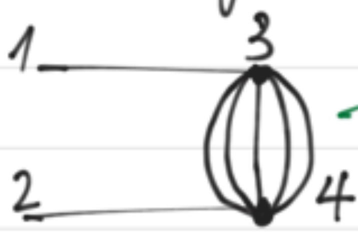
## 2.3. THE 4-POINT FUNCTION.

* In order to understand the spectrum as well as many of the other properties of the (super)SYK model (in 1D as well as its generalization to higher dimensions) we need to know the 4-pt. correlator of the theory.

* To do so we need to compute the kernel describing the propagation of a 2-particle system from $(t_3, \theta_3)$ and $(t_4, \theta_4)$ to $(t_1, \theta_1)$ and $(t_2, \theta_2)$

Notice that this discussion follows in much the same way for the non-susy as for susy so we might as well do it for the susy case.

* As a diagram, we want to compute




for a $q$-fold interaction 3 and 4 are connected by $q-2$ propagators.

* for a model with $\hat{q}$-fold interactions this kernel is

$$K(t_1,\theta_1,t_2,\theta_2;t_3,\theta_3,t_4,\theta_4)$$
$$= (\hat{q}-1)\,j^2\,G(1,3)\,G(2,4)\,G(3,4)^{\hat{q}-2}$$

effective coupling
$j^2 b_\psi^{\hat{q}} = \frac{\tan \pi/\hat{q}}{2\pi}$

propagator
$G(i,j) = \frac{b_\psi \,\mathrm{sgn}(t_i - t_j)}{|\langle i,j\rangle|^{2\Delta}}$

with $\Delta = 1/2\hat{q}$.

$$= (\hat{q}-1)\,\frac{\tan(\pi/2\hat{q})}{2\pi}\,\frac{\mathrm{sgn}(t_1-t_3)\,\mathrm{sgn}(t_2-t_4)\,\mathrm{sgn}(t_3-t_4)}{|\langle 1,3\rangle|^{2\Delta}\,|\langle 2,4\rangle|^{2\Delta}\,|\langle 3,4\rangle|^{2\Delta(\hat{q}-2)}}$$

This should be thought of as a map from $f^{\underline{ns}}$ of 3 & 4 to $f^{\underline{ns}}$ of (1 & 2).

* Superconformal symmetry $\Rightarrow$ eigen$f^{\underline{ns}}$ of K must also be eigen$f^{\underline{ns}}$ of the superCasimirs $\mathcal{C}_{12}/\mathcal{C}_{34}$.

* These come in two kinds

$$S_h^B(t_3,\theta_3;t_4,\theta_4) = \frac{\operatorname{sgn}(t_3-t_4)}{|\langle 3,4\rangle|^{2\Delta-h}}$$

bosonic primary of dimension h.

and

fermionic primary of dim. h.

$$S_h^F(t_3,\theta_3,t_4,\theta_4) = \frac{\theta_3-\theta_4}{|\langle 3,4\rangle|^{2\Delta-h+1/2}}$$

if we insert identical fermionic primaries at 1 and 2.

* Now let's compute some eigenvalues. To evaluate the eigenvalue $k^B(h)$ of $K$ acting on $S_h^B$ we compute the integral

$$\int dt_3\,dt_4\,d\theta_3\,d\theta_4\,K(t_1\ldots\theta_4)\,S_h^B(t_3,\ldots\theta_4)$$

to find

$$k^B(h) = (\hat{q}-1)\frac{\tan\frac{\pi}{2\hat{q}}}{2\pi}\int_{-\infty}^{\infty} dt_3\,dt_4\,d\theta_3\,d\theta_4 \quad \otimes$$

$$\circledast \quad \frac{\mathrm{sgn}(1-t_3)\,\mathrm{sgn}(-t_4)}{|1-t_3|^{2\Delta}\,|t_4|^{2\Delta}\,|t_3-t_4-\theta_3\theta_4|^{1-2\Delta-h}}$$

$$=(\hat{q}-1)(1-2\Delta-h)\frac{\tan \pi/2\hat{q}}{2\pi}\, I_x(h)\, I_y(h)$$

where: $I_x(h)=\int_{-\infty}^{\infty} dx \, \frac{\mathrm{sgn}(1-x)}{|x|^{1-h}|1-x|^{2\Delta}}$

$$I_y(h)=\int_{-\infty}^{\infty} dy \, \frac{\mathrm{sgn}(y(1-y))}{|y|^{2\Delta}|1-y|^{2-2\Delta-h}}$$

* As fun as these integrals are to compute, I will leave them as an exercise for the audience to check that $I_x(h)$ is a sum of 3 B-functions which can be evaluated to give

$$I_x(h)=\frac{1}{\pi}\Big[-\sin((1-h)\pi)+\sin 2\Delta\pi+\sin((1+h-2\Delta)\pi)\Big]\otimes$$

$$\otimes\,\Gamma(h)\,\Gamma(1-2\Delta)\,\Gamma(-h+2\Delta)$$

* Fortunatly, we don't even have to evaluate $I_y(h)$ since

we can simply use an $SL(2,\mathbb{R})$ transformation to permute 0, 1 and $\infty$ to show that $I_y(h) = -I_x(1-h)$

* then $k^B(h)$

$$= -(\hat{q}-1)\,\frac{\sin 2\pi\Delta - \sin \pi h}{\sin 2\pi\Delta}\cdot\frac{\Gamma(h+2\Delta)\,\Gamma(h+2\Delta)}{\Gamma^2(2\Delta)}$$

* This formula passes two important checks:

1) $k^B(0) = -(\hat{q}-1)$ ← this is a general combinatorial feature of melon graphs and shows up in all SYK-like models.

2) $k^B(1) = 1$ ← reflects the existence of a susy-breaking deformation of the solution of the SD eq$^{\underline{n}}$ in the IR limit.

## ASIDE #2 HOW TO COMPUTE THE CENTRAL CHARGE.

* In the CFT's we will study, we will have access to the 4-pt. function (and hence spectrum) of the theory. Here I will show how to compute the central charge of the CFT given this data.

* Essentially, we want to evaluate the contribution of the stress tensor to the 4-pt. $f^{\underline{n}}$.

* To this end, consider a 2D CFT with holomorphic stress tensor satisfying

$$\langle T(x)\, T(y)\rangle = \frac{c}{2(x-y)^4}$$

central charge of the CFT.

and some other spinless operator $O$ of dim. $(h,\tilde{h}) = (\Delta/2, \Delta/2)$ and corresponding 2-pt. $f^{\underline{n}}$

$$\langle O(x,\bar{x})\, O(0,0)\rangle = \frac{b}{|x|^{2\Delta}}$$

* The singular part of the $T \cdot O$ operator product expansion is

$$T(y)\,O(0,0) \sim \frac{h}{y^2} O(0,0) + \frac{1}{y} \partial_y O(0,0)$$

and the $\langle OOT \rangle$ 3-pt. $f^{\underline{n}}$ then follows by conformal invariance as

$$\langle O(x_1,\bar{x}_1)\, O(x_2,\bar{x}_2)\, T(y) \rangle = \frac{b}{|x_1-x_2|^{2\Delta}} \frac{h(x_1-x_2)^2}{(y-x_1)^2 (y-x_2)^2}$$

This is the unique formula that has: 1) the right double poles at $y = x_1, x_2$; 2) is holomorphic everywhere else and 3) vanishes as $y^{-4}$ at infinity.

follows from 2-pt. $f^{\underline{n}}$

$$\Rightarrow O(x_1,\bar{x}_1)\, O(x_2,\bar{x}_2) \sim \frac{b}{|x_1-x_2|^{2\Delta}} \left( 1 + \frac{2h}{c} (x_1-x_2)^2 T(x_2) + \ldots \right)$$

follows from the TT 2pt. $f^{\underline{n}}$ and OOT 3-pt.

* Now, if in addition, the theory has another operator $O'$ with the same dimension and 2-pt. function as $O$ consider the normalized 4-pt. function

$$W(x_1,\bar{x}_1,\ldots,x_4,\bar{x}_4) = \frac{\langle O(x_1,\bar{x}_1)O(x_2,\bar{x}_2)O'(x_3,\bar{x}_3)O'(x_4,\bar{x}_4)\rangle}{\langle O(x_1,\bar{x}_1)O(x_2,\bar{x}_2)\rangle\langle O'(x_3,\bar{x}_3)O'(x_4,\bar{x}_4)\rangle}$$

* In the limit $x_1 \to x_2$ and $x_3 \to x_4$, the stress tensor gives a contribution

$$W \sim \ldots + \underbrace{\frac{2h^2}{c}\frac{(x_1-x_2)^2(x_3-x_4)^2}{(x_2-x_4)^4}}_{W_T = \frac{\Delta^2\chi^2}{2c}} \leftarrow$$

this comes from the $OO$ OPE and the $\langle OO\rangle$ 2pt. f$^{n}$.

* Recall that in the SYK-type models that we are studying the 4-pt. f$^{n}$ $\mathcal{F}$ is usually defined with an extra factor of $N$ so in our cases, the corresponding result will be

$$\mathcal{F}_T = \frac{2h^2N\chi^2}{c} = \frac{N\Delta^2\chi^2}{c}$$

* Let's do an example here to see how this would work in practice. In one of our (naive) 2D bosonic models, using the shadow representation, we will find a four-point $f^{\underline{n}}$

$$\mathcal{F} = \frac{2\Delta}{\pi(2-\Delta)(1-\Delta)^2} \sum_{\ell \text{ even}} \int_{-\infty}^{\infty} \frac{ds}{2\pi} (\ell^2+4s^2) A(s,\ell) F_{s,\ell}(x,\bar{x}) \frac{k(\ell,s)}{1-k(\ell,s)}$$

kernel eigenvalue.

* Don't worry too much about the details of the integrand. What is important is that, in the spin 2 sector the integrand has a simple pole at $(h,\tilde{h}) = (2,0)$ or equivalently $(\ell,s) = (2,-\frac{i}{2})$.

* Now we expand the kernel near this point to get

$$\frac{k(2,s)}{1-k(2,s)} = -\frac{\Delta(2-\Delta)}{2(1-\Delta)(is-\frac{1}{2})} + \text{terms regular at } s=-\tfrac{i}{2}$$

* The stress tensor contribution comes from the residue at this pole. Using the facts that, at $\ell,s = 2,-\frac{i}{2}$ ; $A(s,\ell) = \pi/6$ ; $\ell^2+4s^2=3$ ; $F_{\ell s} = \chi^2+\ldots$ and integrating clockwise around the pole in $s$ gives

$$\mathcal{F}_T = \frac{\Delta^2}{2(1-\Delta)^3}\chi^2$$

and from which we read of the central charge

$$c = (1-\Delta)^3 N = (1-2/q)^3 N$$

* This bosonic model is actually not well defined but the central charge is still sensible:

→ It decreases with decreasing $q$ which is consistent with the c-theorem since we flow from large to small $q$ by perturbing with more relevant smaller $q$ interactions.

→ $c \longrightarrow N$ as $q \to \infty$ consistent with the free-field limit.

→ $c \longrightarrow 0$ as $q \to 2$ and the theory becomes trivial.

## 2.4. 2D SHADOWS

* Recall that in 2-dimensions a CFT has left and right dimensions $h$ and $\tilde{h} \geq 0$ where

$$\Delta = h + \tilde{h} = \text{total scaling dimension}$$

$$J = h - \tilde{h} = \text{spin} \in \begin{cases} \mathbb{Z} & \text{for boson.} \\ \mathbb{Z} + \tfrac{1}{2} & \text{for fermions} \end{cases}$$

and $\Delta \geq |J|$

* We will consider the normalized 4-pt. function

We will take conf. prim. $\Phi_i$ with same $\Delta$ and $J = 0$.

$$W(z_1, \bar{z}_1; \cdots; z_4, \bar{z}_4)$$

$$= \frac{\langle \Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\Phi_3(z_3,\bar{z}_3)\Phi_4(z_4,\bar{z}_4)\rangle}{\langle \Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\rangle\langle \Phi_3(z_3,\bar{z}_3)\Phi_4(z_4,\bar{z}_4)\rangle}$$

* The 2D shadow representation obtains from the insertion of the operator

$$\int d^2z\, V(z,\bar{z})\, \tilde{V}(z,\bar{z})$$

primaries of weight $(h,\tilde{h})$ and complimentary dim. $(1-h, 1-\tilde{h})$

* The normalized 3-pt. fn

$z_{ij} \equiv z_i - z_j$

$$\frac{\langle \Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)V(z_3,\bar{z}_3)\rangle}{\langle \Phi_1(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\rangle} = \frac{z_{12}^{h}}{z_{13}^{h} z_{23}^{h}} \frac{\bar{z}_{12}^{\tilde{h}}}{\bar{z}_{13}^{\tilde{h}} \bar{z}_{23}^{\tilde{h}}}$$

bosonic primary with integer J

ensures single-valued

* Then, the 2D analog of the shadow representation is then

$$\Psi_{h,\tilde{h}}(z_1,\dots,\bar{z}_4) = \int d^2y\, \frac{z_{12}^{h} z_{34}^{1-h} \bar{z}_{12}^{\tilde{h}} \bar{z}_{34}^{1-\tilde{h}}}{z_{y1}^{h} z_{y2}^{h} z_{3y}^{1-h} z_{4y}^{1-h} \bar{z}_{y1}^{\tilde{h}} \bar{z}_{y2}^{\tilde{h}} \bar{z}_{3y}^{1-\tilde{h}} \bar{z}_{4y}^{1-\tilde{h}}}$$

or, as a function of the cross ratio $\chi \equiv \frac{z_{12} z_{34}}{z_{13} z_{24}}$ and its complex conjugate $\bar{\chi}$

$$\Psi_{h,\tilde{h}}(\chi, \bar{\chi}) = \int d^2y \, \frac{\chi^h \bar{\chi}^{\tilde{h}}}{y^h (y-\chi)^h (1-y)^{1-h} \bar{y}^{\tilde{h}} (\bar{y}-\bar{\chi})^{\tilde{h}} (1-\bar{y})^{1-\tilde{h}}}$$

* Most of the symmetries that we identified in 1D have analogues in 2D that can be read off from the integral representation. For example:

$y \to y/y-1$

1) $$\Psi_{h,\tilde{h}}\left(\frac{\chi}{\chi-1}, \frac{\bar{\chi}}{\bar{\chi}-1}\right) = (-1)^{h-\tilde{h}} \, \Psi_{h,\tilde{h}}(\chi, \bar{\chi})$$

this implies that any correlator that is symmetric under $z_1 \leftrightarrow z_2$ can only recieve contributions from $\Psi_{h,\tilde{h}}$ with even spin.

$y \to \chi/y$

2) $$\Psi_{1-h, 1-\tilde{h}}(\chi, \bar{\chi}) = \Psi_{h,\tilde{h}}(\chi, \bar{\chi})$$

this reflects a symmetry of the shadow construction under exchange of the 1st two and last two particles.

* In 2D the special conformal group $SO(2,2) \sim SL_2 \times SL_2$ implying the existence of a holomorphic Casimir

$$C_{12} = -z_{12}^2 \frac{\partial^2}{\partial z_1 \partial z_2} = x^2(1-x)\partial_x^2 - x^2 \partial_x$$

and an antiholomorphic Casimir.

$$\bar{C}_{12} = -\bar{z}_{12}^2 \frac{\partial^2}{\partial \bar{z}_1 \partial \bar{z}_2} = \bar{x}^2(1-\bar{x})\partial_{\bar{x}}^2 - \bar{x}^2 \partial_{\bar{x}}$$

with $\Psi_{h,\tilde{h}}$ satisfying

$C_{12} \Psi_{h,\tilde{h}} = h(h-1)\Psi_{h,\tilde{h}}$ and

$\bar{C}_{12} \Psi_{h,\tilde{h}} = \tilde{h}(\tilde{h}-1)\Psi_{h,\tilde{h}}$

simultaneously. ← this holds because the conformal 3-pt. function is simultaneously an eigenfunction of both Casimirs.

* Single-valuedness of the wavefunction means that

$$\Psi_{h,\tilde{h}}(x,\bar{x}) = A(h,\tilde{h})\, \mathcal{F}_h(x)\, \mathcal{F}_{\tilde{h}}(\bar{x}) + B(h,\tilde{h})\, \mathcal{F}_{1-h}(x)\, \mathcal{F}_{1-\tilde{h}}(\bar{x})$$

conformal blocks

$$\int d^2y \, \frac{1}{y^{2h}(1-y)^{1-h}\,\bar{y}^{2\tilde{h}}(1-\bar{y})^{1-\tilde{h}}} = \frac{1}{2}\frac{\sin(\pi h)}{\cos(\pi h)}\frac{\Gamma(h)^2}{\Gamma(2h)}\frac{\Gamma(\tilde{h})^2}{\Gamma(2\tilde{h})}$$

and $B(h,\tilde{h}) = A(1-h, 1-\tilde{h}) = -\frac{1}{2}\frac{\sin(\pi h)}{\cos(\pi\tilde{h})}\cdot\frac{\Gamma(1-h)^2}{\Gamma(2-2h)}\cdot\frac{\Gamma(1-\tilde{h})^2}{\Gamma(2-2\tilde{h})}$

* The coefficients satisfy several simple identities. Some of those that will be important in our models are:

1) $A(h,\tilde{h})B(h,\tilde{h}) = -\dfrac{\pi^2}{(2h-1)(2\tilde{h}-1)}$

$$\left.\begin{array}{l} \text{2) } A(h,\tilde{h}) = A(\tilde{h},h) \\ \text{3) } B(h,\tilde{h}) = B(\tilde{h},h) \end{array}\right\} \text{these depend on the fact that } J = h-\tilde{h} \in \mathbb{Z}.$$