Some general aspects of Majorana fermions (in particular, neutrinos)

Kazuo Fujikawa

Interdisciplinary Theoretical and Mathematical Sciences Program
RIKEN, Wako 351-0198, Japan

1. Majorana fermion and parity
2. Charge conjugation, $\psi_M(x) = \nu_L(x) + C\overline{\nu_L}^T(x)$
   Pseudo-C $\Rightarrow$ No neutrinoless double-beta decay
3. CP to define Majorana
4. Seesaw and generalized Pauli-Gursey transformation
   (Naturalness of seesaw mechanism)
5. Parity of "emergent" Majorana neutrino in the extension of SM

1. Majorana fermion and parity

Majorana fermion is defined by the Dirac action but with *purely imaginary* Dirac gamma matrices γ^μ.

Then the Dirac equation

$$[i\gamma^\mu\partial_\mu - m]\psi(x) = 0$$

is a real differential equation, and one can impose the reality condition on the solution

$$\psi(x)^\star = \psi(x)$$

which implies the self-conjugate under the charge conjugation

The conventional parity transformation

$$\psi(x) \to \psi^p(t, -\vec{x}) = \gamma^0\psi(t, -\vec{x})$$

with eigenvlues ± 1 cannot maintain the reality condition for the purely imaginary γ^0. Thus the "$i\gamma^0$-parity"

$$\psi(x) \to \psi^p(t, -\vec{x}) = i\gamma^0\psi(t, -\vec{x})$$

is chosen as a natural parity transformation rule for the Majorana fermion.

E. Majorana(1937), B. Kayser(1982).

In the generic representation of the Dirac matrices, the "$i\gamma^0$-parity" satisfies the condition with $C = i\gamma^2\gamma^0$

$$i\gamma^0\psi(t,-\vec{x}) = C\overline{i\gamma^0\psi(t,-\vec{x})}^T$$

for the field which satisfies the *classical* Majorana condition

$$\psi(x) = C\overline{\psi(x)}^T$$

and thus $i\gamma^0$-parity is a natural choice of the parity for the Majorana fermion in this generic representation.

For consistency, we assign the $i\gamma^0$-parity convention to charged leptons also when we assign $i\gamma^0$-parity to neutrinos, although this extra phase is cancelled in the lepton number conserving terms in the charged lepton sector.

2. Charge conjugation C

In the extension of the SM, we often encounter the Majorana neutrino of the form

$$\psi(x) = \nu_L(x) + C\overline{\nu_L}^T(x)$$

This field satisfies the classical Majorana condition

$$\psi(x) = C\overline{\psi}^T(x)$$

and in this operation, we see

$$\nu_L(x) \to C\overline{\nu_L}^T(x), \quad C\overline{\nu_L}^T(x) \to \nu_L(x)$$

It is thus tempting to identify the charge conjugation of $\nu_L(x)$, which is denoted as $\nu_L^c(x)$, by

$$\nu_L^c(x) = C\overline{\nu_L}^T(x)$$

We named this as "Pseudo-C".

KF and A. Tureanu, Eur. Phys. J. C79 (2019) 752.

We note that this pseudo-C differs from the conventional charge conjugation

$$\nu_L^c(x) = C\overline{\nu_R}^T(x)$$

which is related to the absence of the Majorana-Weyl fermion in $d = 4$.

In $d = 4$, charge conjugation changes the chiral operator $\gamma_5 \to -\gamma_5$.

This pseudo-C has several difficulties:

First of all, it is operatorially inconsistent. Suppose that we have an operator which generates the pseudo-C

$$\tilde{\mathcal{C}}\nu_L(x)\tilde{\mathcal{C}}^\dagger = C\overline{\nu_L}^T(x)$$

then we have, by noting $\nu_L(x) = (\frac{1-\gamma_5}{2})\nu_L(x)$,

$$\tilde{\mathcal{C}}\nu_L(x)\tilde{\mathcal{C}}^\dagger = (\frac{1-\gamma_5}{2})\tilde{\mathcal{C}}\nu_L(x)\tilde{\mathcal{C}}^\dagger = (\frac{1-\gamma_5}{2})C\overline{\nu_L}^T(x) = 0$$

since $C\overline{\nu_L}^T(x)$ is right-handed.

Secondly, it would keep the Weyl fermion

$$\int d^4x\mathcal{L} = \int d^4x\overline{\nu_L(x)}i\gamma^\mu\partial_\mu\nu_L(x)$$

formally invariant under the charge conjugation. We know that C is not defined for the Weyl neutrino.

Thirdly, we have a difficulty to define a sensible CP for the pseudo-C.
We know that the parity (mirror reflection) should be of the form

$$\nu_L(t,\vec{x}) \to i\gamma^0\nu_R(t,-\vec{x})$$

We also know that CP should be of the form

$$\nu_L(t,\vec{x}) \to i\gamma^0C\overline{\nu_L}^T(t,-\vec{x})$$

to keep the Weyl neutrino invariant. But it is impossible to define this CP using the pseudo-C.

Furthermore, no neutrinoless double beta decay if the vacuum is invariant under the pseudo-C.

KF. Eur. Phys. J. C80 (2020) 285.

$$\begin{aligned}\int d^4x\mathcal{L}_{\text{Weak}} &= \int d^4x[(g/\sqrt{2})\bar{l}_L(x)\gamma^\mu W_\mu(x)U_{PMNS}\nu_L(x) + h.c.] \\ &= \int d^4x[(g/\sqrt{2})\bar{l}_L(x)\gamma^\mu W_\mu(x)U_{PMNS}\frac{(1-\gamma_5)}{2}\psi_M(x) + h.c.]\end{aligned}$$

in the case of Weinberg's model $\psi_M(x) = \nu_L(x) + C\overline{\nu_L}^T(x)$.

A necessary condition of the neutrinoless double beta decay is that not all the time-ordered correlations of the neutrino mass eigenstates

$$\langle 0|T^\star \nu_L(x)\nu_L(y)|0\rangle = \langle 0|T^\star \frac{(1-\gamma_5)}{2}\psi_M(x)\frac{(1-\gamma_5)}{2}\psi_M(y)|0\rangle$$

vanish in the second order perturbation in $\mathcal{L}_{\text{Weak}}$.

If a unitary operator $\tilde{\mathcal{C}}$ which generates the pseudo-C exists and if the (neutrino) vacuum $|0\rangle$ should be invariant

$$\tilde{\mathcal{C}}^\dagger|0\rangle = |0\rangle$$

and thus $\langle 0|\tilde{\mathcal{C}} = \langle 0|$, one can prove that all of the above correlations vanish

$$\begin{aligned}\langle 0|T^\star \nu_L(x)\nu_L(y)|0\rangle &= \langle 0|T^\star[(\frac{1-\gamma_5}{2})\nu_L](x)\nu_L(y)|0\rangle \\ &= \langle 0|\tilde{\mathcal{C}}T^\star[(\frac{1-\gamma_5}{2})\nu_L](x)\nu_L(y)\tilde{\mathcal{C}}^\dagger|0\rangle \\ &= \langle 0|T^\star[(\frac{1-\gamma_5}{2})C\overline{\nu_L}^T](x)[C\overline{\nu_L}^T](y)|0\rangle \\ &= 0\end{aligned}$$

where we used $\nu_L(x) = (\frac{1-\gamma_5}{2})\nu_L(x)$ and $\tilde{\mathcal{C}}\nu_L(x)\tilde{\mathcal{C}}^\dagger = C\overline{\nu_L(x)}^T$ and the fact that $C\overline{\nu_L}^T(x)$ is right-handed.

How to define Majorana $\psi_M(x) = \nu_L(x) + C\overline{\nu_L}^T(x)$?

KF. Eur. Phys. J. C80 (2020) 285.

1. The use of CP to define the Majorana

We know CP for the chiral fermion

$$\nu_L(x) \to i\gamma^0 C\overline{\nu_L}^T(t, -\vec{x})$$

Then

$$\psi_M(x) \to i\gamma^0 C\overline{\psi_M}^T(t, -\vec{x}) = i\gamma^0 \psi_M(t, -\vec{x})$$

for the fermion which satisfies the classical Majorana condition $C\overline{\psi_M}^T(x) = \psi_M(x)$ We thus define the Majorana by CP symmetry

$$\psi_M(x) \to \psi_M^{cp}(t, -\vec{x}) = i\gamma^0 \psi_M(t, -\vec{x})$$

We do not assign C to the Majorana $\psi_M(x)$, since $\psi_M(x) \to C\overline{\nu_R}^T(x) + \nu_R(x)$ under C.

If one wishes to have C and P for the Majorana neutrino $\psi_M(x)$, one may define **modified symmetry**

$$\mathcal{C}_M = 1, \quad \mathcal{P}_M = CP$$

Then

$$\mathcal{C}_M \psi_M(x) \mathcal{C}_M^\dagger = \psi_M(x), \quad \mathcal{P}_M \psi_M(x) \mathcal{P}_M^\dagger = i\gamma^0 \psi_M(t, -\vec{x})$$

For seesaw model, the use of generalized Pauli-Gursey

KF. Phys.Lett. B789 (2019) 76

We start with a model Lagrangian

$$\begin{aligned}\mathcal{L} &= \overline{\nu}_L(x)i\gamma^\mu\partial_\mu\nu_L(x) + \overline{\nu}_R(x)i\gamma^\mu\partial_\mu\nu_R(x)\\ &- \overline{\nu}_L(x)m_D\nu_R(x) - (1/2)\nu_L^T(x)Cm_L\nu_L(x)\\ &- (1/2)\nu_R^T(x)Cm_R\nu_R(x) + h.c.\end{aligned} \tag{1}$$

We write the mass term as

$$(-2)\mathcal{L}_{mass} = \begin{pmatrix} \overline{\nu_R} & \overline{\nu_R^C} \end{pmatrix} \begin{pmatrix} m_R & m_D \\ m_D^T & m_L \end{pmatrix} \begin{pmatrix} \nu_L^C \\ \nu_L \end{pmatrix} + h.c., \tag{2}$$

where we defined

$$\nu_L^C \equiv C\overline{\nu_R}^T, \quad \nu_R^C \equiv C\overline{\nu_L}^T \tag{3}$$

We diagonalize the *complex symmetric* mass matrix using a 6×6 unitary matrix (**Autonne-Takagi factorization**)

$$U^T \begin{pmatrix} m_R & m_D \\ m_D^T & m_L \end{pmatrix} U = \begin{pmatrix} M_1 & 0 \\ 0 & -M_2 \end{pmatrix}, \tag{4}$$

where M_1 and M_2 are 3×3 real diagonal matrices.

In passing, we comment on **Naturalness of seesaw**:

KF and A. Tureanu, Phys. Lett. B767 (2017) 199.

We denote that

$$M_1, M_2 = \sqrt{(m_R/2)^2 + m_D^2} \pm m_R/2 \simeq m_R \text{ or } m_D^2/m_R$$

for the special case of a single generation with real m_D, m_R, and $m_L = 0$.

Hierarchy issue has been analyzed using the dimensional regularization which is free of quadratic divergences (hierarchy and quadratic divergence are inependent notions.)

In this interpretation, for the typical $m_R = 10^4 \sim 10^{15}$ GeV, a hitherto unrecognized fine tuning of the order $m_\nu/m_R = 10^{-15} \sim 10^{-26}$.

If SUSY is discovered at some energy below GUT scale, this naive estimate is modified.

“Naturalness” is a subjective concept.

Coming back to seesaw, we thus have

$$(-2)\mathcal{L}_{mass} = \begin{pmatrix} \overline{\tilde{\nu}_R} & \overline{\tilde{\nu}_R^C} \end{pmatrix} \begin{pmatrix} M_1 & 0 \\ 0 & -M_2 \end{pmatrix} \begin{pmatrix} \tilde{\nu}_L^C \\ \tilde{\nu}_L \end{pmatrix} + h.c., \tag{5}$$

where we defined

$$\begin{pmatrix} \nu_L^C \\ \nu_L \end{pmatrix} = U \begin{pmatrix} \tilde{\nu}_L^C \\ \tilde{\nu}_L \end{pmatrix}, \qquad \begin{pmatrix} \nu_R \\ \nu_R^C \end{pmatrix} = U^\star \begin{pmatrix} \tilde{\nu}_R \\ \tilde{\nu}_R^C \end{pmatrix}. \tag{6}$$

We then have

$$\begin{aligned} \mathcal{L} &= \overline{\tilde{\nu}_L}(x) i \not\partial \tilde{\nu}_L(x) + \overline{\tilde{\nu}_R}(x) i \not\partial \tilde{\nu}_R(x) \\ &- (1/2)\{\tilde{\nu}_R^T C M_1 \tilde{\nu}_R - \tilde{\nu}_L^T C M_2 \tilde{\nu}_L\} + h.c.. \end{aligned} \tag{7}$$

Conventionally, one defines

$$\begin{aligned} \psi_+(x) &= \tilde{\nu}_R + \tilde{\nu}_L^C = \tilde{\nu}_R + C\overline{\tilde{\nu}_R}^T, \\ \psi_-(x) &= \tilde{\nu}_L - \tilde{\nu}_R^C = \tilde{\nu}_L - C\overline{\tilde{\nu}_L}^T \end{aligned}$$

and one obtains

$$\begin{aligned} \mathcal{L} &= (1/2)\{\overline{\psi_+}(x) i \not\partial \psi_+(x) + \overline{\psi_-}(x) i \not\partial \psi_-(x)\} \\ &- (1/2)\{\overline{\psi_+} M_1 \psi_+ + \overline{\psi_-} M_2 \psi_-\} \end{aligned}$$

One may use CP to define the Majorana $\psi_\pm(x)$, as we discussed.

Alternatively, generalized Pauli-Gursey transformation: canonical transformation. *Arbitrary* 6×6 unitary U and generic transformation

$$\begin{pmatrix} \nu_L^C \\ \nu_L \end{pmatrix} = U \begin{pmatrix} \tilde{\nu}_L^C \\ \tilde{\nu}_L \end{pmatrix}, \quad \begin{pmatrix} \nu_R \\ \nu_R^C \end{pmatrix} = U^\star \begin{pmatrix} \tilde{\nu}_R \\ \tilde{\nu}_R^C \end{pmatrix} \tag{8}$$

The kinetic part of the Lagrangian is form invariant under this transformation.

We thus consider a further 6×6 real generalized Pauli-Gursey transformation O by

$$\begin{pmatrix} \tilde{\nu}_L^C \\ \tilde{\nu}_L \end{pmatrix} = O \begin{pmatrix} N_L^C \\ N_L \end{pmatrix}, \quad \begin{pmatrix} \tilde{\nu}_R \\ \tilde{\nu}_R^C \end{pmatrix} = O \begin{pmatrix} N_R \\ N_R^C \end{pmatrix}, \tag{9}$$

By choose a specific 6×6 orthogonal transformation

$$O = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{10}$$

where 1 stands for a 3×3 unit matrix, we have

$$\begin{aligned} \mathcal{L} &= (1/2)\{\overline{N}(x) i \not\partial N(x) + \overline{N^C}(x) i \not\partial N^C(x)\} \\ &- (1/4)\{\overline{N}(M_1 + M_2) N + \overline{N^C}(M_1 + M_2) N^C\} \\ &- (1/4)[\overline{N}(M_1 - M_2) N^C + \overline{N^C}(M_1 - M_2) N] \end{aligned} \tag{11}$$

which is invariant under C, P and CP

$$\begin{aligned} C &: N(x) \leftrightarrow N^C(x) = C\overline{N}^T(x), \\ P &: N(x) \rightarrow i\gamma^0 N(t, -\vec{x}), \quad N^C(x) \rightarrow i\gamma^0 N^C(t, -\vec{x}), \\ CP &: N(x) \rightarrow i\gamma^0 N^C(t, -\vec{x}), \quad N^C(x) \rightarrow i\gamma^0 N(t, -\vec{x}). \end{aligned} \tag{12}$$

Note that **only the Dirac-type particles** $N(x)$ and $N^C(x)$ with well-defined C, P and CP properties appear after this Pauli-Gursey: γ_5 disappears.

We now make a renaming of variables

$$\psi_+(x) = \frac{1}{\sqrt{2}}(N(x) + N^C(x)), \quad \psi_-(x) = \frac{1}{\sqrt{2}}(N(x) - N^C(x)), \tag{13}$$

and we obtain

$$\begin{aligned}\mathcal{L} &= (1/2)\{\overline{\psi_+}(x) i \not{\partial}\psi_+(x) + \overline{\psi_-}(x) i \not{\partial}\psi_-(x)\} \\ &- (1/2)\{\overline{\psi_+} M_1 \psi_+ + \overline{\psi_-} M_2 \psi_-\}.\end{aligned} \tag{14}$$

After this renaming of variables, we find the transformation laws of $\psi_\pm(x)$ induced by those of N and N^C,

$$\begin{aligned} C &: \psi_+(x) \to \psi_+(x), \quad \psi_-(x) \to -\psi_-(x), \\ P &: \psi_+ \to i\gamma^0\psi_+(t, -\vec{x}), \quad \psi_-(x) \to i\gamma^0\psi_-(t, -\vec{x}), \\ CP &: \psi_+(x) \to i\gamma^0\psi_+(t, -\vec{x}), \quad \psi_-(x) \to -i\gamma^0\psi_-(t, -\vec{x}) \end{aligned} \tag{15}$$

which naturally keep the Lagrangian invariant.

One can define Majorana fermions in a natural manner by a suitable choice of generalized Pauli-Gursey transformation in the seesaw model.

Parity in the extension of SM and "emergent" Majorana neutrino

If $i\gamma^0$ parity is mandatory for the Majorana, how to formulate the Majorana emeging in an extension of SM, such as in the model of Weinberg?

One can in fact formulate the emergent Majorana neutrino consistently using either γ^0 parity or $i\gamma^0$ parity.

The basic idea is that

$$\psi_M(x) = e^{i\alpha}\nu_L(x) + e^{-i\alpha}C\overline{\nu_L}^T(x)$$

with arbitrary real α satisfies the Majorana condition

$$\psi_M(x) = C\overline{\psi_M(x)}^T(x)$$

A suitable choice of α can compensate the different parity, γ^0 parity or $i\gamma^0$ parity, in the starting theory.

KF. Phys. Rev. D102 (2020) 105001.