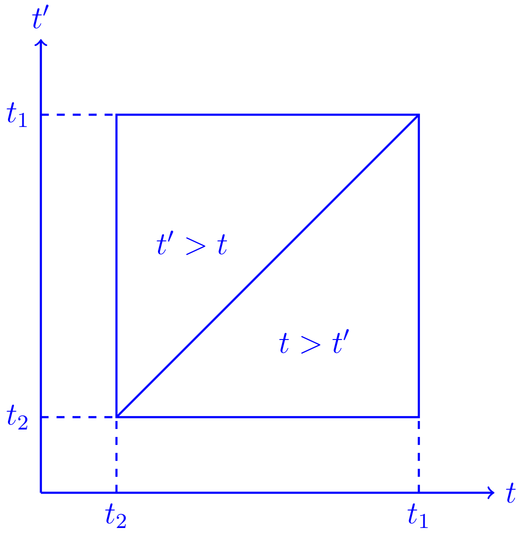
$$U(t_1, t_2) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{T}\left[\left(-i \int_{t_2}^{t_1} H_I(t) dt\right)^n\right]$$

where I changed the order of the summation and the time-ordering.

Now, consider the term for $n = 2$, for simplicity. It has the following form:

$$U(t_1,t_2)|_{n=2} = \frac{1}{2!}\mathrm{T}\left[\left(-i\int_{t_2}^{t_1} H_I(t)dt\right)^2\right] = \frac{1}{2!}\mathrm{T}\left[\left(-i\int_{t_2}^{t_1} H_I(t)dt\right)\left(-i\int_{t_2}^{t_1} H_I(t')dt'\right)\right]$$

Note that $H_I(t) = \int d^3x(\lambda/24)\phi_I(t,\vec{x})^4$. The integration variable t and t' take values between t_1 and t_2. In the region where $t > t'$, $\mathrm{T}[H_I(t)H_I(t')] = H_I(t)H_I(t')$. But in the region where $t' > t$, $\mathrm{T}[H_I(t)H_I(t')] = H_I(t')H_I(t)$. (See the following figure).



Thus

$$\begin{aligned} U(t_1,t_2)|_{n=2} &= \frac{1}{2!}\mathrm{T}\left[(-i)^2\int_{t_2}^{t_1} H_I(t)dt\int_{t_2}^{t} H_I(t')dt' + (-i)^2\int_{t_2}^{t_1} H_I(t)dt\int_{t}^{t_1} H_I(t')dt'\right] \\ &= \frac{1}{2!}(-i)^2\int_{t_2}^{t_1} dt\int_{t_2}^{t_1} dt'\Big(\theta(t-t')H_I(t)H_I(t') + \theta(t'-t)H_I(t')H_I(t)\Big) \end{aligned}$$

Now, the 2nd term is the same as the 1st term ($t \leftrightarrow t'$), and hence

$$U(t_1,t_2)|_{n=2} = (-i)^2\int_{t_2}^{t_1} dt\int_{t_2}^{t} dt' H_I(t)H_I(t')$$

Similarly,

$$\begin{aligned} U(t_1,t_2)|_{n=3} &= (-i)^3\int_{t_2}^{t_1} dt\int_{t_2}^{t} dt'\int_{t_2}^{t'} dt'' H_I(t)H_I(t')H_I(t'') \\ &\cdots \\ U(t_1,t_2)|_{n} &= (-i)^n\int_{t_2}^{t_1} dt\int_{t_2}^{t} dt'\int_{t_2}^{t'} dt''\cdots\int_{t_2}^{t^{(n-1)}} dt^{(n)} H_I(t)H_I(t')H_I(t'')\cdots H_I(t^{(n)}) \end{aligned}$$

Note that the factor $1/n!$ cancels the combinatorial factor of exchanging variables. Summing over all terms, we have the explicit form:

$$U(t_1,t_2)=\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{T}\left[\left(-i\int_{t_2}^{t_1}H_I(t)dt\right)^n\right]=\sum_{n=0}^{\infty}U(t_1,t_2)|_n.$$

Using the above explicit expression for $U(t_1,t_2)|_n$, one can easily show that

$$\frac{\partial}{\partial t_1}U(t_1,t_2)|_n=-iH_I(t_1)U(t_1,t_2)|_{n-1}$$

$$\therefore\quad\frac{\partial}{\partial t_1}U(t_1,t_2)=-iH_I(t_1)U(t_1,t_2).$$

comment: You should have explained what is ϵ when introducing the Feynman propagator. $\epsilon>0$ and $\epsilon\to 0$, right? (You didn't write it explicitly).

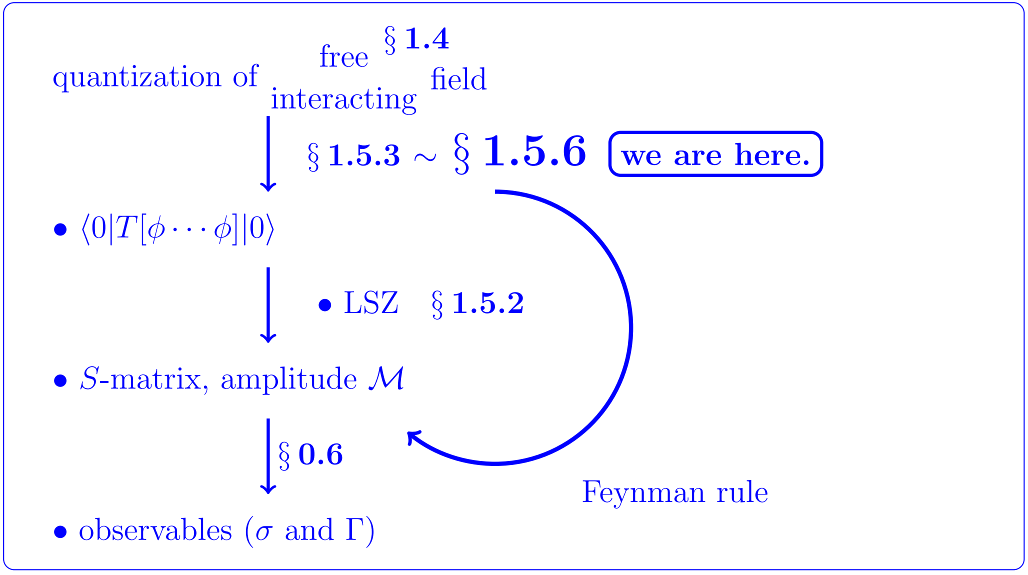
A: You are right! Thanks! I added an explanation in the note, and I will comment on it in the next week.

comment: There is a typo in the formula of the Feynman propagator. $e^{+ip\cdot(x_1-x_2)}$ should be $e^{-ip\cdot(x_1-x_2)}$.

A: Thanks! corrected.

——— on June 12, from here. ———

Outline



quantization of free / interacting field §**1.4**

§**1.5.3** ~ § **1.5.6** **we are here.**

- $\langle 0|T[\phi\cdots\phi]|0\rangle$

- LSZ §**1.5.2**

- S-matrix, amplitude $\mathcal{M}$

§**0.6**

Feynman rule

- observables (σ and Γ)

Comments on the lecture last week (corrected/added in the note on the web)

(i) There was a typo in Feynman propagator. $e^{+ip\cdot(x_1-x_2)}\to e^{-ip\cdot(x_1-x_2)}$.

(ii) In the formula of the Feynman propagator, the parameter ϵ is a small constant which represent the contour in the complex plane. $\epsilon > 0$ and $\epsilon \to 0$.

(iii) In the case of $x_2^0 > x_1^0$, the contour should be closed at $\mathrm{Im}p^0 > 0$.

(iv) The time-ordering $\mathrm{T}[\exp(\int H_I)]$ in Eq.(6) in §1.5.5 can be expressed more explicitly. (See the lecture note on the web.)

- ▶ $n = 3$

For $x_3^0 > x_1^0, x_2^0$,

$$\begin{aligned}
\mathrm{T}(\phi_1\phi_2\phi_3) &= \phi_3\mathrm{T}(\phi_1\phi_2) \\
&= \phi_3\mathrm{N}(\phi_1\phi_2) + \phi_3\wick{\c{\phi}_1\c{\phi}_2} \\
&= \varphi_3\mathrm{N}(\phi_1\phi_2) + \varphi_3^\dagger\mathrm{N}(\phi_1\phi_2) + \phi_3\wick{\c{\phi}_1\c{\phi}_2} \\
&\quad \begin{cases} \varphi_3\mathrm{N}(\phi_1\phi_2) = N(\phi_1\phi_2\varphi_3) + \wick{\c{\phi}_1\c{\phi}_3}\phi_2 + \phi_1\wick{\c{\phi}_2\c{\phi}_3} \\ \varphi_3^\dagger\mathrm{N}(\phi_1\phi_2) = N(\phi_1\phi_2\varphi_3^\dagger) \end{cases} \\
&= \mathrm{N}(\phi_1\phi_2\phi_3) + \wick{\c{\phi}_1\c{\phi}_2}\phi_3 + \wick{\c{\phi}_1\phi_2\c{\phi}_3} + \phi_1\wick{\c{\phi}_2\c{\phi}_3}\,.
\end{aligned}$$

(Similar for $x_1^0 > x_2^0, x_3^0$ and $x_2^0 > x_1^0, x_3^0$.)

- ▶ $n = 4$

$$\mathrm{T}(\phi_1\phi_2\phi_3\phi_4) = \mathrm{N}(\phi_1\phi_2\phi_3\phi_4) + \underbrace{\wick{\c{\phi}_1\c{\phi}_2}\mathrm{N}(\phi_3\phi_4) + \wick{\c{\phi}_1\c{\phi}_3}\mathrm{N}(\phi_2\phi_4) + \cdots}_{\text{6 terms}} + \underbrace{\wick{\c1{\phi}_1\c1{\phi}_2\c2{\phi}_3\c2{\phi}_4} + \cdots}_{\text{3 terms}}$$

- ▶ In general

Wick's theorem

$$\begin{aligned}
\mathrm{T}(\phi_1\cdots\phi_n) &= \mathrm{N}(\phi_1\cdots\phi_n) \\
&\quad + \sum_{\text{pairs}} \wick{\c{\phi}_i\c{\phi}_j}\ \mathrm{N}(\phi_1\cdot\underset{\hat{i}}{\cdots}\underset{\hat{j}}{\cdots}\cdot\phi_n) \\
&\quad + \sum_{\text{2 pairs}} \wick{\c1{\phi}_i\c1{\phi}_j\c2{\phi}_k\c2{\phi}_\ell}\ \mathrm{N}(\phi_1\cdot\underset{\hat{i}\,\hat{j}\,\hat{k}\,\hat{\ell}}{\cdots\cdots}\cdot\phi_n) \\
&\quad + \cdots \\
&\quad + \begin{cases} \displaystyle\sum_{\frac{n}{2}\text{ pairs}} \wick{\c{\phi}_i\c{\phi}_j}\cdots\wick{\c{\phi}_p\c{\phi}_q} & (n = \text{ even}) \\ \displaystyle\sum_{\frac{n-1}{2}\text{ pairs}} \wick{\c{\phi}_i\c{\phi}_j}\cdots\wick{\c{\phi}_p\c{\phi}_q}\phi_r & (n = \text{ odd})\,. \end{cases}
\end{aligned}$$

(**Problem.** Prove it by induction.)

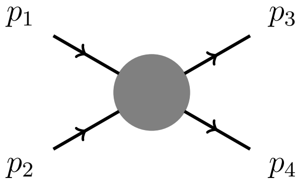
- Therefore,

$$\boxed{{}_I\langle 0|\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_n)\right]|0\rangle_I = \begin{cases} \displaystyle\sum_{n/2\text{ pairs}} \wick{\c{\phi}_i \c{\phi}_j} \cdots \wick{\c{\phi}_p \c{\phi}_q} & (n = \text{ even}) \\ 0 & (n = \text{ odd})\,. \end{cases}}$$

§ 1.5.7 Summary, Feynman rules, examples

- Let's calculate the cross section for 2 → 2 scattering in the ϕ^4 theory,

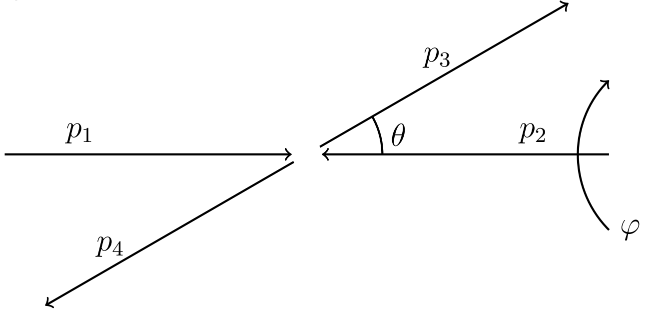
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{24}\phi^4\,.$$



- From § 0.6,

$$\sigma(p_1,p_2\to\phi\phi) = \frac{1}{2E_1\cdot 2E_2|v_1-v_2|}\underbrace{\int d\Phi_2}_{\text{final state}}\,|\mathcal{M}(p_1,p_2\to p_3,p_4)|^2\ \underbrace{\times\frac{1}{2}}_{\substack{\text{identical}\\\text{final}\\\text{particles}}}$$

Consider in the center-of-mass frame



$$\begin{aligned} &\vec{p}_1 = -\vec{p}_2 \\ &\vec{p}_3 = -\vec{p}_4 \\ &|\vec{p}_1| = |\vec{p}_2| = |\vec{p}_3| = |\vec{p}_4| \\ &E_1 = E_2 = E_3 = E_4 \\ &= \sqrt{|\vec{p}_1|^2+m^2} \end{aligned}$$

Then

$$|v_1 - v_2| = \left|\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2}\right| = 2\frac{\vec{p}_1}{E_1}$$

$$\int d\Phi_2 |\mathcal{M}|^2 = \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta^{(4)}(\sum p_1 + p_2 - p_3 - p_4)|\mathcal{M}|^2$$

$$= \cdots$$

(In the center-of-mass frame, this is simplified as. . . [Problem: Show it.])

$$= \cdots$$

$$= \frac{1}{8\pi}\frac{\vec{p}_3}{E_3}\int \frac{d\Omega}{4\pi}|\mathcal{M}|^2. \qquad \left(\int d\Omega \equiv \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta\right)$$

- Therefore,

$$\sigma(p_1, p_2 \to \phi\phi) = \frac{1}{128\pi}\frac{1}{E_1^2}\int \frac{d\Omega}{4\pi}|\mathcal{M}|^2 \qquad \text{———————(1).}$$

- On the other hand, $\mathcal{M}$ is given by

$$\langle p_3, p_4; \text{out}|p_1, p_2; \text{in}\rangle = (2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4) \cdot i\mathcal{M}(p_1, p_2 \to p_3, p_4) \qquad \text{———————(2).}$$

and $\langle p_3, p_4; \text{out}|p_1, p_2; \text{in}\rangle$ is given by, from the LSZ formula § 1.5.2,

$$\langle p_3, p_4; \text{out}|p_1, p_2; \text{in}\rangle$$
$$= \underbrace{\prod_{i=1,2}\left[i\int d^4x_i e^{-ip_i\cdot x_i}\left(\Box_i + m^2\right)\right] \times \prod_{i=3,4}^{n}\left[i\int d^4x_i e^{+ip_i\cdot x_i}\left(\Box_i + m^2\right)\right]}_{\text{We call it “LSZ factor”}}$$
$$\times \langle 0|\mathrm{T}\Big(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\Big)|0\rangle$$

where, from § 1.5.5,

$$\langle 0|\phi(x_1)\cdots\phi(x_4)|0\rangle = \frac{{}_I\langle 0|\mathrm{T}\left[\phi_I(x_1)\cdots\phi_I(x_4)\exp\left(-i\int\frac{\lambda}{24}\phi_I(x)^4\right)\right]|0\rangle_I}{{}_I\langle 0|\mathrm{T}\left[\exp\left(-i\int\frac{\lambda}{24}\phi_I(x)^4\right)\right]|0\rangle_I} \qquad \text{———————(3).}$$

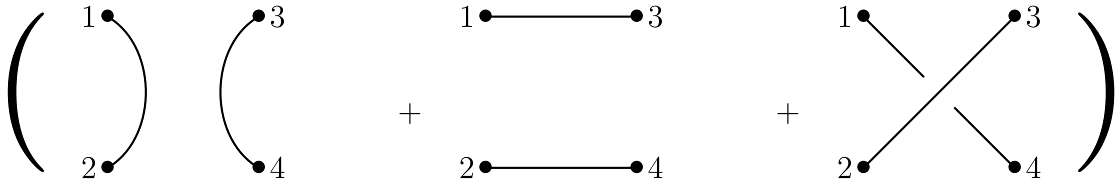
- Namely,

$$\underline{\langle p_3, p_4; \text{out}|p_1, p_2; \text{in}\rangle = (\text{LSZ factor}) \times (3)}.$$

We can expand it with respect to λ.

- $\mathcal{O}(\lambda^0)$ term of (3)'s denominator = ${}_I\langle 0|0\rangle_I = 1$.
- $\mathcal{O}(\lambda^0)$ term of (3)'s numerator = ${}_I\langle 0|\mathrm{T}(\phi_1\phi_2\phi_3\phi_4)|0\rangle_I$. ($\phi_i = \phi_I(x_i)$).
  from Wick's theorem in § 1.5.6,

$$= \quad \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} \quad + \quad \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} \quad + \quad \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}$$



where

$$\wick{\c\phi_1\c\phi_2} = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x_1-x_2)} \equiv D_F(x_1-x_2).$$

Now, (LSZ factor) $\times D_F(x_1-x_2) =?$

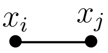
- In general,

$$\begin{aligned}
&(\text{LSZ factor}) \times D_F(x_1-x_2)\\
&= i\int d^4x_i e^{\mp ip_i\cdot x_i}\underbrace{\left(\Box_i + m^2\right)D_F(x_i-y)}\\
&\qquad\qquad\qquad\quad \left(\begin{aligned} &= \int\frac{d^4p}{(2\pi)^4}\frac{i(-p^2+m^2)}{p^2-m^2+i\epsilon}e^{-ip\cdot(x_1-x_2)}\\ &= -i\delta^{(4)}(x_i-y)\end{aligned}\right)\\
&= e^{\mp ip_i\cdot y} \qquad \text{———————————}(4).
\end{aligned}$$

$\boxed{\textbf{POINT}}$ LSZ factor cancels the $D_F(x_i-y)$ factor of the external line.

- In the case of $\wick{\c\phi_1\c\phi_2} = D_F(x_1-x_2)$, both x_1 and x_2 are at the external lines, so

$$\begin{aligned}
&i\int d^4x_i e^{-ip_2\cdot x_2}\left(\Box_2+m^2\right)\underbrace{i\int d^4x_i e^{-ip_1\cdot x_1}\left(\Box_1+m^2\right)D_F(x_1-x_2)}_{e^{-ip_1\cdot x_2}}\\
&= i\int d^4x_i e^{-ip_2\cdot x_2}\underbrace{\left(-p_1^2+m^2\right)}_{=0!}e^{-ip_1\cdot x_2}\\
&= 0.
\end{aligned}$$

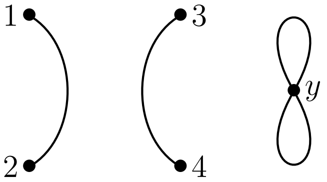
$\boxed{\textbf{POINT}}$ If two external points are directly connected,  = 0.

- Thus, $\mathcal{O}(\lambda^0)$ term in (3)'s numerator = 0.
- $\mathcal{O}(\lambda)$ term in (3)'s numerator

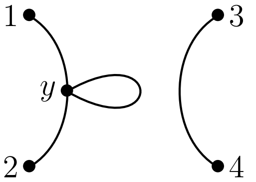
$$= {}_I\langle 0|\mathrm{T}\left(\phi_1\phi_2\phi_3\phi_4(-i)\int d^4y\frac{\lambda}{24}\phi_y\phi_y\phi_y\phi_y\right)|0\rangle_I \qquad \phi_i=\phi_I(x_i),\phi_y=\phi_I(y)$$

(4 pairs = 105 combinations)

$= \text{terms including } \overset{\text{ext.}}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{\text{ext.}}{\bullet} \quad (=0)$

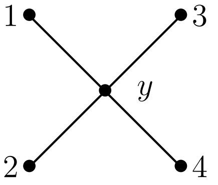


$\phi_1\phi_2\phi_3\phi_4\cdot\phi_y\phi_y\phi_y\phi_y$ etc, 9 terms



$\phi_1\phi_2\phi_3\phi_4\cdot\phi_y\phi_y\phi_y\phi_y$ etc, 72 terms

$$+\quad \phi_1\phi_2\phi_3\phi_4\frac{-i\lambda}{24}\int d^4y\phi_y\phi_y\phi_y\phi_y \qquad 4!=24\text{ terms} \qquad \text{(in total 105 terms)}$$



$\left(24 \text{ of } \frac{\lambda}{4!}=\frac{\lambda}{24} \text{ was introduced for this.}\right)$

$$=(-i\lambda)\int d^4yD_F(x_1-y)D_F(x_2-y)D_F(x_3-y)D_F(x_4-y).$$

- Thus, from (4),

(LSZ factor) × ($\mathcal{O}(\lambda)$ term in (3)'s numerator)

$$=(-i\lambda)\int d^4ye^{-ip_1\cdot y}e^{-ip_2\cdot y}e^{+ip_3\cdot y}e^{+ip_4\cdot y}$$

$$=(-i\lambda)(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4) \qquad \text{———————————}(5).$$

- Since

$$\langle \text{out}|\text{in}\rangle = \frac{(\text{LSZ factor}) \times (3)\text{'s numerator}}{(3)\text{'s demominator}} = \frac{\overbrace{\mathcal{O}(\lambda^0)}^{=0} + \overbrace{\mathcal{O}(\lambda)}^{=(5)} + \mathcal{O}(\lambda^2) + \cdots}{1 + \mathcal{O}(\lambda) + \cdots},$$

  (5) is the leading term of $\langle \text{out}|\text{in}\rangle$.

$$\langle p_3, p_4; \text{out}|p_1, p_2; \text{in}\rangle = (-i\lambda)(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4) + \mathcal{O}(\lambda^2)$$

- Thus, from (2), we eventually obtain the amplitude at the leading order,

$$\underline{i\mathcal{M}(p_1, p_2 \to p_3, p_4) = -i\lambda} \qquad + \mathcal{O}(\lambda^2)$$

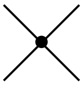
- and substituting it to (1), the cross section

$$\begin{aligned} \sigma(p_1, p_2 \to \phi\phi) &= \frac{1}{128\pi}\frac{1}{E_1^2}\underbrace{\int \frac{d\Omega}{4\pi}}_{=1}\underbrace{|\mathcal{M}|^2}_{=\lambda^2} \\ &= \underline{\frac{\lambda^2}{128\pi} \cdot \frac{1}{E_1^2}}. \\ \\ &= 2.5 \times 10^{-3}\lambda^2\text{GeV}^{-2}\left(\frac{\text{GeV}}{E_1}\right)^2 \\ &= 1.0 \times 10^{-30}\lambda^2\text{cm}^2\left(\frac{\text{GeV}}{E_1}\right)^2. \end{aligned}$$

-

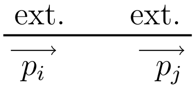
Feynman rules for $\mathcal{M}$

$i\mathcal{M} =$ diagrams $=$



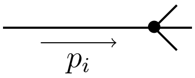
$+ \cdots$

(1) diagram with



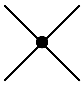
$= 0$.

(2) external line



$= 1$.

(3) vertex



$= -i\lambda$.

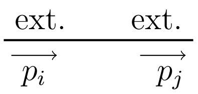
(cont'd)

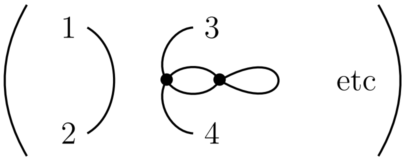
- Higher order terms:

  $\mathcal{O}(\lambda^2)$ term in (3)'s numerator

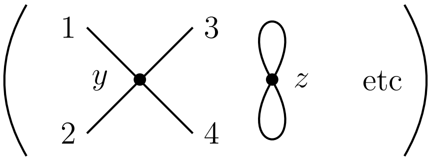
  $$= {}_I\langle 0|\mathrm{T}\left(\phi_1\phi_2\phi_3\phi_4\frac{(-i)^2}{2}\int d^4y\frac{\lambda}{24}\phi_y^4\int d^4z\frac{\lambda}{24}\phi_z^4\right)|0\rangle_I$$

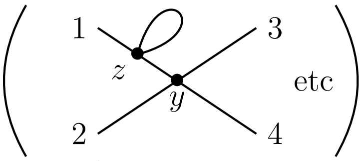
  (6 pairs → 10395 combinations!) term in (3)'s numerator

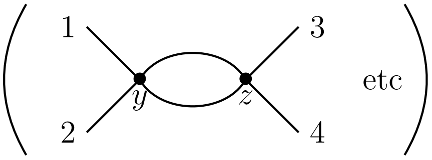
  = terms with 



$\to 0$

  + terms with bubbles 

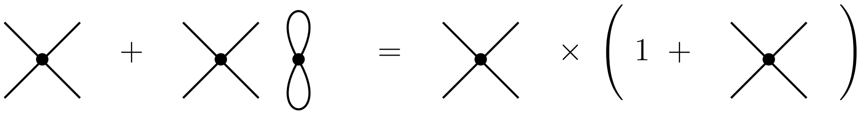
  + terms with loops at external lines 

  + other loops 

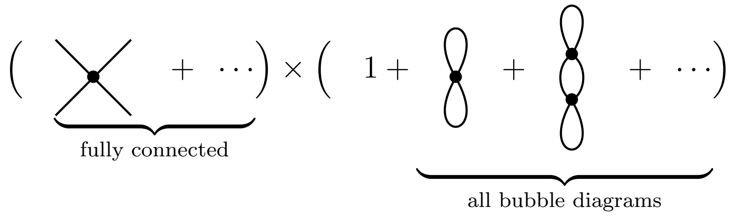
.

- In general, terms with loop diagrams are often divergent, and requires "renormalization". Here, we just give qualitative discussion.

- Terms with bubbles are, together with the leading order term,

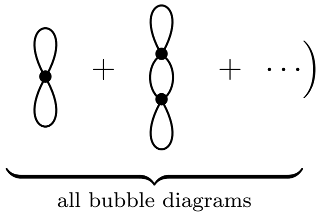


  In general,

  (3)'s numerator = 

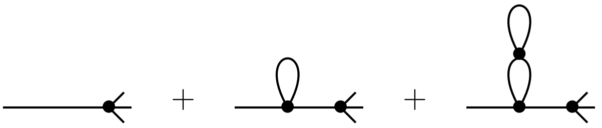
On the other hand,

$$(3)\text{'s denominator} = {}_I\langle 0|\mathrm{T}\left(\exp\left[-i\int\frac{\lambda}{24}\phi_I^4\right]\right)|0\rangle_I = \Big(\ 1+$$



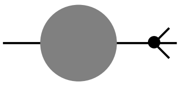
Therefore, all bubble diagrams canceled out between numerator and denominator.

- ► Loops in the external line



+ ⋯ (all these diagrams)

=



$$= \frac{iZ}{p^2 - m^2}.$$

The factor Z is absorbed by the field renormalization. (In general, the mass "m" here is also different from the parameter "m" in the Lagrangian.) We do not discuss the more details here.

- ►

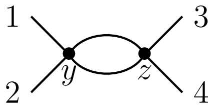
Feynman rules (cont'd)

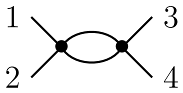
(4) Ignore the bubble diagrams.

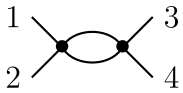
(5) We can also ignore the loops in the external lines if we take into account the renormalization.

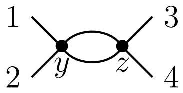
- ► The other loops.

Example:



The  term in $\langle\text{out}|\text{in}\rangle$

$= (\text{LSZ factor}) \times$ the  term in (3)'s numerator

$= (\text{LSZ factor}) \times$ the  term in ${}_I\langle 0|\mathrm{T}\left(\phi_1\phi_2\phi_3\phi_4\frac{(-i)^2}{2}\int d^4y\frac{\lambda}{24}\phi_y^4\int d^4z\frac{\lambda}{24}\phi_z^4\right)|0\rangle_I$

$$= (\text{LSZ factor}) \times \frac{1}{2}\left(\frac{-i\lambda}{24}\right)^2\int d^4y\int d^4z\ \phi_1\phi_2\cdot\phi_y\phi_y\phi_y\phi_y\cdot\phi_z\phi_z\phi_z\phi_z\cdot\phi_3\phi_4$$

$\times\, 12$ $(1,2 \leftrightarrow y^4$ combinations$)$

$\times\, 12$ $(3,4 \leftrightarrow z^4$ combinations$)$

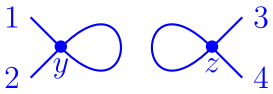
$\times\, 2$ (remaining $y^2 \leftrightarrow z^2$ combinations)

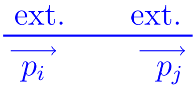
$\times\, 2$ (replacing $y \leftrightarrow z$)

$$= (\text{LSZ factor}) \times \frac{1}{2}(-i\lambda)^2\int d^4y\int d^4z\ D_F(x_1-y)D_F(x_2-y)D_F(x_3-z)D_F(x_4-z)D_F(y-z)^2$$

——————— on June 12, up to here. ———————

Questions after the lecture: (only some of them)

Q: What about a diagrams like ??

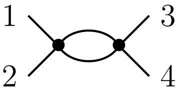
A: Good question! In fact, after renormalization of fields are take into account, this class of diagrams also corresponds to a diagram with , and its contribution to $i\mathcal{M}$ is zero.

comment: There is a typo in the calculation of the normal ordering in § 1.5.6. $\varphi^\dagger(x_1)\varphi^\dagger(x_2) \to \varphi^\dagger(x_1)\varphi(x_2)$.

A: Thanks! corrected.

——————— on June 19, from here. ———————

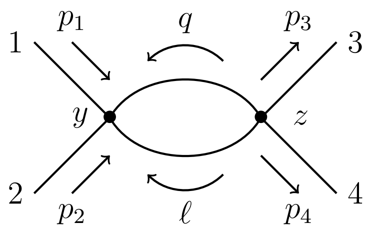
From (4), (LSZ factor) $\times D_F(x_i - y) = e^{\mp i p_i y}$, and hence

The  term in $\langle \text{out}|\text{in}\rangle$

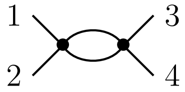
$$
\begin{aligned}
&= (\text{LSZ factor}) \times \frac{1}{2}(-i\lambda)^2 \int d^4y \int d^4z \; D_F(x_1-y)D_F(x_2-y)D_F(x_3-z)D_F(x_4-z)D_F(y-z)^2 \\
&= \frac{1}{2}(-i\lambda)^2 \int d^4y \int d^4z \; e^{-ip_1\cdot y}e^{-ip_2\cdot y}e^{+ip_3\cdot z}e^{+ip_4\cdot z}D_F(y-z)^2 \\
&= \frac{1}{2}(-i\lambda)^2 \int d^4y \int d^4z \; e^{-ip_1\cdot y}e^{-ip_2\cdot y}e^{+ip_3\cdot z}e^{+ip_4\cdot z} \\
&\qquad \times \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}e^{-iq\cdot(y-z)} \int \frac{d^4\ell}{(2\pi)^4}\frac{i}{\ell^2-m^2+i\epsilon}e^{-i\ell\cdot(y-z)}
\end{aligned}
$$

Here

$$
\begin{cases}
\displaystyle\int d^4y e^{-i(p_1+p_2+q+\ell)\cdot y} = (2\pi)^4\delta^{(4)}(p_1+p_2+q+\ell) \\
\displaystyle\int d^4z e^{+i(p_3+p_4+q+\ell)\cdot y} = (2\pi)^4\delta^{(4)}(p_3+p_4+q+\ell)
\end{cases}
$$

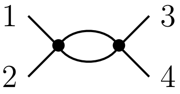


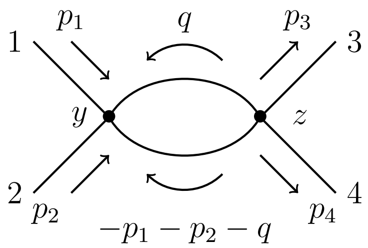
which represents the $\boxed{\text{momentum conservation}}$ at each vertex.

Thus, the  term in $\langle \text{out}|\text{in}\rangle$

$$
\begin{aligned}
&= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}\int \frac{d^4\ell}{(2\pi)^4}\frac{i}{\ell^2-m^2+i\epsilon} \\
&\qquad \times (2\pi)^4\delta^{(4)}(p_1+p_2+q+\ell)(2\pi)^4\delta^{(4)}(p_3+p_4+q+\ell) \\
&= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}\cdot\frac{i}{(-p_1-p_2-q)^2-m^2+i\epsilon}\underbrace{(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)}_{\text{the factor in (2)}}
\end{aligned}
$$

and finally, from (2),

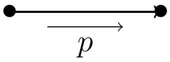
The  term in $i\mathcal{M} = \underline{\frac{1}{2}(-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}\cdot\frac{i}{(-p_1-p_2-q)^2-m^2+i\epsilon}}\,.$



-

Feynman rules (cont'd)

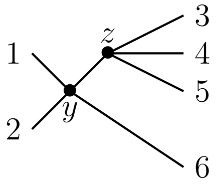
(6) Momentum conservation at each vertex.

(7) Internal line  $= \dfrac{i}{p^2 - m^2 + i\epsilon}$.

(8) Loop momentum should be integrated by $\displaystyle\int \frac{d^4p}{(2\pi)^4}$.

(9) Multiply the "symmetry factor" (such as 1/2 in the above example).

- Note that, if there is no loop, there is no symmetry factor and all coefficients cancel:

Example: 2 → 4 scattering. 

$$= (\mathrm{LSZ}\times)\frac{1}{2}\left(\frac{-i\lambda}{24}\right)^2 \int d^4y \int d^4z\, \phi_1\phi_2 \cdot \phi_y\phi_y\phi_y\phi_y \cdot \phi_z\phi_z\phi_z\phi_z \cdot \phi_3\phi_4\phi_5\phi_6$$

$$\times\, 4!\ (y \text{ contraction})$$

$$\times\, 4!\ (z \text{ contraction})$$

$$\times\, 2\ (y \leftrightarrow z)$$

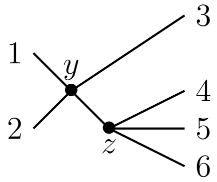
$$= (\mathrm{LSZ}\times)(-i\lambda)^2 \int d^4y \int d^4z D_F(x_1 - y)D_F(x_2 - y)D_F(x_6 - y)$$

$$\times\, D_F(x_3 - z)D_F(x_4 - z)D_F(x_5 - z)D_F(y - z)$$

$$= \cdots$$

$$= \underbrace{(-i\lambda)^2 \frac{i}{(p_1 + p_2 - p_6)^2 - m^2 - i\epsilon}}_{\text{the corresponding term in } i\mathcal{M}} \times (2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4 - p_5 - p_6).$$

- Different diagrams give different terms: for instance,

The  term in $i\mathcal{M} \;=\; (-i\lambda)^2 \dfrac{i}{(p_1 + p_2 - p_3)^2 - m^2 - i\epsilon}$.

§ 2 Fermion (spin 1/2) Field

$\boxed{\text{Goal}}$: To construct the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi,$$

solve the EOM (Dirac equation)

$$\mathcal{L} = (i\gamma^\mu \partial_\mu - m)\psi = 0,$$

and quantize the Dirac field.

§ 2.1 Representations of the Lorentz group

§ 2.1.1 Lorentz Transformation of coordinates (again) (see § 1.1.1)

- $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$, where

$$g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\nu\sigma}, \qquad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\begin{aligned} \iff\ & \Lambda^\mu{}_\rho g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma} \\ \iff\ & \underline{\Lambda^T g \Lambda = g}. \end{aligned}$$

- Exmaples

$\underline{\boxed{\text{rotations}} \text{ around } x,\ y,\ z \text{ axes}}$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_1 & \sin\theta_1 \\ & & -\sin\theta_1 & \cos\theta_1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & \cos\theta_2 & & -\sin\theta_2 \\ & & 1 & \\ & \sin\theta_2 & & \cos\theta_2 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & \cos\theta_3 & \sin\theta_3 & \\ & -\sin\theta_3 & \cos\theta_3 & \\ & & & 1 \end{pmatrix}.$$

$\underline{\boxed{\text{boosts}} \text{ in the } x,\ y,\ z \text{ directions}}$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\eta_1 & \sinh\eta_1 & & \\ \sinh\eta_1 & \cosh\eta_1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \cosh\eta_2 & & \sinh\eta_2 & \\ & 1 & & \\ \sinh\eta_2 & & \cosh\eta_2 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \cosh\eta_3 & & & \sinh\eta_3 \\ & 1 & & \\ & & 1 & \\ \sinh\eta_3 & & & \cosh\eta_3 \end{pmatrix}.$$

$$\cosh\eta = \frac{e^\eta + e^{-\eta}}{2} = \gamma$$

$$\sinh\eta = \frac{e^\eta - e^{-\eta}}{2} = \beta\gamma = \sqrt{\gamma^2 - 1}$$

§ 2.1.2 infinitesimal Lorentz Transformation and generators of Lorentz group (in the 4-vector basis)

- Consider an infinitesimal Lorentz Transformation:

$$\begin{aligned} & \Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \qquad (\omega^\mu{}_\nu \ll 1), \\ \text{or} \qquad & \Lambda = I + \omega. \end{aligned}$$

where I is the identity matrix. (I changed the notation from the blackboard.)
Then, from $\Lambda^T g \Lambda = g$,

$$\begin{aligned} & (I+\omega)^T g (I+\omega) = g \\ \therefore\ & \omega^T g + g\omega = 0 \qquad (\text{up to } \mathcal{O}(\omega^2)) \\ \therefore\ & \underbrace{\omega^\rho{}_\mu\, g_{\rho\nu}}_{\substack{\| \\ g_{\nu\rho}\omega^\rho{}_\mu \\ \| \\ \omega_{\nu\mu}}} + \underbrace{g_{\mu\rho}\, \omega^\rho{}_\nu}_{\equiv\, \omega_{\mu\nu}} = 0 \\ \therefore\ & \underline{\omega_{\nu\mu} = -\omega_{\mu\nu} \qquad \text{anti-symmetric}} \end{aligned}$$

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

6 independent degrees
$\Updownarrow$
3 rotations and 3 boosts

- In fact, the matrix $\omega^\mu{}_\nu = g^{\mu\rho}\omega_{\rho\nu}$ can be written as

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & \eta_1 & \eta_2 & \eta_3 \\ \eta_1 & 0 & \theta_3 & -\theta_2 \\ \eta_2 & -\theta_3 & 0 & \theta_1 \\ \eta_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix} \qquad \text{———————(1)}$$

Note that
$\omega^0{}_\nu = g^{00}\omega_{0\nu} = \omega_{0\nu}, \qquad \omega_{0i} = \eta_i = -\omega_{i0}.$
$\omega^i{}_\nu = g^{ij}\omega_{j\nu} = -\omega_{i\nu}, \qquad -\omega_{ij} = \epsilon_{ijk}\theta_k = \omega_{ji}.$

and the rotations and boosts in § 2.1.1 can be expanded as

<u>rotation around x axis</u>

$$\Lambda = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_1 & \sin\theta_1 \\ & & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \theta_1 \\ & & -\theta_1 & 0 \end{pmatrix} + \mathcal{O}(\theta_1^2)$$

<u>boost in the x direction</u>

$$\Lambda = \begin{pmatrix} \cosh\eta_1 & \sinh\eta_1 & & \\ \sinh\eta_1 & \cosh\eta_1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & \eta_1 & & \\ \eta_1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \mathcal{O}(\eta_1^2)$$

- The matrix $\omega^\mu{}_\nu$ in (1) can also be written as

$$\omega^\mu{}_\nu = i\left[\theta_i (J_i)^\mu{}_\nu + \eta_i (K_i)^\mu{}_\nu\right]$$

where

$$(J_1)^\mu{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix}, \; (J_2)^\mu{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{pmatrix}, \; (J_3)^\mu{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{pmatrix},$$

$$(K_1)^\mu{}_\nu = \begin{pmatrix} 0 & -i & & \\ -i & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \; (K_2)^\mu{}_\nu = \begin{pmatrix} 0 & & -i & \\ & 0 & & \\ -i & & 0 & \\ & & & 0 \end{pmatrix}, \; (K_3)^\mu{}_\nu = \begin{pmatrix} 0 & & & -i \\ & 0 & & \\ & & 0 & \\ -i & & & 0 \end{pmatrix}.$$

These 6 matrices are the generators of the Lorentz group in the 4-vector basis.

- Any group elements can be uniquely written as

$$\Lambda = \exp\left(i\theta_i J_i + i\eta_i K_i\right)$$

up to some discrete transformations. (cf. § 2.1.A)

(We omit the proof.
For example, for $\theta_1 \neq 0, \theta_2 = \theta_3 = \eta_i = 0$,

$$\begin{aligned} \Lambda &= \exp\left(i\theta_1 J_1\right) \\ &= \exp\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \theta_1 \\ & & -\theta_1 & 0 \end{pmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \theta_1 \\ & & -\theta_1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_1 & \sin\theta_1 \\ & & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \end{aligned}$$
)

For $\theta_i, \eta_i \ll 1$,

$$\Lambda = I_{4\times 4} + i\underbrace{(\theta_i J_i + \eta_i K_i)}_{\omega} + \mathcal{O}(\theta_i, \eta_i)^2$$

- The generators J_i and K_i satisfy the following commutation relations

$$\boxed{\begin{aligned} &[J_i, J_j] = i\epsilon_{ijk} J_k \\ &[J_i, K_j] = i\epsilon_{ijk} K_k \\ &[K_i, K_j] = -i\epsilon_{ijk} J_k \end{aligned}} \qquad \left(\begin{array}{l} \text{Lie algebra} \\ \text{of Lorentz group } SO(1,3) \end{array}\right)$$

In § 2.1.3, we will see the same commutation relations hold for generators of general representations of Lorentz group.

Problem. Show the above commutation relations.

§ 2.1.A Other (disconnected) Lorentz transformations

- The above discussion implicitly assumes that Λ is continuously connected to the identity element I by infinitesimal Lorentz transformations (LTs).

$$\begin{tikzcd} I \arrow[r] & I+\omega_1 \arrow[r] & I+\omega_2 \arrow[r] & \cdots \arrow[r, dotted] & \Lambda = (1+\omega_n)\cdots(1+\omega_2)(1+\omega_1)I \end{tikzcd}$$

  But there are also LTs which cannot be connected to I by infinitesimal LTs.

- From $g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\nu\sigma}$,

$$\text{(i)}\ \det g \cdot (\det\Lambda)^2 = \det g \quad \therefore \underline{\det\Lambda = \pm 1}.$$

$$\text{(ii)}\ g_{00} = g_{00}\Lambda^0{}_0\Lambda^0{}_0 + g_{ij}\Lambda^i{}_0\Lambda^j{}_0$$

$$1 = (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 \quad \therefore \underline{(\Lambda^0{}_0)^2 = 1 + (\Lambda^i{}_0)^2 \geq 1}.$$

- From (i) LTs are divided into two sets:

$$\begin{cases} \det\Lambda = +1\ ; & \text{“proper” LTs} \\ \det\Lambda = -1\ ; & \text{“improper” LTs} \end{cases}$$

  - Proper LTs form a subgroup of Lorentz group.
    (If $\det\Lambda_1 = \det\Lambda_2 = 1$, $\det(\Lambda_1\Lambda_2) = 1$.)
  - Proper LTs and improper LTs are disconnected.
    (Infinitesimal LTs cannot make $\det\Lambda = 1 \to \det\Lambda = -1$.)

- From (ii) LTs are also divided as:

$$\begin{cases} \Lambda^0{}_0 \geq 1\ ; & \text{“orthochronous” LTs} \\ \Lambda^0{}_0 \leq -1\ ; & \text{“anti-orthochronous” LTs} \end{cases}$$

  - Orthochronous LTs form a subgroup. (**Problem**: Show it.)
  - Orthochronous LTs and anti-orthochronous LTs are disconnected.

-

	$\det\Lambda = +1$	$\det\Lambda = -1$
$\Lambda^0{}_0 \geq 1$	connected to $I = \begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix}$	connected to $P = \begin{pmatrix} 1 &&& \\ & -1 && \\ && -1 & \\ &&& -1 \end{pmatrix}$
$\Lambda^0{}_0 \leq -1$	connected to $PT = \begin{pmatrix} -1 &&& \\ & -1 && \\ && -1 & \\ &&& -1 \end{pmatrix}$	connected to $I = \begin{pmatrix} -1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix}$

- In the following, we consider only proper-orthochronous ($\det\Lambda = +1$ and $\Lambda^0{}_0 \geq 1$) LTs, which are connected to I. (They form a subgroup.)

§ 2.1.3 Lorentz transformations of fields, and representations of Lorentz group.

- In § 1.2, we have shown the scalar action

$$S = \int dt L = \int d^4x \mathcal{L}[\phi(x), \partial_\mu \phi(x)]$$

is invariant under the LTs of the scalar field,

$$\underline{\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)}.$$

We'd like to generalize it as

$$\boxed{\Phi_a(x) \to \Phi'_a(x) = D_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x)} \quad a, b = 1, \cdots N$$

$$\text{or} \quad \boxed{\Phi'(x') = D(\Lambda)\Phi(x)}, \qquad x' = \Lambda x$$

- The matrix $D(\Lambda)$ ($N \times N$ matrix) must be a representation of the Lorentz group.

$$\boxed{D(\Lambda_2\Lambda_1) = D(\Lambda_2)D(\Lambda_1)}$$

(Proof:) For two successive LTs,

$$\begin{aligned}
& x \xrightarrow[\Lambda_1]{} x' \xrightarrow[\Lambda_2]{} x'' \\
& \Phi'(x') = D(\Lambda_1)\Phi(x) \\
& \Phi''(x'') = D(\Lambda_2)\Phi(x') = D(\Lambda_2)D(\Lambda_1)\Phi(x)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
x'' &= \Lambda_2 x' = \Lambda_2(\Lambda_1 x) = (\Lambda_2\Lambda_1)x \\
\therefore \Phi''(x'') &= D(\Lambda_2\Lambda_1)\Phi(x)
\end{aligned}$$

Thus

$$D(\Lambda_2\Lambda_1) = D(\Lambda_2)D(\Lambda_1) \qquad \blacksquare$$

- What kind of representations does the Lorentz group have?
($\Longleftrightarrow$ What kind of fields (particles) are allowed in relativistic QFT?)

$$\begin{cases}
\text{scalar field} : D(\Lambda) = 1 & (1 \times 1 \text{ matrix}) \\
\text{spinor field} : D(\Lambda) = ?? & (2 \times 2 \text{ or } 4 \times 4 \text{ matrix. (We construct it now!))} \\
\text{vector field} : D(\Lambda) = \Lambda^\mu{}_\nu & (4 \times 4 \text{ matrix})
\end{cases}$$

- Consider an infinitesimal LT

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu},$$

parametrized by 6 small parameters $\omega_{\mu\nu} \ll 1$ (or $\theta_i, \eta_i \ll 1$).
For $\omega_{\mu\nu} = 0$, $\Lambda = I_{4\times 4}$ (no transformation), and

$$D_{ab}(\Lambda) = D_{ab}(I) = \delta_{ab} \qquad (N \times N \text{ matrix})$$

Thus we can expand $D(\Lambda)$ as

$$D(I+\omega) = I_{N\times N} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}, \qquad M^{\mu\nu} = -M^{\nu\mu}$$

$$\text{or} \qquad D_{ab}(I_{4\times 4}+\omega) = \delta_{ab} + \frac{i}{2}\omega_{\mu\nu}\left[M^{\mu\nu}\right]_{ab}$$

where the $N \times N$ matrix $\left[M^{\mu\nu}\right]_{ab}$ are the 6 generators of Lorentz group in this representation.

- The 6 generators $M^{\mu\nu}$ satisfy the commutation relations of Lorentz algebra. Let's show it.

——————— on June 19, up to here. ———————

Questions after the lecture: (only some of them)

Q: What is the factor 1/2 in the equation $D(I+\omega) = I_{N\times N} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}$?

A: Here, the summation goes as $D(I+\omega) = I_{N\times N} + \frac{i}{2}\sum_{\mu,\nu=0}^{3}\omega_{\mu\nu}M^{\mu\nu}$.

Since both $\omega_{\mu\nu}$ and $M^{\mu\nu}$ are anti-symmetric, $\omega_{01}M^{01} = \omega_{10}M^{10}$ etc. Therefore, its explicit expression is

$$D(I+\omega) = I_{N\times N} + i\left(\omega_{01}M^{01} + \omega_{02}M^{02} + \omega_{03}M^{03} + \omega_{12}M^{12} + \omega_{23}M^{23} + \omega_{31}M^{31}\right)$$

——————— on June 26, from here. ———————

Last week,

$$\begin{cases} \S 2.1 & \text{Rep. of L. group.} \\ \S 2.1.1 & x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} \\ \S 2.1.2 & \Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + i[\theta_i(J_i)^{\mu}{}_{\nu} + \eta_i(K_i)^{\mu}{}_{\nu}] \\ \S 2.1.\text{A} & \\ \S 2.1.3 & \Phi'(x') = D(\Lambda)\Phi(x) \end{cases}$$

$$\underbrace{D(I+\omega)}_{N\times N} = \underbrace{I}_{N\times N} + \frac{i}{2}\sum_{\mu,\nu}\omega_{\mu\nu}\underbrace{M^{\mu\nu}}_{N\times N} + \mathcal{O}(\omega^2)$$

today →

► Consider three successive LTs,

$$D(\Lambda_3\Lambda_2\Lambda_1) = D(\Lambda_3)D(\Lambda_2)D(\Lambda_1) \qquad \text{—————————(1)}$$

with

$$\begin{cases} \Lambda_1 = I + \omega \\ \Lambda_2 = I + \bar{\omega} \qquad\qquad \bar{\omega}_{\mu\nu}, \omega_{\mu\nu} \ll 1 \\ \Lambda_3 = I - \omega \end{cases}$$

Then,

$$\begin{aligned} \text{LHS of (1)} &= D\Big((I-\omega)(I+\bar{\omega})(I+\omega)\Big) \\ &= D(I + \bar{\omega} - \omega^2 - \omega\bar{\omega} + \bar{\omega}\omega - \omega\bar{\omega}\omega) \\ &= I + \frac{i}{2}(\bar{\omega} - \omega^2 - \omega\bar{\omega} + \bar{\omega}\omega)_{\alpha\beta}M^{\alpha\beta} + \mathcal{O}(\bar{\omega}, \omega^2, \omega\bar{\omega})^2 \\ \text{RHS of (1)} &= D(I-\omega)D(I+\bar{\omega})D(I+\omega) \\ &= \left(I - \frac{i}{2}\omega M + \mathcal{O}(\omega)^2\right)\left(I + \frac{i}{2}\bar{\omega}M + \mathcal{O}(\bar{\omega})^2\right)\left(I + \frac{i}{2}\omega M + \mathcal{O}(\omega)^2\right) \end{aligned}$$

Comparing both sides

$$\begin{aligned} \mathcal{O}(1), \mathcal{O}(\omega), \mathcal{O}(\bar{\omega}) \text{ eqs.} &\to \text{trivial} \\ \mathcal{O}(\omega^2), \mathcal{O}(\bar{\omega}^2) \text{ eqs.} &\to \text{no closed relations among } M^{\mu\nu}. \\ \mathcal{O}(\omega\bar{\omega}) \text{ eq.} &\to \frac{i}{2}(-\omega\bar{\omega} + \bar{\omega}\omega)_{\alpha\beta}M^{\alpha\beta} = \frac{1}{4}\left(\omega M \cdot \bar{\omega}M - \bar{\omega}M \cdot \omega M\right) \\ &\qquad \text{—————————(2)} \end{aligned}$$

Now,

$$\begin{aligned} 4 \times [\text{RHS of (2)}] &= \omega_{\mu\nu}M^{\mu\nu}\bar{\omega}_{\rho\sigma}M^{\rho\sigma} - \bar{\omega}_{\rho\sigma}M^{\rho\sigma}\omega_{\mu\nu}M^{\mu\nu} \\ &= \omega_{\mu\nu}\bar{\omega}_{\rho\sigma}\left[M^{\mu\nu}, M^{\rho\sigma}\right] \\ 4 \times [\text{LHS of (2)}] &= 2i(\bar{\omega}\omega - \omega\bar{\omega})_{\alpha\beta}M^{\alpha\beta} \\ &= 2i(\bar{\omega}_{\alpha\gamma}\omega^{\gamma}{}_{\beta} - \omega_{\alpha\gamma}\bar{\omega}^{\gamma}{}_{\beta})M^{\alpha\beta} \\ &= 2i(\bar{\omega}_{\alpha\gamma}g^{\gamma\delta}\omega_{\delta\beta} - \omega_{\alpha\gamma}g^{\gamma\delta}\bar{\omega}_{\delta\beta})M^{\alpha\beta} \\ &= 2i\ \omega_{\mu\nu}\bar{\omega}_{\rho\sigma}(\underbrace{g^{\sigma\mu}M^{\rho\nu}}_{-g^{\mu\sigma}M^{\nu\rho}} - g^{\nu\rho}M^{\mu\sigma}) \\ &= i\ \omega_{\mu\nu}\bar{\omega}_{\rho\sigma}\left((-g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}) - (\mu \leftrightarrow \nu)\right) \qquad (\because \omega_{\mu\nu} = -\omega_{\nu\mu}) \\ &= i\ \omega_{\mu\nu}\bar{\omega}_{\rho\sigma}\underbrace{(-g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma})}_{\text{anti-symmetric under } \mu \leftrightarrow \nu,\ \rho \leftrightarrow \sigma.} \end{aligned}$$

Comparing the $\omega_{\mu\nu}\bar{\omega}_{\rho\sigma}$ components in the both sides, we obtain

$$\boxed{[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right)} \qquad \text{———————}(3)$$

Lie algebra of Lorentz group

Defining

$$\begin{cases} D(J_i) \equiv -\dfrac{1}{2}\epsilon_{ijk}M^{jk} \iff M^{ij} = -\epsilon_{ijk}D(J_k) \\ D(K_i) \equiv M^{0i} \end{cases} \qquad \text{———————}(4),$$

$$(3) \iff \boxed{\begin{array}{l} [D(J_i), D(J_j)] = i\epsilon_{ijk}D(J_k) \\ [D(J_i), D(K_j)] = i\epsilon_{ijk}D(K_k) \\ [D(K_i), D(K_j)] = -i\epsilon_{ijk}D(J_k) \end{array}} \text{————}(5)$$

The same as those in § 2.1.2 !

- ► Recall that $M^{\mu\nu} \iff D(J_i), D(K_i)$ are the generators defined by

$$D(I_{4\times4} + \omega) = I_{N\times N} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu},$$

which represents infinitesimal rotations and boosts. Using the notation $\omega_{0i} = \eta_i$ and $\omega_{ij} = -\epsilon_{ijk}\theta_k$ for the 6 small parameters (see § 2.1.2), the above eq. becomes

$$D(I_{4\times4} + \omega) = I_{N\times N} + i\left[\theta_i D(J_i) + \eta_i D(K_i)\right]$$

which is the same form as the infinitesimal LT of coordinates in § 2.1.2.

- ► Now, define

$$\boxed{\begin{array}{l} D(A_i) = \dfrac{1}{2}\left(D(J_i) + iD(K_i)\right) \\[2ex] D(B_i) = \dfrac{1}{2}\left(D(J_i) - iD(K_i)\right) \end{array}}$$

(Note that $D(B_i) \neq D(A_i)^\dagger$, because $D(K_i)^\dagger \neq D(K_i)$ (see discussion later).)

Then

$$(5) \iff \boxed{\begin{array}{l} [D(A_i), D(A_j)] = i\epsilon_{ijk}D(A_k) \\ [D(B_i), D(B_j)] = i\epsilon_{ijk}D(B_k) \\ [D(A_i), D(B_j)] = 0 \end{array}} \text{————}(6)$$

This is the algebra of SU(2)×SU(2), and therefore we can classify the representations of Lorentz group by using representations of SU(2).

- Before going ahead, let's summarize the discussion so far:
    - LTs of fields: $\Phi'(x') = D(\Lambda)\Phi(x)$ with $x' = \Lambda x$.
    - For infinitesimal LTs, $D(\Lambda) = D(I_{4\times 4} + \omega) = I_{N\times N} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}$
    - Three equivalent ways of representing the 6 generators:

$$\begin{array}{rcccccc} & M^{\mu\nu} & \Longleftrightarrow & D(J_i), D(K_i) & \Longleftrightarrow & D(A_i), D(B_i) \\ \text{satisfying} & (3) & \Longleftrightarrow & (5) & \Longleftrightarrow & (6) \end{array}$$

    - So far, $D(\Lambda)$, $M^{\mu\nu}$, $D(J_i)$, $D(K_i)$, $D(A_i)$, $D(B_i)$, are all generic $N \times N$ matrices.
- What's the generic representation which satisfy (6) ?

$$[D(A_i), D(A_j)] = i\epsilon_{ijk}D(A_k)$$

We know this from QM !!

$$\underline{\left[\hat{j}_i, \hat{j}_j\right] = i\epsilon_{ijk}\hat{j}_k.}$$

Starting from this, we could show that generic representation is

$$\begin{aligned} \underline{\text{spin-}j\text{ state}}: \quad & |j, m\rangle \\ & j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots \\ & m = \underbrace{-j, -j+1, \cdots j-1, j}_{2j+1} \\ \text{where} \quad & \begin{cases} \hat{j}^2\,|j,m\rangle = j(j+1)\,|j,m\rangle \\ \hat{j}_3\,|j,m\rangle = m\,|j,m\rangle \end{cases} \quad . \end{aligned}$$

- $\boxed{\text{NOTE}}$ In QM, we have used the fact that $\hat{j}_i$ are Hermitian, $\hat{j}_i^\dagger = \hat{j}_i$. Here, $D(A_i)$ and $D(B_i)$ are not Hermitian, but we can derive similar result assuming finite dimensional representation. Let's see this.
- Representation of "A-spin".
For simplicity, we denote

$$A_i = D(A_i) \qquad (N \times N \text{ matrix}).$$

Define

$$\begin{cases} A^2 = A_1^2 + A_2^2 + A_3^2, \\ A_\pm = A_1 \pm iA_2. \end{cases}$$

From (6), we can show

$$\begin{aligned}
[A^2, A_3] &= 0 \quad \text{———} \ \text{(i)}, \\
[A^2, A_\pm] &= 0 \quad \text{———} \ \text{(ii)}, \\
[A_3, A_\pm] &= \pm A_\pm \quad \text{———} \ \text{(iii)}, \\
A^2 &= A_3(A_3+1) + A_- A_+ \\
&= A_3(A_3-1) + A_+ A_- \quad \text{———} \ \text{(iv)}.
\end{aligned}$$

From (i), there exists a simultaneous eigenvector of A^2 and A_3; $\Phi_{\lambda,\mu}$,[3]

$$\begin{aligned}
\begin{pmatrix} & & \\ & A^2 & \\ & & \end{pmatrix} \begin{pmatrix} \\ \Phi_{\lambda,\mu} \\ \\ \end{pmatrix} &= \lambda \begin{pmatrix} \\ \Phi_{\lambda,\mu} \\ \\ \end{pmatrix} \\
\underbrace{\begin{pmatrix} & & \\ & A_3 & \\ & & \end{pmatrix}}_{N\times N} \begin{pmatrix} \\ \Phi_{\lambda,\mu} \\ \\ \end{pmatrix} &= \mu \left.\begin{pmatrix} \\ \Phi_{\lambda,\mu} \\ \\ \end{pmatrix}\right\} N
\end{aligned}$$

Then, from (ii) and (iii), the vector

$$\Phi_{\lambda,\mu\pm1} \equiv A_\pm \Phi_{\lambda,\mu}$$

satisfy

$$\begin{cases} A^2 \Phi_{\lambda,\mu\pm1} = \lambda \Phi_{\lambda,\mu\pm1} \\ A_3 \Phi_{\lambda,\mu\pm1} = (\mu\pm1)\Phi_{\lambda,\mu\pm1}. \end{cases}$$

Continuing further, the vector $\Phi_{\lambda,\mu\pm n} = (A_\pm)^n \Phi_{\lambda,\mu}$ satisfy

$$\begin{cases} A^2 \Phi_{\lambda,\mu\pm n} = \lambda \Phi_{\lambda,\mu\pm1} \\ A_3 \Phi_{\lambda,\mu\pm n} = (\mu\pm n)\Phi_{\lambda,\mu\pm1}. \end{cases}$$

Now, assuming <u>finite dimensional representation</u>, there must be upper and lower bounds on A_3's eigenvalue

$$\begin{cases} \mu_{\text{max}} = \mu + n_+, \\ \mu_{\text{min}} = \mu + n_-, \end{cases}$$

with

$$\begin{cases} A_+ \Phi_{\lambda,\mu_{\text{max}}} = 0 \quad \text{———} \ \text{(v)}, \\ A_- \Phi_{\lambda,\mu_{\text{min}}} = 0 \quad \text{———} \ \text{(vi)}. \end{cases}$$

---

[3] (At this stage, since A^2 and A_3 are not Hermitian, the eigenvalues λ and μ are not necessarily real numbers, but we will see they are real. I thank the student who pointed out this!)

From (iv) and (v),

$$\underbrace{A^2}_{\to\lambda}\Phi_{\lambda,\mu_{\max}} = \Big(\underbrace{A_3(A_3+1)}_{\to\mu_{\max}(\mu_{\max}+1)} + A_-\underbrace{A_+}_{\to 0}\Big)\Phi_{\lambda,\mu_{\max}}$$

$$\therefore \quad \lambda = \mu_{\max}(\mu_{\max}+1) \quad \text{———} \text{ (vii)},$$

Similarly from (iv) and (vi),

$$\lambda = \mu_{\min}(\mu_{\min}+1) \quad \text{———} \text{ (viii)},$$

From (vii)−(viii),

$$(\mu_{\max}+\mu_{\min})(\mu_{\max}-\mu_{\min}+1) = 0$$

Since $\mu_{\max} - \mu_{\min} = n_+ + n_- \equiv n =$ non-negative integer (and therefore real), $\mu_{\max} - \mu_{\min} + 1 > 0$, and we have $\underline{\mu_{\max} = -\mu_{\min}}$. Together with $\mu_{\max} - \mu_{\min} = n$, we thus have

$$\begin{cases} \mu_{\max} = \dfrac{n}{2} = -\mu_{\min} \\ \\ \lambda = \dfrac{n}{2}\left(\dfrac{n}{2}+1\right) \end{cases}$$

We have obtained the irreducible representation of the "A-spin".

$$\begin{cases} A^2\Phi_a^{(A)} = A(A+1)\ \Phi_a^{(A)} \\ A_3\Phi_a^{(A)} = a\ \Phi_a^{(A)} \end{cases}$$

where

$$A = 0, \frac{1}{2}, 1, \frac{3}{3}, \cdots$$

$$a = \underbrace{-A, -A+1, \cdots A-1, A}_{(2A+1)\text{ components}}$$

- Since we have two SU(2)s, $D(A_i)$ and $D(B_i)$, any irreducible representations of Lorentz group are parametrized by a set of two numbers:

$$(A, B) \qquad A, B = \text{integer or half-intetger}$$

- The corresponding field is

$$\Phi_{a,b}^{(A,B)} \qquad (2A+1)(2B+1) \text{ components}$$

$$\begin{cases} a = -A, -A+1, \cdots A-1, A \\ b = -B, -B+1, \cdots B-1, B \end{cases}$$

transforming as

$$\begin{cases} D(A^2)\Phi_{a,b}^{(A,B)} &= A(A+1)\Phi_{a,b}^{(A,B)} \\ D(A_3)\Phi_{a,b}^{(A,B)} &= a\Phi_{a,b}^{(A,B)} \\ D(B^2)\Phi_{a,b}^{(A,B)} &= B(B+1)\Phi_{a,b}^{(A,B)} \\ D(B_3)\Phi_{a,b}^{(A,B)} &= b\Phi_{a,b}^{(A,B)} \end{cases} \qquad \text{———————}(7)$$

where $D(A^2) = \sum_{i=1}^{3} D(A_i)^2$, $D(B^2) = \sum_{i=1}^{3} D(B_i)^2$ and we omitted the transformation of the argument x.

-

$$\begin{aligned} (A,B) =&(0,0) \qquad \text{scalar fields} \\ (A,B) =&\left(0,\frac{1}{2}\right),\left(\frac{1}{2},0\right) \qquad \text{spinor fields} \\ (A,B) =&\left(\frac{1}{2},\frac{1}{2}\right) \qquad \text{vector fields} \end{aligned}$$

- Scalar fields

$$\begin{aligned} &A = B = 0, \\ &a = b = 0 \\ &\underline{\Phi^{(0,0)} \qquad \text{1-components}} \\ &D(A_i) = D(B_i) = 0 \\ \therefore &D(\Lambda) = I + \frac{i}{2}\omega_{\mu\nu}\underbrace{M^{\mu\nu}}_{=0} = I \\ \therefore &\Phi'(x') = D(\Lambda)\Phi(x) = \Phi(x) \end{aligned}$$

§ 2.1.4 Spinor Fields

- Consider fields with $\boxed{(A,B) = \left(0,\frac{1}{2}\right)}$.

$$\begin{aligned} &(2A+1)(2B+1) = 1 \times 2 = 2 \text{ components} \\ &\Phi_b^{(0,1/2)}, \qquad b = -1/2, 1/2. \end{aligned}$$

Thus, $D(A_i), D(B_i) \Longleftrightarrow D(J_i), D(K_i) \Longleftrightarrow M^{\mu\nu}$ and $D(\Lambda)$ are 2×2 matrices.

From (6) and (7),

$$\begin{cases} D(A_i) = 0 \qquad (2\times 2), \\ D(B_i) = \frac{1}{2}\sigma_i \end{cases} \text{———— (8).}$$

σ_i : Pauli matrices $\quad \sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$

$$\left(\begin{array}{l} \because \text{For } \Phi = \begin{pmatrix} \Phi_{1/2} \\ \Phi_{-1/2} \end{pmatrix}, \\ D(B_3)\Phi = \begin{pmatrix} 1/2 & \\ & -1/2 \end{pmatrix}\Phi,\ D(B_+)\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Phi,\ D(B_-)\Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\Phi. \end{array} \right)$$

Thus,

$$(8) \Longleftrightarrow \begin{cases} D(J_i) = D(A_i) + D(B_i) = \frac{1}{2}\sigma_i \\ D(K_i) = -iD(A_i) + iD(B_i) = i\frac{1}{2}\sigma_i \end{cases}$$

$$\Longleftrightarrow \begin{cases} M^{ij} = -\frac{1}{2}\epsilon_{ijk}\sigma_k \\ M^{0i} = i\frac{1}{2}\sigma_i \end{cases}$$

- ▶ Denoting the 2-component field $\Phi_b^{(0,1/2)}$ as

$$\left(0, \frac{1}{2}\right)\text{-field: } \boxed{\xi_\alpha(x) \quad \alpha = 1, 2},$$

its Lorentz transformation is given by

$$\xi'_\alpha(x') = D(\Lambda)_\alpha{}^\beta \xi_\beta(x),$$

$$D(\Lambda)_\alpha{}^\beta = \exp\Big(i\theta_i D(J_i) + i\eta_i D(K_i)\Big)_\alpha{}^\beta$$

$$= \exp\left(i\theta_i \frac{1}{2}\sigma_i - \eta_i \frac{1}{2}\sigma_i\right)_\alpha{}^\beta$$

- ▶ $\boxed{\text{Example}}$:
  - $D(\Lambda)$ for a rotation around the z-axis

$$D(\Lambda) = \exp\left(i\theta_3\frac{1}{2}\sigma_3\right) = \exp\begin{pmatrix} \frac{i}{2}\theta_3 & \\ & -\frac{i}{2}\theta_3 \end{pmatrix} = \begin{pmatrix} e^{\frac{i}{2}\theta_3} & \\ & e^{-\frac{i}{2}\theta_3} \end{pmatrix} \quad \text{——(i)}$$

  - $D(\Lambda)$ for a boost in the z-direction

$$D(\Lambda) = \exp\left(-\eta_3\frac{1}{2}\sigma_3\right) = \exp\begin{pmatrix} -\frac{1}{2}\eta_3 & \\ & \frac{1}{2}\eta_3 \end{pmatrix} = \begin{pmatrix} e^{-\frac{1}{2}\eta_3} & \\ & e^{\frac{1}{2}\eta_3} \end{pmatrix} \quad \text{——(ii)}$$

Comment on the unitarity.

$D(J_i) = \frac{1}{2}\sigma_i$ are Hermitian, $D(J_i)^\dagger = D(J_i)$,
but $D(K_i) = i\frac{1}{2}\sigma_i$ are anti-Hermitian, $D(K_i)^\dagger = -D(K_i)$.
Thus, the spinor representation of the Lorentz group $D(\Lambda)$ is NOT unitary in general.
(For instance, the rotation (i) is unitary, $D(\Lambda)^\dagger D(\Lambda) = I$,
but the boost (ii) is not unitary, $D(\Lambda)^\dagger D(\Lambda) \neq I$.)

In general, there are no non-trivial finite dimensional unitary representation of the Lorentz group.

Is this OK?

→ No problem, as far as Lorentz transformation of states ~~are unitary:~~
~~$\langle\beta|\alpha\rangle \to \langle\beta|U(\Lambda)^\dagger U(\Lambda)|\alpha\rangle = \langle\beta|\alpha\rangle$.~~ have proper transformation.[4]

(Maybe more on this later, if we have time...)

——— on June 26, up to here. ———

——— on July 3, from here. ———

Last week,

- §2.1: Rep. of L. group.
- §2.1.3: $\Phi'(x') = D(\Lambda)\Phi(x)$,
  generic irreducible rep. $\Phi(x)_{a,b}^{(A,B)}$, $a = -A, \cdots A-1, A$, $b = -B, \cdots B-1, B$.
- §2.1.4: Spinor fields, $\Phi(x)_b^{(0,1/2)}(x)$, or $\xi_\alpha(x)$. 2-component field.
  $D(\Lambda) = \exp(i\theta_i(\sigma_i/2) - \eta_i(\sigma_i/2))$.

today →

▶ Similarly, for spinor fields with $\boxed{(A,B) = \left(\frac{1}{2}, 0\right)}$, $\Phi_a^{(1/2,0)}$, from (6) and (7),

$$\begin{cases} D(A_i) = \frac{1}{2}\sigma_i \\ D(B_i) = 0 \qquad (2\times 2) \end{cases} \Longleftrightarrow \begin{cases} D(J_i) = \frac{1}{2}\sigma_i \\ D(K_i) = -i\frac{1}{2}\sigma_i \end{cases} \Longleftrightarrow \begin{cases} M^{ij} = -\frac{1}{2}\epsilon_{ijk}\sigma_k \\ M^{0i} = -i\frac{1}{2}\sigma_i \end{cases}$$

---

[4] (corrected after the lecture)

- To summarize, there are two kinds of 2-component spinor fields with $(A, B) = (0, 1/2)$ and $(1/2, 0)$, and their Lorentz transformations are given by

$$
\begin{aligned}
&(A,B)\\
&\left(0,\frac{1}{2}\right) : \ \psi_L \to D_L(\Lambda)\psi_L = \exp\left(i\theta_i\frac{1}{2}\sigma_i - \eta_i\frac{1}{2}\sigma_i\right)\psi_L = \left(I + i\theta_i\frac{1}{2}\sigma_i - \eta_i\frac{1}{2}\sigma_i + \cdots\right)\psi_L\\
&\left(\frac{1}{2},0\right) : \ \psi_R \to D_R(\Lambda)\psi_R = \exp\left(i\theta_i\frac{1}{2}\sigma_i + \eta_i\frac{1}{2}\sigma_i\right)\psi_R = \left(I + i\theta_i\frac{1}{2}\sigma_i + \eta_i\frac{1}{2}\sigma_i + \cdots\right)\psi_R
\end{aligned}
$$

(sorry that I changed the notation $\xi \to \psi_L$.)
(We omit the argument x and $x' = \Lambda x$ for simplicity.)

Their infinitesimal transformations are

$$
\begin{cases}
\delta\psi_L = \dfrac{1}{2}(i\theta_i - \eta_i)\sigma_i\psi_L\\[2ex]
\delta\psi_R = \dfrac{1}{2}(i\theta_i + \eta_i)\sigma_i\psi_R
\end{cases}
\qquad \text{———} \ (9).
$$

- Note that

$$
\begin{aligned}
&\psi_L \sim \left(0,\frac{1}{2}\right) \qquad \psi_R \sim \left(\frac{1}{2},0\right)\\
\longleftrightarrow \quad &\psi_L^* \sim \left(\frac{1}{2},0\right) \qquad \psi_R^* \sim \left(0,\frac{1}{2}\right)
\end{aligned}
$$

From (9), $\delta\psi_L^* = \dfrac{1}{2}(-i\theta_i - \eta_i)\sigma_i^*\psi_L^*$

by using $\epsilon\sigma_i = -\sigma_i^*\epsilon$ where $\epsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$$
\begin{aligned}
\epsilon\delta\psi_L^* &= \frac{1}{2}(-i\theta_i - \eta_i)\epsilon\sigma_i^*\psi_L^*\\
\delta(\epsilon\psi_L^*) &= \frac{1}{2}(i\theta_i + \eta_i)\sigma_i(\epsilon\psi_L^*)
\end{aligned}
$$

Thus, $\epsilon\psi_L^*$ transforms in the same way as ψ_R in (9).

- **Comment on spinor indices**

The indices of 2-component spinors are often denoted by undotted and dotted labels:

$$(\psi_L)_\alpha, \qquad (\psi_R)_{\dot\alpha}$$

together with invariant tensors $\epsilon^{\alpha\beta}$, $\epsilon^{\dot\alpha\dot\beta}$, and extended Pauli matrices $\sigma^\mu_{\alpha\dot\beta}$ (see below). In particular, the spinor contraction such as $\psi\xi \equiv \psi_\alpha\xi^\alpha = \psi_\alpha\epsilon^{\alpha\beta}\xi_\beta$ are very convenient (once you get used to it), but in this lecture, we do not use them.

§ 2.1.5 Lorentz transformations of spinor bilinears

- We have seen the Lorentz transformations of spinor fields ψ_L and ψ_R. In order to construct a Lorentz invariant Lagrangian, let's consider Lorentz transformations of spinor bilinears, such as

$$\psi_L^\dagger \psi_R = (\psi_{L1}^*, \psi_{L2}^*) \begin{pmatrix} \psi_{R1} \\ \psi_{R2} \end{pmatrix} = \psi_{L1}^* \psi_{R1} + \psi_{L2}^* \psi_{R2}.$$

$$\psi_L^\dagger \sigma_3 \psi_L = (\psi_{L1}^*, \psi_{L2}^*) \sigma_3 \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \end{pmatrix} = \psi_{L1}^* \psi_{L1} - \psi_{L2}^* \psi_{L2}.$$

In general, we can think of various combinations

$$\left\{ \psi_L^T, \psi_R^T, \psi_L^\dagger, \psi_R^\dagger \right\} \times (2 \times 2 \text{ matrix}) \times \{\psi_L, \psi_R, \psi_L^*, \psi_R^*\}.$$

They can be classified according to SU(2)×SU(2).

$$\psi_L, \psi_R^* \cdots \left(0, \frac{1}{2}\right)$$
$$\psi_R, \psi_L^* \cdots \left(\frac{1}{2}, 0\right)$$

§ $\boxed{\left(0, \frac{1}{2}\right) \otimes \left(0, \frac{1}{2}\right)}$

- If there is only ψ_L field, the possible bilinear terms transforming as $\left(0, \frac{1}{2}\right) \otimes \left(0, \frac{1}{2}\right)$ are

$$\psi_L^T \cdot (2 \times 2 \text{ matrics}) \cdot \psi_L.$$

Among them, $\underline{\psi_L^T \epsilon \, \psi_L \text{ is Lorentz invariant.}}$

$$\begin{aligned} \because \ \delta(\psi_L^T \epsilon \, \psi_L) &= (\delta \psi_L^T) \epsilon \psi_L + \psi_L^T \epsilon (\delta \psi_L) \\ &= \left(\frac{1}{2} (i\theta_k - \eta_k) \psi_L^T \sigma_k^T \right) \epsilon \psi_L + \psi_L^T \epsilon \left(\frac{1}{2} (i\theta_k - \eta_k) \sigma_k \psi_L \right) \quad (\because \ (9)) \\ &= \frac{1}{2} (i\theta_k - \eta_k) \psi_L^T \underbrace{(\sigma_k^T \epsilon + \epsilon \sigma_k)}_{=0} \psi_L \\ &= 0 \end{aligned}$$

- If there is only ψ_R field, similarly, $\underline{\psi_R^\dagger \epsilon \, \psi_R^* \text{ is Lorentz invariant.}}$

▶ If there are both ψ_L and ψ_R field, we can also think

$$\psi_R^\dagger \cdot (2 \times 2 \text{ matrics}) \cdot \psi_L.$$

Among them, $\underline{\psi_R^\dagger \psi_L \text{ is Lorentz invariant}}$.

$$\begin{aligned} \because\ \delta(\psi_R^\dagger \psi_L) &= (\delta\psi_R^\dagger)\psi_L + \psi_R^\dagger(\delta\psi_L) \\ &= \left(\frac{1}{2}(-i\theta_k + \eta_k)\psi_R^\dagger \sigma_k\right)\psi_L + \psi_R^\dagger\left(\frac{1}{2}(i\theta_k - \eta_k)\sigma_k \psi_L\right) \quad (\because\ (9)) \\ &= 0 \end{aligned}$$

<u>Comments</u>

(i) In terms of SU(2)×SU(2), the above terms, $\psi_L^T \epsilon\, \psi_L$, $\psi_R^\dagger \epsilon\, \psi_R^*$ and $\psi_R^\dagger \psi_L$ correspond to

$$\left(0, \frac{1}{2}\right) \otimes \left(0, \frac{1}{2}\right) = \underbrace{(0,0)}_{\text{this term.}} \oplus (0,1)$$

(ii) One might think that

$$\psi_L^T \epsilon \psi_L = (\psi_1, \psi_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_1\psi_2 - \psi_2\psi_1$$

vanishes. However, if ψ_i are anti-commuting (as in quantized fermion field), $\psi_1\psi_2 = -\psi_2\psi_1$ and hence $\psi_L^T \epsilon \psi_L$ does not vanish.

(iii) $\psi_L^T \epsilon\, \psi_L$ and $\psi_R^\dagger \epsilon\, \psi_R^*$ terms correspond to <u>Majorana mass terms</u>, and $\psi_R^\dagger \psi_L$ corresponds to <u>Dirac mass term</u>.
If we consider a <u>charged fermion</u> (such as electron and positron), only the Dirac mass term is allowed.

$$\left(\begin{array}{l} \text{Field } \Phi \text{ is charged (under conserved symmetry)} \\ \Longleftrightarrow \text{Lagrangian is invariant under } \Phi \to e^{i\alpha}\Phi. \\ \psi_L^T \epsilon\, \psi_L \text{ is not invariant under } \psi_L \to e^{i\alpha}\psi_L, \\ \text{while } \psi_R^\dagger \psi_L \text{ is invariant under } \psi_L \to e^{i\alpha}\psi_L, \psi_R \to e^{i\alpha}\psi_R. \end{array}\right)$$

In the following we consider a charged fermion and hence only the Dirac mass term $\psi_R^\dagger \psi_L$.

(Neutrinos may have Majorana mass term (maybe Majorana fermion). Still unknown.)

§ $\boxed{\left(\frac{1}{2}, 0\right) \otimes \left(\frac{1}{2}, 0\right)}$

- Similarly,

$$\psi_L^\dagger\psi_R, \quad \psi_R^T\epsilon\psi_R, \quad \psi_L^\dagger\epsilon\psi_L^*$$

are Lorentz invariant, corresponding to

$$\left(\frac{1}{2},0\right)\otimes\left(\frac{1}{2},0\right) = \underbrace{(0,0)}_{\text{this term.}} \oplus (1,0)$$

We only consider the Dirac mass term $\psi_L^\dagger\psi_R$.

§ $\boxed{\left(0,\frac{1}{2}\right)\otimes\left(\frac{1}{2},0\right)}$

- Consider

$$\psi_R^\dagger\cdot(2\times 2 \text{ matrics})\cdot\psi_R.$$

There are 4 independent combinations, which can be taken as

$$\psi_R^\dagger\psi_R, \qquad \psi_R^\dagger\sigma_i\psi_R \quad (i=1,2,3).$$

They transform as

$$\begin{aligned}
\delta(\psi_R^\dagger\psi_R) &= (\delta\psi_R^\dagger)\psi_R + \psi_R^\dagger(\delta\psi_R)\\
&= \cdots\\
&= \eta_k(\psi_R^\dagger\psi_k\psi_R),\\
\delta(\psi_R^\dagger\sigma_i\psi_R) &= (\delta\psi_R^\dagger)\sigma_i\psi_R + \psi_R^\dagger\sigma_i(\delta\psi_R)\\
&= \cdots\Big(\text{using } [\sigma_i,\sigma_j] = 2\epsilon_{ijk}\sigma_k \text{ and } \{\sigma_i,\sigma_j\} = \sigma_i\sigma_j+\sigma_j\sigma_i = 2\delta_{ij}I\Big)\\
&= \eta_i(\psi_R^\dagger\psi_R) + \epsilon_{ijk}\theta_k(\psi_R^\dagger\psi_j\psi_R).
\end{aligned}$$

Combining them

$$\delta\begin{pmatrix}\psi_R^\dagger\psi_R\\ \psi_R^\dagger\sigma_1\psi_R\\ \psi_R^\dagger\sigma_2\psi_R\\ \psi_R^\dagger\sigma_3\psi_R\end{pmatrix} = \begin{pmatrix}0 & \eta_1 & \eta_2 & \eta_3\\ \eta_1 & 0 & \theta_3 & -\theta_2\\ \eta_2 & -\theta_3 & 0 & \theta_1\\ \eta_3 & \theta_2 & -\theta_1 & 0\end{pmatrix}\begin{pmatrix}\psi_R^\dagger\psi_R\\ \psi_R^\dagger\sigma_1\psi_R\\ \psi_R^\dagger\sigma_2\psi_R\\ \psi_R^\dagger\sigma_3\psi_R\end{pmatrix}.$$

This is nothing but the transformation of Lorentz 4-vector! (See eq.(1) of § 2.1.2.)
Defining

$$\boxed{\sigma^\mu = (I,\sigma_i) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}$$

the above equation can be written as

$$\underline{\delta(\psi_R^\dagger\sigma^\mu\psi_R) = \omega^\mu{}_\nu(\psi_R^\dagger\sigma^\nu\psi_R)}$$

► Similarly, defining

$$\boxed{\bar{\sigma}^\mu = (I, -\sigma_i)}$$

One can show

$$\underline{\delta(\psi_L^\dagger \bar{\sigma}^\mu \psi_L) = \omega^\mu{}_\nu (\psi_L^\dagger \bar{\sigma}^\nu \psi_L)}$$

<u>Comments</u>

(i) In terms of SU(2)×SU(2), the above argument means

$$\left(0, \frac{1}{2}\right) \otimes \left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

is a Lorentz 4-vector.

(ii) For finite Lorentz transformation, they transform as

$$\begin{aligned} {\psi'_R}^\dagger(x')\sigma^\mu \psi'_R(x') &= \psi_R^\dagger(x) D_R(\Lambda)^\dagger \sigma^\mu D_R(\Lambda)\psi_R(x) \qquad (\because \psi'_R(x') = D_R(\Lambda)\psi_R(x)) \\ &= \Lambda^\mu{}_\nu \psi_R^\dagger(x)\sigma^\nu \psi_R(x), \end{aligned}$$

namely

$$D_R(\Lambda)^\dagger \sigma^\mu D_R(\Lambda) = \Lambda^\mu{}_\nu \sigma^\nu$$

where

$$\begin{aligned} D_R(\Lambda) &= \exp\left(i\theta_k \frac{1}{2}\sigma_k + \eta_k \frac{1}{2}\sigma_k\right) \\ \Lambda^\mu{}_\nu &= \exp\left(i\theta_i J_i + i\eta_i K_i\right) \end{aligned}$$

with $(J_i)^\mu{}_\nu$ and $(K_i)^\mu{}_\nu$ given in § 2.1.2.
Similarly,

$$\begin{aligned} {\psi'_L}^\dagger(x')\bar{\sigma}^\mu \psi'_L(x') &= \psi_L^\dagger(x) \underbrace{D_L(\Lambda)^\dagger \bar{\sigma}^\mu D_L(\Lambda)}_{\Lambda^\mu{}_\nu \bar{\sigma}^\nu} \psi_L(x) \\ &= \Lambda^\mu{}_\nu \psi_L^\dagger(x)\bar{\sigma}^\nu \psi_L(x). \end{aligned}$$

(iii) The other combinations, $\psi_L^T \epsilon \sigma^\mu \psi_R$ and $\psi_R^\dagger \bar{\sigma}^\mu \epsilon \psi_L^*$, also transform as Lorentz 4-vector, but we do not consider them. (They are not invariant under $\psi_L \to e^{i\alpha}\psi_L, \psi_R \to e^{i\alpha}\psi_R$.)

► Now we have obtained Lorentz scalars and vectors from spinor bilinears, ready to construct the Dirac Lagrangian.

§ 2.2 Free Dirac Field

§ 2.2.1 Lagrangian

- In § 2.1.5, we have seen

$$\psi_R^\dagger \psi_L, \quad \psi_L^\dagger \psi_R$$

  are Lorentz invariant. They can be the Lagrangian terms.

- On the other hand,

$$\psi_R^\dagger \sigma^\mu \psi_R, \quad \psi_L^\dagger \bar{\sigma}^\mu \psi_L$$

  are Lorentz vector. They can be combined with ∂_μ to make the Lagrangian (more precisely, the action) Lorentz invariant. For instance,

$$\int d^4x \; \psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$$

  is Lorentz invariant, because

$$\begin{aligned}
\int d^4x \; \psi_R^\dagger(x) \sigma^\mu \partial_\mu \psi_R(x) \to & \int d^4x \; {\psi'_R}^\dagger(x) \sigma^\mu \partial_\mu \psi'_R(x) \\
=& \int d^4x' \; {\psi'_R}^\dagger(x') \sigma^\mu \partial'_\mu \psi'_R(x') \\
& (x' = \Lambda x, \quad \text{change of integration variable. } \partial'_\mu = \frac{\partial}{\partial x'^\mu}) \\
=& \int d^4x' \; {\psi_R}^\dagger(x) D_R(\Lambda)^\dagger \sigma^\mu \partial'_\mu D_R(\Lambda) \psi_R(x) \\
=& \int d^4x' \; {\psi_R}^\dagger(x) \Lambda^\mu{}_\nu \sigma^\nu \partial'_\mu \psi_R(x) \\
& \left(\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial}{\partial x^\rho} = (\Lambda^{-1})^\rho{}_\mu \partial_\rho, \quad d^4x' = d^4x \right) \\
=& \int d^4x \; {\psi_R}^\dagger(x) \Lambda^\mu{}_\nu \sigma^\nu (\Lambda^{-1})^\rho{}_\mu \partial_\rho \psi_R(x) \\
=& \int d^4x \; {\psi_R}^\dagger(x) \sigma^\nu \partial_\nu \psi_R(x).
\end{aligned}$$

- We can also think other combinations with ∂_μ to make Lorentz invariant terms, but
  - $\partial_\mu(\psi_R^\dagger \sigma^\mu \psi_R)$: total derivative and not a viable Lagrangian term.
  - $(\partial_\mu \psi_R^\dagger) \sigma^\mu \psi_R$: equivalent to $\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$ up to total derivative.

- Similarly,

$$\int d^4x\, \psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L$$

is Lorentz invariant.

- Combining them all, we obtain the Lagrangian of the free Dirac field:

$$\boxed{\mathcal{L} = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L - m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R).}$$

where we have added a factor of i in the derivative term to make the Lagrangian Hermitian:

$$\begin{aligned}(i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R)^\dagger &= -i(\partial_\mu \psi_R)^\dagger \sigma^\mu \psi_R \\ &= i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + \text{total derivative}\end{aligned}$$

The above Lagrangian can be written in terms of **four-component Dirac spinor** and **gamma matrices (Dirac matrices)**

$$\begin{aligned}&\boxed{\mathcal{L} = \overline{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi} \\ &= (\psi_R^\dagger, \psi_L^\dagger)\left[\begin{pmatrix} 0 & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & 0\end{pmatrix} - \begin{pmatrix} m & \\ & m\end{pmatrix}\right]\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}\end{aligned}$$

where

$$\text{Dirac Spinor: } \Psi \equiv \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$$

$$\text{gamma matrices: } \gamma^\mu \equiv \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}$$

$$\overline{\Psi} \equiv \Psi^\dagger \gamma^0 = (\psi_L^\dagger, \psi_R^\dagger)\begin{pmatrix} & I \\ I & \end{pmatrix} = (\psi_R^\dagger, \psi_L^\dagger).$$

———— on July 3, up to here. ————

———— on July 10, from here. ————

Last week,

§ 2.1 Rep. of L. group.

§ 2.2 Free Dirac Field

§ 2.2.1 Lagrangian $\mathcal{L} = \overline{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$

announcement

extra class (補講) on July 31. (to finish the quantization of Dirac field)

- The γ matrices

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} = \left\{ \begin{pmatrix} & I_{2\times2} \\ I_{2\times2} & \end{pmatrix}, \begin{pmatrix} & \sigma_1 \\ -\sigma_1 & \end{pmatrix}, \begin{pmatrix} & \sigma_2 \\ -\sigma_2 & \end{pmatrix}, \begin{pmatrix} & \sigma_3 \\ -\sigma_3 & \end{pmatrix} \right\}$$

satisfy

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_{4\times4}} \qquad \text{Clifford algebra in 4d}$$

Problem: Show it.

Comments:

(i) There are representations (bases) of γ matrices which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$.
$\left(\text{e.g., } \underline{\text{Dirac rep.}} \;\; \gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix}, \gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix}\right)$
The above rep. $\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}$ is called Weyl (chiral) rep..

(ii) We sometimes use a notation "Feynman slash":

$$\not{a} = \gamma^\mu a_\mu,$$

for a four vector a^μ. The Dirac Lagrangian is written as

$$\boxed{\mathcal{L} = \overline{\Psi}(i\not{\partial} - m)\Psi.}$$

(iii) A convenient identity

$$\begin{aligned} \not{a}\not{a} &= \gamma^\mu a_\mu \gamma^\nu a_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} a_\mu a_\nu \\ &= g^{\mu\nu} I \; a_\mu a_\nu = a^2 I \end{aligned}$$

- The Lorentz transformation of the 4-component Dirac field is given by

$$\begin{aligned} \Psi'(x') &= \begin{pmatrix} \psi'_L(x') \\ \psi'_R(x') \end{pmatrix} \\ &= \begin{pmatrix} D_L(\Lambda) & \\ & D_R(\Lambda) \end{pmatrix} \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \\ &\equiv \exp\left(\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right) \Psi(x) \end{aligned}$$

where, from $D_{L/R}(\Lambda)$ in § 2.1.4, the generators $S^{\mu\nu}$ are given by

$$\begin{cases} S^{0i} = \begin{pmatrix} \frac{i}{2}\sigma_i & \\ & -\frac{i}{2}\sigma_i \end{pmatrix} \\ S^{ij} = \begin{pmatrix} -\frac{1}{2}\epsilon_{ijk}\sigma_k & \\ & -\frac{1}{2}\epsilon_{ijk}\sigma_k \end{pmatrix} \end{cases} \qquad \text{(block-diagonal: reducible rep.)}$$

They can be written as

$$S^{\mu\nu} = \frac{-i}{4}[\gamma^\mu, \gamma^\nu]$$

and satisfy

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\mu\rho}S^{\nu\sigma} - g^{\nu\rho}S^{\mu\sigma} - g^{\mu\sigma}S^{\nu\rho} + g^{\nu\sigma}S^{\mu\rho}\right).$$

- Note that $\Psi^\dagger$ and $\overline{\Psi}$ transform as

$$\begin{aligned}
\Psi'^\dagger(x') &= \Psi^\dagger(x)\exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\dagger\mu\nu}\right). \\
\overline{\Psi}'(x') &= \Psi^\dagger(x)\exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\dagger\mu\nu}\right)\gamma^0 \\
&= \Psi^\dagger(x)\gamma^0\exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \qquad (\because S^{\dagger\mu\nu}\gamma^0 = \gamma^0 S^{\mu\nu}) \\
&= \overline{\Psi}(x)\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)
\end{aligned}$$

Thus, $\Psi^\dagger\Psi$ is not Lorentz invariant (note that $S^{\dagger\mu\nu} \neq S^{\mu\nu}$), while $\overline{\Psi}\Psi$ is Lorentz invariant.

§ 2.2.2 Dirac equation and its solution

- From the Lagrangian $\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi$, the EOM (Euler-Lagrange eq.) is

$$\begin{aligned}
0 &= \partial_\mu\left(\frac{\delta}{\delta(\partial_\mu\Psi_a^\dagger)}\mathcal{L}\right) - \frac{\delta}{\delta\Psi_a^\dagger}\mathcal{L} \\
&= 0 - \left[\gamma^0(i\partial\!\!\!/ - m)\right]_{ab}\Psi_b.
\end{aligned}$$

$$\therefore \boxed{(i\partial\!\!\!/ - m)\Psi(x) = 0 \qquad \textbf{Dirac equiation}}$$

Commnets

(i) The other Euler-Lagrange eq. $0 = \partial_\mu\left(\frac{\delta}{\delta(\partial_\mu\Psi_a)}\mathcal{L}\right) - \frac{\delta}{\delta\Psi_a}\mathcal{L}$ gives the same eq.

(ii) In terms of 2-component spinors, it is

$$\begin{pmatrix} -mI & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & -mI \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

(The mass term mixes left- and right-handed spinors. For massless fermion, ψ_L and ψ_R are different particles.)

- Let's solve it. First of all, if $\Psi(x)$ is a solution of Dirac eq., then it also satisfies the Klein-Gordon eq.

$$\begin{aligned} 0 &= (-i\not\partial - m)_{ab}(i\not\partial - m)_{bc}\Psi_c \\ &= (\underbrace{\not\partial\not\partial}_{=\partial^\mu\partial_\mu I} - m^2)_{ac}\Psi_c \\ &= (\Box + m^2)\Psi_a . \end{aligned}$$

- As in § 1.4.1, consider Fourier transform of $\Psi(x)$ with respect to $\vec{x}$,

$$\Psi_a(\vec{x}, t) = \int d^3p \; \widetilde{\Psi}_a(\vec{p}, t) e^{i\vec{p}\cdot\vec{x}}$$

From $(\Box + m^2)\Psi_a(x) = 0$,

$$\int d^3p \left(\ddot{\widetilde{\Psi}}_a(\vec{p}, t) + \widetilde{\Psi}_a(\vec{p}, t) \underbrace{(\vec{p}^2 + m^2)}_{E_p^2} \right) e^{i\vec{p}\cdot\vec{x}} = 0$$

$$\begin{aligned} &(\text{inverse FT}) \rightarrow \quad \ddot{\widetilde{\Psi}}_a(\vec{p}, t) + E_p^2 \widetilde{\Psi}_a(\vec{p}, t) = 0 \\ &\therefore\ \widetilde{\Psi}_a(\vec{p}, t) = u_a(\vec{p}) e^{-iE_pt} + w_a(\vec{p}) e^{+iE_pt}. \qquad (u_a(\vec{p}), w_a(\vec{p}) : \text{4-component spinor}) \\ &\therefore \Psi_a(\vec{x}, t) = \int d^3p \; \left(u_a(\vec{p}) e^{-iE_pt} + w_a(\vec{p}) e^{+iE_pt} \right) e^{i\vec{p}\cdot\vec{x}} \\ &\qquad = \int d^3p \left(u_a(\vec{p}) e^{-ip\cdot x} + \underbrace{w_a(-\vec{p})}_{\equiv v_a(\vec{p})} e^{+ip\cdot x} \right)_{p^0 = E_p} \qquad \text{———} \ (1) \end{aligned}$$

- Eq. (1) satisfies the necessary condition $(\Box + m^2)\Psi_a(x) = 0$, but not sufficient. From Dirac eq.

$$\begin{aligned} & 0 = (i\not\partial - m)_{ab}\Psi_b(x) \\ & \quad = \int d^3p \left((\not p - m)_{ab} u_b(\vec{p}) e^{-ip\cdot x} + (-\not p - m)_{ab} v_b(\vec{p}) e^{+ip\cdot x} \right)_{p^0 = E_p} . \\ (\text{inverse FT}) \rightarrow \quad & 0 = (\not p - m)_{ab} u_b(\vec{p}) e^{-iE_pt} + (-\gamma^0 p_0 - \gamma^i(-p_i) - m)_{ab} v_b(-\vec{p}) e^{+iE_pt} \end{aligned}$$

This should be satisfied for any t. Thus,

$$\begin{cases} (\not p - m)_{ab} u_b(\vec{p}) = 0 \\ (-\not p - m)_{ab} v_b(\vec{p}) = 0 \end{cases} \qquad (p^0 = E_p),$$

i.e., $u(\vec{p})$ and $v(\vec{p})$ are eigenvectors of $\not{p}$ with eigenvalues m and $-m$ respectively.

$$\begin{pmatrix} & & \\ & \not{p} & \\ & & \end{pmatrix}\begin{pmatrix} \\ u(\vec{p}) \\ \\ \end{pmatrix} = m \begin{pmatrix} \\ u(\vec{p}) \\ \\ \end{pmatrix},$$

$$\begin{pmatrix} & & \\ & \not{p} & \\ & & \end{pmatrix}\begin{pmatrix} \\ v(\vec{p}) \\ \\ \end{pmatrix} = -m \begin{pmatrix} \\ v(\vec{p}) \\ \\ \end{pmatrix}. \qquad \text{———— (2)}$$

- In fact, the eigenvalues of the matrix $\not{p}$ are

$$\det(\not{p} - xI) = \cdots = (x^2 - m^2)^2$$
$$\to x = m, m, -m, -m,$$

corresponding to two independent $u(\vec{p})$ and two independent $v(\vec{p})$, satisfying (2).

- We can also think the "helicity" (= projection of the spin onto the direction of momentum):

$$h(p) = \frac{1}{2|\vec{p}|}\begin{pmatrix} \vec{p}\cdot\vec{\sigma} & \\ & \vec{p}\cdot\vec{\sigma} \end{pmatrix}.$$

whose eigenvalues are ±1/2. Since $[\not{p}, h(p)] = 0$, simultaneous eigenvectors of $\not{p}$ and $h(p)$ can be taken:

	$u_+(p)$	$u_-(p)$	$v_+(p)$	$v_-(p)$
$\not{p}$	m	m	$-m$	$-m$
$h(p)$	$1/2$	$-1/2$	$-1/2$	$1/2$

e.g., $\begin{cases} \not{p}u_+(p) = mu_+(p) \\ h(p)u_+(p) = \frac{1}{2}u_+(p) \end{cases}$

- The explicit form of $u(\vec{p})$ and $v(\vec{p})$ can be written as

$$u_\pm(p) = \begin{pmatrix} \sqrt{p^0 \mp |\vec{p}|}\,\eta_\pm \\ \sqrt{p^0 \pm |\vec{p}|}\,\eta_\pm \end{pmatrix}, \quad v_\pm(p) = \begin{pmatrix} \sqrt{p^0 \pm |\vec{p}|}\,\epsilon\eta_\pm^* \\ -\sqrt{p^0 \mp |\vec{p}|}\,\epsilon\eta_\pm^* \end{pmatrix},$$

with

$$\begin{cases} \eta_+ = \dfrac{1}{\sqrt{2(1-\hat{p}^3)}}\begin{pmatrix} \hat{p}^1 - i\hat{p}^2 \\ 1 - \hat{p}^3 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix} \\ \eta_- = \dfrac{1}{\sqrt{2(1-\hat{p}^3)}}\begin{pmatrix} 1 - \hat{p}^3 \\ -\hat{p}^1 - i\hat{p}^2 \end{pmatrix} = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{+i\phi} \end{pmatrix} \end{cases} \qquad \begin{cases} \hat{p}^1 = p^1/|\vec{p}| = \sin\theta\cos\phi \\ \hat{p}^2 = p^2/|\vec{p}| = \sin\theta\sin\phi \\ \hat{p}^3 = p^3/|\vec{p}| = \cos\theta \end{cases}$$

satisfying $(\vec{p}\cdot\vec{\sigma})\eta_\pm = \pm|\vec{p}|\eta_\pm, \quad \eta_s^\dagger\eta_{s'} = \delta_{ss'}$

They are normalized as

$$\begin{cases} \bar{u}_s(p)u_{s'}(p) = 2m\delta_{ss'} \\ \bar{v}_s(p)v_{s'}(p) = -2m\delta_{ss'} \\ \bar{u}_s(p)v_{s'}(p) = 0 \end{cases},$$

and satisfy

$$\sum_{s=\pm} u_s(p)\bar{u}_s(p) = \not{p} + m$$
$$\sum_{s=\pm} v_s(p)\bar{v}_s(p) = \not{p} - m$$

► To summarize, there are four independent solutions to the Dirac equation,

$$\Psi(x) = \int d^3p \left(u_+(p)e^{-ip\cdot x},\ u_-(p)e^{-ip\cdot x},\ v_+(p)e^{+ip\cdot x},\ v_-(p)e^{+ip\cdot x}\right)_{p^0=E_p}.$$

§ 2.2.3 Quantization of Dirac field

$$\mathcal{L} = \overline{\Psi}(i\not{\partial} - m)\Psi.$$
$$\Psi_a \overset{\text{conjugate}}{\longleftrightarrow} \Pi_{\Psi a} = \frac{\delta\mathcal{L}}{\delta\dot{\Psi}_a} = (\overline{\Psi}i\gamma^0)_a = i\Psi_a^\dagger\ (= i\Psi_a^*)$$

Comments

(i) $\Psi_a \longleftrightarrow \Pi_{\Psi a} = i\Psi_a^*$

but then, $\Psi_a^* \longleftrightarrow$? $\quad (\Pi_{\Psi^* a} = \dfrac{\delta\mathcal{L}}{\delta\dot{\Psi}_a^*} = 0\ ???)$

One should do the quantization of constrained sysytem with "Dirac bracket".

In general, if $\det\left(\dfrac{\partial^2\mathcal{L}(q,\dot{q})}{\partial\dot{q}_i\partial\dot{q}_j}\right) = 0$, p_i and q_i are not independent.

In such a case,

$$\begin{tikzcd}
\text{Poisson bracket} \arrow[rr, "\times"] \arrow[dr] & & \text{quantization} \\
& \text{Dirac bracket} \arrow[ur] &
\end{tikzcd}$$

Here, we skip the details and do naive quantization with Ψ_a and $\Pi_{\Psi a}$.

(ii) When Ψ_a and $\Pi_{\Psi a}$ are anti-commuting, right-derivative and left-derivative gives opposite sign. Here, $\Pi_{\Psi a}$ is defined with right-derivative.

(If A and B are anti-commuting, $(BA)\dfrac{\overleftarrow{\partial}}{\partial A} = B$, while $\dfrac{\overrightarrow{\partial}}{\partial A}(BA) = -B$.

Quantization with Commutation relation vs Anti-commutation relation

$$\Psi_a \longleftrightarrow \Pi_{\Psi a} = i\Psi_a^\dagger$$

Quantization with equal-time commutation relation

$$[\Psi_a(x), \Pi_{\Psi b}(y)]_{x^0=y^0} = [\Psi_a(x), i\Psi_b^\dagger(y)]_{x^0=y^0} = i\delta^{(3)}(\vec{x}-\vec{y})\delta_{ab}$$

does NOT work. Instead, quantization with equal-time anti-commutation relation

$$\{\Psi_a(x), \Pi_{\Psi b}(y)\}_{x^0=y^0} = \{\Psi_a(x), i\Psi_b^\dagger(y)\}_{x^0=y^0} = i\delta^{(3)}(\vec{x}-\vec{y})\delta_{ab}$$

works. Let's see it.

- First of all, expand the $\Psi(x)$ with the solutions of Dirac eq.

$$u_\pm(p)e^{-ip\cdot x}, \quad v_\pm(p)e^{+ip\cdot x}$$

as

$$\Psi_a(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=\pm} \left(a_s(\vec{p})u_{s,a}(p)e^{-ip\cdot x} + d_s(\vec{p})v_{s,a}(p)e^{+ip\cdot x}\right)_{p^0=E_p}.$$

$$\Psi_a^\dagger(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=\pm} \left(a_s^\dagger(\vec{p})u_{s,a}^\dagger(p)e^{+ip\cdot x} + d_s^\dagger(\vec{p})v_{s,a}^\dagger(p)e^{-ip\cdot x}\right)_{p^0=E_p}.$$

Here, $\Psi(x)$, $a_s(\vec{p})$, and $d_s(\vec{p})$ are the quantum operators. At this moment $a_s(\vec{p})$ and $d_s(\vec{p})$ are just expansion coefficients.

- The following can be shown:

$$\begin{cases} [\Psi_a(x), \Psi_b^\dagger(y)]_{x^0=y^0} & = \delta^{(3)}(\vec{x}-\vec{y})\delta_{ab} \\ \text{others} & = 0 \end{cases} \qquad \text{——— (1)}$$

$$\Updownarrow$$

$$\begin{cases} [a_r(\vec{p}), a_s^\dagger(\vec{q})] & = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs} \\ [d_r(\vec{p}), d_s^\dagger(\vec{q})] & = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs} \\ \text{others} & = 0 \end{cases} \qquad \text{——— (1)'} \dashrightarrow \text{(problematic)}$$

$$\begin{cases} \{\Psi_a(x), \Psi_b^\dagger(y)\}_{x^0=y^0} & = \delta^{(3)}(\vec{x}-\vec{y})\delta_{ab} \\ \text{others} & = 0 \end{cases} \qquad \text{——— (2)}$$

$$\Updownarrow$$

$$\begin{cases} \{a_r(\vec{p}), a_s^\dagger(\vec{q})\} & = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs} \\ \{d_r(\vec{p}), d_s^\dagger(\vec{q})\} & = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs} \\ \text{others} & = 0 \end{cases} \qquad \text{——— (2)'} \dashrightarrow \text{(OK)}$$

——————— on July 10, up to here. ———————

(In order to finish the quantization of Dirac field, I will have an extra class (補講) on July 31. It's after the exam and the reports, and irrelevant to the grades. . .)

——————— on July 31, from here. ———————

- Let's first show (1)' $\Longrightarrow$ (1) and (2)' $\Longrightarrow$ (2). Hereafter, we use a notation

$$[A, B\} = \begin{cases} [A, B] = AB - BA \\ \{A, B\} = AB + BA \end{cases}$$

to discuss the two cases simultaneously.

- First of all, from (1)' (2)', $[a, a\} = [d, d\} = [a, d\} = 0$, and therefore $[\Psi, \Psi\} = 0$. Similarly, $[\Psi^\dagger, \Psi^\dagger\} = 0$.

- The remaining is $[\Psi, \Psi^\dagger\}$, and

$$\begin{aligned}
[\Psi_a(t, \vec{x}), \Psi_b^\dagger(t, \vec{y})\} &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}} \\
&\times \sum_{r=\pm}\sum_{s=\pm} \Big([a_r(\vec{p}), a_s^\dagger(\vec{q})\}\; u_{a,r}(p)u_{b,s}^\dagger(q)e^{-ip\cdot x}e^{iq\cdot y} \\
&\qquad + [d_r(\vec{p}), d_s^\dagger(\vec{q})\}\; v_{a,r}(p)v_{b,s}^\dagger(q)e^{ip\cdot x}e^{-iq\cdot y} \Big)_{x^0=y^0=t} \\
&\qquad \left(\text{using (1)', and performing } \int d^3p \text{ and } \sum_{r=\pm} \right) \\
&= \int \frac{d^3q}{(2\pi)^3}\frac{1}{2E_q} \sum_{s=\pm} \left(u_{a,s}(q)u_{b,s}^\dagger(q)e^{i\vec{q}\cdot(\vec{x}-\vec{y})} + v_{a,s}(q)v_{b,s}^\dagger(q)e^{-i\vec{q}\cdot(\vec{x}-\vec{y})} \right) \\
&\qquad \left(\text{from § 2.2.2, } \begin{matrix} \sum_s u_{a,s}(q)u_{b,s}^\dagger(q) = [(\not{q} + m)\gamma^0]_{ab} \\ \sum_s v_{a,s}(q)v_{b,s}^\dagger(q) = [(\not{q} - m)\gamma^0]_{ab} \end{matrix} \right) \\
&= \int \frac{d^3q}{(2\pi)^3}\frac{1}{2E_q} \left(e^{i\vec{q}\cdot(\vec{x}-\vec{y})}[(\not{q} + m)\gamma^0]_{ab} + e^{-i\vec{q}\cdot(\vec{x}-\vec{y})}[(\not{q} - m)\gamma^0]_{ab} \right) \\
&= \int \frac{d^3q}{(2\pi)^3}\frac{1}{2E_q} e^{i\vec{q}\cdot(\vec{x}-\vec{y})}[(\gamma^0q_0 - \gamma^iq_i + m + \gamma^0q_0 - \gamma^i(-q_i) - m)\gamma^0]_{ab} \\
&= \int \frac{d^3q}{(2\pi)^3}\frac{1}{2E_q} e^{i\vec{q}\cdot(\vec{x}-\vec{y})}\; 2q^0 \cdot \delta_{ab} \\
&= \delta^{(3)}(\vec{x} - \vec{y})\delta_{ab} \qquad \blacksquare
\end{aligned}$$

- Thus, (1)' $\Longrightarrow$ (1) and (2)' $\Longrightarrow$ (2) are shown.

- The other way round, (1) $\Longrightarrow$ (1)' and (2) $\Longrightarrow$ (2)' can be shown, by the following steps:

(i) By using the explicit forms of $u(p)$ and $v(p)$ in § 2.2.2, show that

$$\begin{cases} u_r^\dagger(\vec{p})u_s(\vec{p}) = 2E_p\delta_{rs} \\ v_r^\dagger(\vec{p})v_s(\vec{p}) = 2E_p\delta_{rs} \\ u_r^\dagger(\vec{p})v_s(-\vec{p}) = 0 \\ v_r^\dagger(\vec{p})u_s(-\vec{p}) = 0 \end{cases} \qquad \text{———————} \quad (3)$$

(ii) Using (3), show that

$$\begin{cases} a_s(\vec{p}) = \dfrac{1}{\sqrt{2E_p}} \displaystyle\sum_a u^\dagger_{s,a}(\vec{p}) \int d^3x e^{ip\cdot x}\Psi_a(x)\Bigg|_{p^0=E_p} \\ d_s(\vec{p}) = \dfrac{1}{\sqrt{2E_p}} \displaystyle\sum_a v^\dagger_{s,a}(\vec{p}) \int d^3x e^{-ip\cdot x}\Psi_a(x)\Bigg|_{p^0=E_p} \end{cases} \qquad \text{———————} \quad (4).$$

(One can also show that the RHS is independent of x^0.)

(iii) Using (3) (4), show (1) $\Longrightarrow$ (1)' and (2) $\Longrightarrow$ (2)'.

- ► So far we have shown (1) $\Longleftrightarrow$ (1)' and (2) $\Longleftrightarrow$ (2)'.
- ► On the other hand, the Hamiltonian is given by

$$H = \int d^3x \left(\Pi_\Psi \dot{\Psi} - \mathcal{L}\right)$$

(note that Π_Ψ is defined with right-derivative, and hence $H\dfrac{\overleftarrow{\partial}}{\partial\dot{\Psi}} = 0$.)

$$= \int d^3x \Big(i\Psi^\dagger\dot{\Psi} - \overline{\Psi}\underbrace{(i\partial\!\!\!/ - m)\Psi}_{=0}\Big)$$

$$= \int d^3x \; i\Psi^\dagger\dot{\Psi}$$

$$= \cdots \text{(using (3))}$$

$$= \underline{\int \frac{d^3p}{(2\pi)^3} E_p \sum_{s=\pm}\left(a_s^\dagger(\vec{p})a_s(\vec{p}) - d_s^\dagger(\vec{p})d_s(\vec{p})\right)} \qquad \text{———————} \quad (5).$$

Note that

(i) this is the case both for quantization with $[\bullet,\bullet]$, (1)$\Longleftrightarrow$(1)' and that with $\{\bullet,\bullet\}$, (2)$\Longleftrightarrow$(2)',

(ii) and there is a minus sign in front of $d^\dagger d$.

- ► From (5) and (1)'(2)', one can show:

$$\begin{cases} [H, a_s^\dagger(\vec{p})] = E_p a_s^\dagger(\vec{p}) \\ [H, a_s(\vec{p})] = -E_p a_s(\vec{p}) \\ [H, d_s^\dagger(\vec{p})] = -E_p d_s^\dagger(\vec{p}) \\ [H, d_s(\vec{p})] = E_p d_s(\vec{p}) \end{cases}$$

for both of the cases with $[\bullet, \bullet]$, (1)$\Longleftrightarrow$(1)' and $\{\bullet, \bullet\}$, (2)$\Longleftrightarrow$(2)'.

- ► Now, if we would quantize with $[\bullet, \bullet]$, (1)$\Longleftrightarrow$(1)', then $\underline{d_s^\dagger(\vec{p})\text{ would decrease energy.}}$

$$\begin{aligned} H\Big(d_s^\dagger(\vec{p})\,|X\rangle\Big) &= \Big(d_s^\dagger(\vec{p})H + [H, d_s^\dagger(\vec{p})]\Big)\,|X\rangle \\ &= (E_X - E_p)\Big(d_s^\dagger(\vec{p})\,|X\rangle\Big), \end{aligned}$$

and one could construct a state with infinitely negative energy.

$$H\Big(d_1^\dagger d_2^\dagger \cdots |X\rangle\Big) = (\underbrace{E_X - E_1 - E_2 - \cdots}_{\to -\infty})\Big(d_1^\dagger d_2^\dagger \cdots |X\rangle\Big).$$

Note that, $\underline{\text{we cannot change the roles of } d \text{ and } d^\dagger}$, because

$$\big[d(\vec{p}), d^\dagger(\vec{q})\big] = (2\pi)^3\delta^{(3)}(\vec{p} - \vec{q}),$$

fixes that $d^\dagger$ (d) is the creation (annihilation):

$$\begin{aligned} & d(\vec{p})d^\dagger(\vec{p}) - d^\dagger(\vec{p})d(\vec{p}) = (2\pi)^3\delta^{(3)}(0), \\ \therefore\ & \Big|\Big|d^\dagger(\vec{p})\,|X\rangle\Big|\Big|^2 - \Big|\Big|d(\vec{p})\,|X\rangle\Big|\Big|^2 = (2\pi)^3\delta^{(3)}(0)\langle X|X\rangle \geq 0. \end{aligned}$$

(If we would define $\widetilde{d} = d^\dagger$, $\widetilde{d}^\dagger = d$, and define the vacuum by $\widetilde{d}\,|0\rangle = 0$, then $-||\widetilde{d}^\dagger(\vec{p})\,|X\rangle\,||^2 \geq 0$, inconsistent!)

- ► On the other hand, if we quantize with $\{\bullet, \bullet\}$, (2)$\Longleftrightarrow$(2)', we still have

$$\begin{cases} [H, d_s^\dagger(\vec{p})] = -E_p d_s^\dagger(\vec{p}) \\ [H, d_s(\vec{p})] = E_p d_s(\vec{p}) \end{cases}$$

but now we can $\underline{\text{exchange the roles of creation and annihilation operator.}}$

$$\begin{aligned} b^\dagger(\vec{p}) &\equiv d(\vec{p}) \\ b(\vec{p}) &\equiv d^\dagger(\vec{p}) \end{aligned}$$

because

$$\begin{aligned} &\left\{d(\vec{p}), d^\dagger(\vec{q})\right\} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\\ &= dd^\dagger + d^\dagger d\\ &= b^\dagger b + bb^\dagger\\ &= \left\{b(\vec{q}), b^\dagger(\vec{p})\right\}\end{aligned}$$

and also we can define the vacuum state by $b\left|0\right\rangle = 0$.

Comment

$b(\vec{p})\left|0\right\rangle = 0$ means that, in terms of original d and $d^\dagger$, $d^\dagger(\vec{p})\left|0\right\rangle = 0$.
In terms of the original "vacuum" $\left|0_d\right\rangle$ with $d(\vec{p})\left|0_d\right\rangle = 0$, the vacuum $\left|0\right\rangle$ can be understood as

$$\left|0\right\rangle \propto \prod_{\text{all } \vec{p}} d^\dagger(\vec{p})\left|0_d\right\rangle ,$$

which leads to $d^\dagger(\vec{p})\left|0\right\rangle = 0$ because $d^\dagger(\vec{p})^2 = 0$. This is related to the idea of the "Dirac sea".

- The Hamiltonian then becomes

$$\begin{aligned} H &= \int \frac{d^3p}{(2\pi)^3} E_p \sum_{s=\pm}\Big(a_s^\dagger(\vec{p})a_s(\vec{p}) \underbrace{-d_s^\dagger(\vec{p})d_s(\vec{p})}_{\substack{= -b_s(\vec{p})b_s^\dagger(\vec{p}) \\ = +b_s^\dagger(\vec{p})b_s(\vec{p}) - (2\pi)^3\delta^{(3)}(0)}}\Big)\\ &= \int \frac{d^3p}{(2\pi)^3} E_p \sum_{s=\pm}\Big(a_s^\dagger(\vec{p})a_s(\vec{p}) + b_s^\dagger(\vec{p})b_s(\vec{p})\Big) - \int d^3p\, E_p\delta^{(3)}(0)\end{aligned}$$

We neglect the (infinite) constant term, as in the scalar case.

- To summarize, quantization with anti-commutation works, and we have

$$\Psi_a(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=\pm} \left(a_s(\vec{p})u_{s,a}(p)e^{-ip\cdot x} + b_s^\dagger(\vec{p})v_{s,a}(p)e^{+ip\cdot x}\right)_{p^0=E_p}$$

$$\begin{cases} \{\Psi_a(x), \Psi_b^\dagger(y)\}_{x^0=y^0} & = \delta^{(3)}(\vec{x}-\vec{y})\delta_{ab} \\ \text{others} & = 0 \end{cases} \Longleftrightarrow \begin{cases} \{a_r(\vec{p}), a_s^\dagger(\vec{q})\} & = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs} \\ \{b_r(\vec{p}), b_s^\dagger(\vec{q})\} & = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs} \\ \text{others} & = 0 \end{cases}$$

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_{s=\pm}\Big(a_s^\dagger(\vec{p})a_s(\vec{p}) + b_s^\dagger(\vec{p})b_s(\vec{p})\Big)$$

The anti-commutation relation implies Fermi-Dirac statistics;

$$a_r^\dagger(\vec{p})a_s^\dagger(\vec{q}) = -a_s^\dagger(\vec{q})a_r^\dagger(\vec{p}), \quad \text{in particular} \quad \left(a_r^\dagger(\vec{p})\right)^2 = 0 \quad \boxed{\text{Pauli blocking}}$$

▶ Particle and Anti-particle

$$\begin{aligned} \text{U(1) Symmetry: } &\Psi \to \Psi e^{i\alpha} \\ \to \text{Current: } &j^\mu = \overline{\Psi}\gamma^\mu\Psi \\ \to \text{Charge: } &Q = \int d^3x j^0 \\ &= \cdots \\ &= \int \frac{d^3p}{(2\pi)^3}\sum_s \left(a_s^\dagger(\vec{p})a(\vec{p}) - b_s^\dagger(\vec{p})b(\vec{p})\right) \quad (+\text{constant}) \end{aligned}$$

and hence

$$\begin{cases} [Q, a_s^\dagger(\vec{p})] = +a_s^\dagger(\vec{p}) \\ [Q, b_s^\dagger(\vec{p})] = -b_s^\dagger(\vec{p}) \end{cases}$$

Namely,

- $a_s^\dagger(\vec{p})$ increases the charge by one. (Particle creation)
- $b_s^\dagger(\vec{p})$ decreases the charge by one. (Anti-particle creation)

▶ One particle state

$$\begin{aligned} |\psi;\vec{p},r\rangle &= \sqrt{2E_p}a_r^\dagger(\vec{p})\,|0\rangle \ : \text{particle} \\ |\bar{\psi};\vec{p},r\rangle &= \sqrt{2E_p}b_r^\dagger(\vec{p})\,|0\rangle \ : \text{anti-particle} \end{aligned}$$

Normalization:

$$\begin{aligned} \langle\psi;\vec{p},r|\psi;\vec{q},s\rangle &= \sqrt{2E_p}\sqrt{2E_q}\langle 0|a_r(\vec{p})a_s^\dagger(\vec{q})\rangle 0 \\ &= \sqrt{2E_p}\sqrt{2E_q}\langle 0|\ \underbrace{\{a_r(\vec{p})a_s^\dagger(\vec{q})\}}_{(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs}} -a_s^\dagger(\vec{p})a_r(\vec{p})\rangle 0 \\ &= (2\pi)^3 2E_p\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs}. \\ \text{Similarly } \langle\bar{\psi};\vec{p},r|\bar{\psi};\vec{q},s\rangle &= (2\pi)^3 2E_p\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs}. \end{aligned}$$

Lorentz transformation:
(check it by yourself. . .)

▶ (That's all for this semester. Thank you for your attendance!)

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