# LECTURES ON DISORDERED QUANTUM MECHANICS AND (SUPERSYMMETRIC) CONFORMAL FIELD THEORY.

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## OUTLINE :

1. THE SYK MODEL OF DISORDERED QUANTUM MECHANICS
2. ANOTHER LOOK AT THE SYK MODEL
3. INTEGRAL REPRESENTATIONS AND THE SHADOW FORMALISM.
4. DISORDED SCFT'S IN (1+1)-D

# LECTURE 1.

* The SYK model is a many body Hamiltonian with N sites in 0+1 dimensions ie. a Quantum Mechanics problem.

* Each site is populated by a Majorana fermion satisfying the Clifford algebra

$$\{\psi_i(t), \psi_j(t)\} = \delta_{ij} \qquad i,j = 1, \dots, N$$

and

$$[\psi_i] = 0 \leftarrow \text{mass dimension.}$$

* Note that:

(i) $i \neq j \Rightarrow \psi_i \psi_j = -\psi_j \psi_i$

(ii) $i = j \Rightarrow \psi_i^2 = \frac{1}{2}$

* The Hamiltonian defining the model is

antisymmetric in all indices.

all-to-all interaction.

$$H = \sum_{jklm} \frac{J_{jklm}}{4!} \psi_j \psi_k \psi_l \psi_m$$

* Since $[H] = -1$ and the fermions are dimensionless $\Rightarrow [J_{jklm}] = -1$

$\Rightarrow$ The coupling is relevant and the theory is strongly coupled in the IR and weakly coupled in the UV.

* What makes the SYK model special is that $J_{jklm}$ is a random variable that is time independent.

$J_{jklm}$ being $\nearrow$ random $\rightsquigarrow$ disorderd

$\searrow$ time indep $\rightsquigarrow$ quenched.

* The SYK Hamiltonian is a 4-body Hamiltonian with quenched disorder.

* As a random variable the couplings $J_{jklm}$ are drawn from some ensemble according to the probability density:

$$P(J_{jklm}) = \mathcal{N} \exp\left(-N^3 J_{ijkl} J_{ijkl} / 12 J^2\right)$$

Normalization factor fixed by normalizing $P$ to unity

Gaussian distribution with $\overline{J_{ijkl}} = 0$

* $$\frac{1}{3!} \overline{J_{ijkl} J_{mnop}} = \begin{cases} 0 & \text{if } mnop \text{ is not a perm. of } ijkl. \\ J^2/N^3 & \text{if } mnop \text{ is an even perm.} \end{cases}$$

* The Lagrangian of the model is easily computed as

$$L = \underbrace{\frac{1}{2}\psi_j \dot{\psi}_j}_{L_0} - \underbrace{\frac{J_{jklm}}{4!}\psi_j \psi_k \psi_l \psi_m}_{L_{int}}$$

with
$$\int \mathcal{D}\psi_k \, e^{-S_0} \psi_i(t) \psi_j(0) = \delta_{ij} G_0(t)$$

where the free Green's function

$$G_0(t) = \frac{1}{2}\,\mathrm{sgn}(t)$$

and $S_0 = \int dt \, L_0$

* Now we want to compute the time ordered 2-pt. correlation function

$$\langle T(\psi_i(t_1)\psi_j(t_2))\rangle$$
$$= \int \mathcal{D}\psi_k \, e^{-S} \psi_i(t_1)\psi_j(t_2)$$

this is now the full interacting theory.

* To compute the 2-pt. function we make a perturbative expansion about the non interacting theory:

$$\int \mathcal{D}\psi_k \, e^{-S} \psi_a(t_1)\psi_b(t_2)$$

$$= \int \mathcal{D}\psi_k \, e^{-\int dt\, L_0} \Big( 1 - \int dt\, L_{int} + \frac{1}{2!}\Big(\int dt\, L_{int}\Big)^2$$
$$+ \ldots \Big) \psi_a(t_1)\psi_b(t_2)$$

* Let's evaluate each of these contributions in turn:

$$① = \int \mathcal{D}\psi_k \, e^{-S_0} \psi_a(t_1) \psi_b(t_2) = \delta_{ab} G_0(t_1 - t_2)$$

$$② = \int \mathcal{D}\psi_k \, e^{-S_0} \frac{J_{jklm}}{4!} \int dt \, \psi_j(t) \psi_k(t) \psi_l(t) \psi_m(t)$$

$$\otimes \psi_a(t_1) \psi_b(t_2) = 0$$

$\overline{J_{ijkl}} = 0$ and
Wick contraction on the $\psi$'s set 2 indices on J equal.

$$③ = \frac{1}{2} \int dt \int dt' \Big\langle \psi_a(t_1) \psi_b(t_2) \otimes$$

$$\otimes J_{jklm} \psi_j(t) \psi_k(t) \psi_l(t) \psi_m(t) \otimes$$

$$\otimes J_{npqr} \psi_n(t') \psi_p(t') \psi_q(t') \psi_r(t') \Big\rangle_0$$

quantum average over free theory.

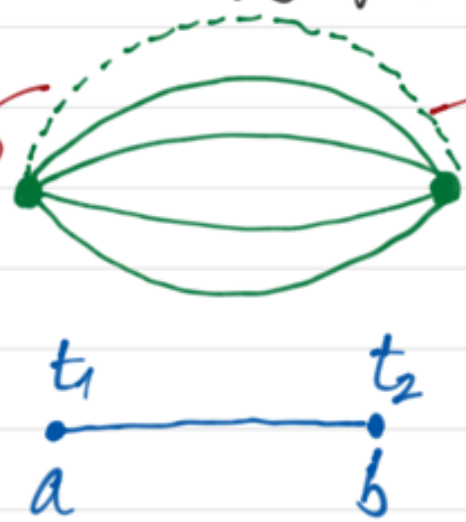
* Now let's think about the Wick contractions on the fermion indices:

$$a \quad b \qquad j \quad k \quad l \quad m \quad n \quad p \quad q \quad r$$

corresponds to

$$\# \int dt \int dt' \, \delta_{ab} \, J^{\,2}_{jklm} \, G_0(t_1-t_2)\left(G_0(t-t')\right)^4$$

dotted line keeps track of the J contraction.

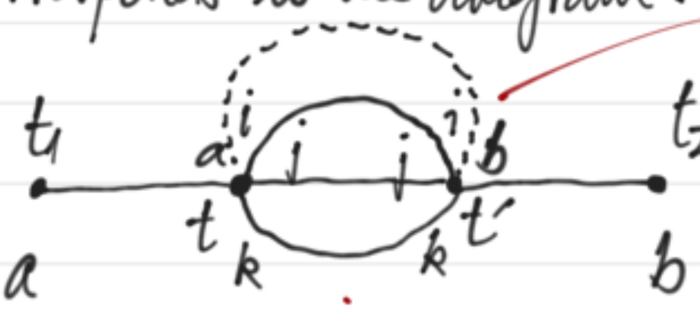



the solid lines in the Feynman diagram keep track of the fermion contractions

this is a disconnected diagram that is subtracted off by properly normalizing so that $\langle 1 \rangle = 1$. We will discard all such diagrams.

$$a \quad b \quad j \quad k \quad l \quad m \quad n \quad p \quad q \quad r$$

corresponds to the diagram:




Notice that the disorder average equates the indices of the two verti

Combinatoric factor that counts the # of such diagrams.

$$= \# \, \delta^{ab} \delta^{ii} \delta^{jj} \delta^{kk} G_0(t_1 - t) G_0(t' - t_2) \otimes$$

$$\otimes \left( G_0(t-t') \right)^3 \cdot \frac{J^2}{N^3} \sim O(N^0)$$

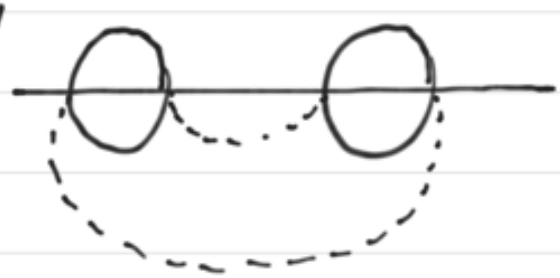
* The diagram above was order 1 in N and will be leading in the perturbation expansion. But what about

* What happens at higher orders in the expansion of $\langle T(\psi_a(t_1)\psi_b(t_2))\rangle$?

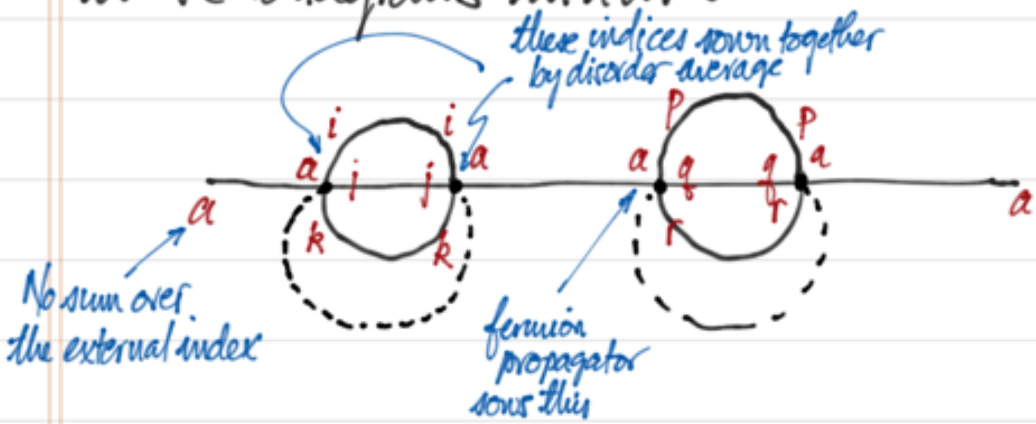
* The term cubic (and any odd order actually) in $S_{int}$ will vanish because there will always be an odd power of $J_{ijkl}$.

* However the quartic term will produce something interesting. There, we will encounter 4 vertices with diagrams like

and

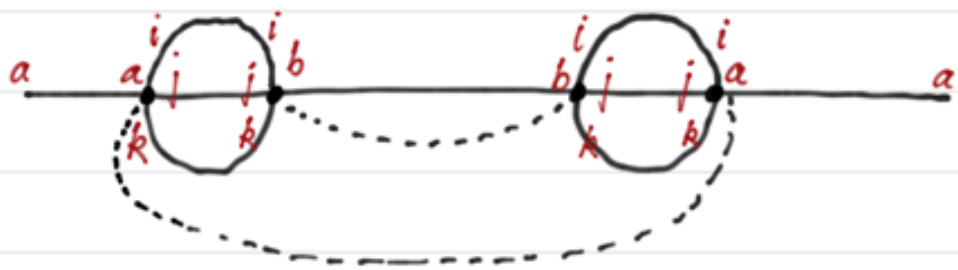


* Let's work out the $N$-dependence of each of these diagrams in turn:




contributes $\left(J^2/N^3\right)^2 \cdot N^6 \sim J^4$

while:




contributes $\left(\frac{J^2}{N^3}\right)^2 N^4 \sim J^4/N^2$

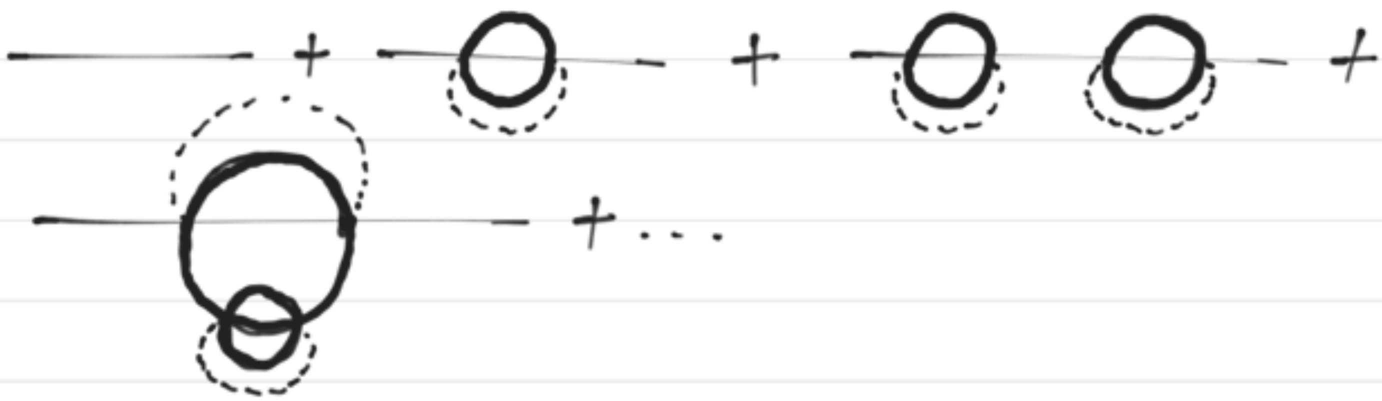
2 disorder averages taken

summing over $i, j, k$ and $b$

and is then suppressed at large N.

* The upshot is that the disorder average must always be taken on the same sunset diagrams. (This is why the melonic graphs in the Gurau-Witten theory have the same large N limit as the SYK model.)

* In summary then the large N perturbation series in diagrams looks like:



* In order to solve the theory in the large $N$ limit, we — need to sum all these diagrams!

* Fortunately, this can be done. The result is an integral equation satisfied by the Green's function called the Schwinger-Dyson equation.

* Specifically, if

$$\langle T(\psi_a(t_2)\psi_b(t_1))\rangle = \delta_{ab}\, G(t_1-t_2)$$

then the Green's function satisfies:

This is the full 2-pt. function

This is the bare 2-pt. function.

$$G(t_1-t_2)=G_0(t_1-t_2)+J^2\int dt\,dt'\,G_0(t_1-t)\otimes$$
$$\otimes\left(G(t-t')\right)^3 G(t'-t_2)$$

* Let's expand on this to show how it recursively captures the full series of sunset diagrams:

* Ordering the Schwinger-Dyson equation in $J$ we read off that

$$G(t_1-t_2)=G_0(t_1-t_2)+O(J^2)$$

Now substitute into the RHS of the SD equation:

$$G(t_1-t_2)=G_0(t_1-t_2)+J^2\int dt\,dt'\,G_0(t_1-t)$$
$$\left(G_0(t-t')\right)^3 G_0(t'-t_2)+O(J^4)$$

etc.

* In terms of the fermion self-energy

$$\Sigma(t-t') \equiv J^2 \big(G(t-t')\big)^3$$

the SD equation reads

$$G(t_1-t_2) = G_0(t_1-t_2) + \int dt\,dt'\, G_0(t_1-t)\,\Sigma(t-t')\,G(t'-t_2)$$

* While this equation cannot be solved exactly, it can be solved in the IR where $J \to \infty$ (since $J$ is a relevant coupling).

* In this limit $\Sigma \to \infty \Rightarrow G(t) \to 0$ and the SD equation becomes

$$0 = G_0(t_1-t_2) + \int dt\, G_0(t_1-t) \int dt'\, \Sigma(t-t') \otimes$$

$$\otimes\, G(t'-t_2)$$

The structure of this equation requires that

$$\int dt' \, \Sigma(t-t') G(t'-t_2) = -\delta(t-t_2)$$

* This equation is actually conformally invariant and reflects an emergent conformal symmetry of the SYK model. To show this we reparameterize

$$t \to f(t)$$

$$\Rightarrow G(t-t') \to \left[\frac{df(t)}{dt} \frac{df(t')}{dt'}\right]^{1/4} G(f(t)-f(t'))$$

and

$$\Sigma(t-t') \to \left[\frac{df(t)}{dt} \frac{df(t')}{dt'}\right]^{3/4} \Sigma(f(t)-f(t'))$$

Then choose the particular reparameterization $f(t) = \alpha t$ so that

$$G(t-t') \to \sqrt{\alpha}\, G(\alpha(t-t'))$$

* This implies that

$$G(t-t') = b/(t-t')^{1/2} = b\,\mathrm{sgn}(t-t')/|t-t'|^{1/2}$$

so that substituting into the conformal limit requires that the proportionality constant b satisfies

$$J^2 b^4 \pi = \frac{1}{4} \tan(\pi/4)$$

* More generally, for a q-fermion model

$$G(t-t') = \frac{b\,\mathrm{sgn}(t-t')}{|t-t'|^{2\Delta}} \quad \text{where } \Delta = 1/q.$$

and $J^2 b^q \pi = \left(\frac{1}{2} - \Delta\right) \tan \pi\Delta$.

* To summarize then:

In the IR: $J^2 \gg 1 \Rightarrow G(\tau) \sim \frac{\mathrm{sgn}(\tau)}{|\tau|^{1/2}}$

and the fermions have scaling dimension $\Delta = 1/4$

- In the UV: the theory is free since $J^2 \ll 1$ and

$G(\tau) \sim \mathrm{sgn}(\tau)$

while the fermions have scaling dimension $\Delta = 0$.
(this can be seen by scaling time and seeing that the 2-pt. function is invariant)

* A key feature of the SYK model is the fact that it is chaotic

* A classical chaotic system is one in which a small fluctuation leads to an exponentially large effect at some later time (the so-called Butterfly effect)

Question: How is such an effect characterized in a quantum system?

* Can a small perturbation $W$, say, have a macroscopic effect on the system?

* Does this occur for any small perturbation or only very specific ones?

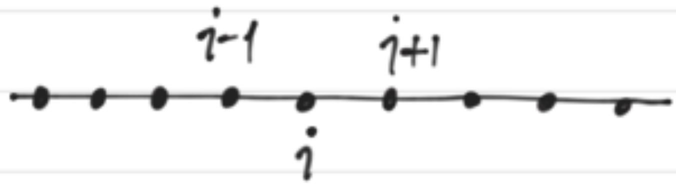
* How should we characterize this?

* Let's try to understand the evolution of a simple local operator in some generic quantum system.

$$W(t) = e^{iHt} \, W \, e^{-iHt}$$

* The butterfly effect will be identified with an exponential growth of the operator $W(t)$.

* To make things concrete lets consider the quantum spinchain:




$\sigma_x \equiv X ; \sigma_y \equiv Y$
$\sigma_z \equiv Z$

The Hamiltonian for this system is given by

$$H = -\sum_i Z_i Z_{i+1} + g X_i + h Z_i$$

Ising term external transverse magnetic field. parallel field.

* Let's perturb the spinchain by inserting a field $W = Z_1$ at site 1. At time $t$ this will evolve to

$$Z_1(t) = e^{iHt} Z_1 e^{-iHt}$$

$$= Z_1 - it[H, Z_1] - \frac{t^2}{2!}[H,[H,Z_1]]$$

$$-\frac{it^3}{3!}[H,[H,[H,Z_1]]] + \dots$$

* Let's look at the individual commutators:

$[H, Z_1] \sim Y_1$

$[H,[H,Z_1]] \sim [H, Y_1] \sim X_1 Z_2 + Z_1 + X_1$

this is a string of length 2.

$\vdots$

* $[H,[H,[H,[H,[H,[H,Z_1]]]]]]$

$\sim Y_1 + X_1 Y_2 + 5 \text{ more}$

$+ X_1 X_2 Y_3 + 12 \text{ more}$

$+ X_1 X_2 X_3 Y_4 + 3 \text{ more}$

operator strings grow rapidley.

* In the operator then

$Z_1(t) = \ldots + \alpha(t) X_1 X_2 + \beta(t) X_1 Z_2 + \gamma(t) Y_1 Y_2 + \ldots$

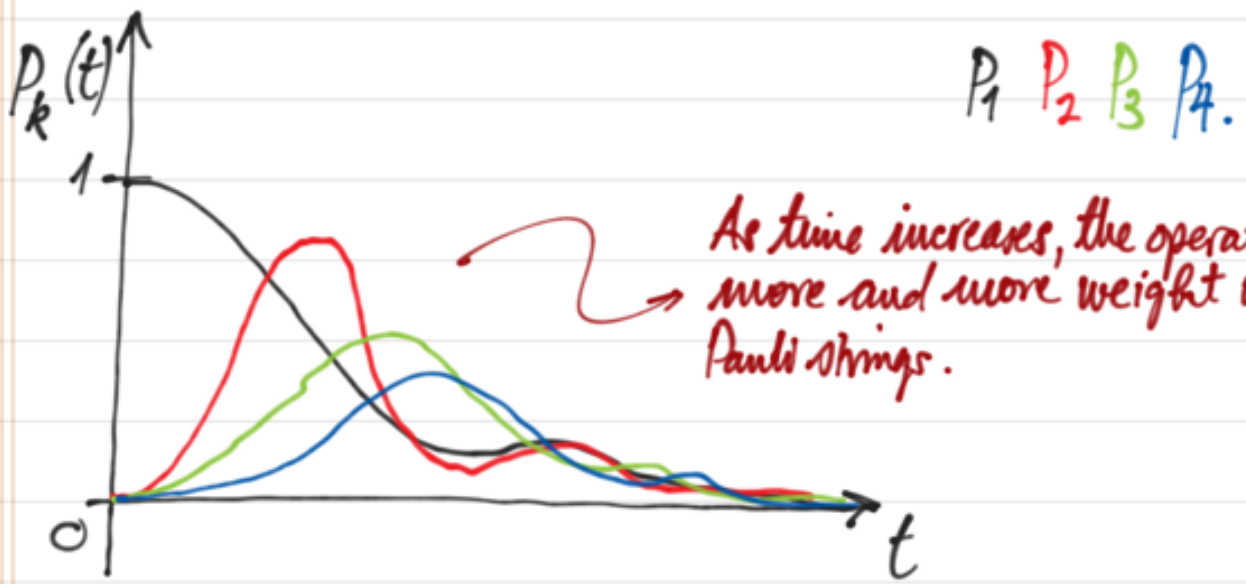
* Let's define the time-dependent variable

$$P_2(t) \equiv \alpha^2(t) + \beta^2(t) + \gamma^2(t)$$

that measures the length of the part of $Z_1(t)$ made up of length 2 strings.
(We can similarly define an object $P_n(t)$).

* How do the $P_i(t)$'s behave?

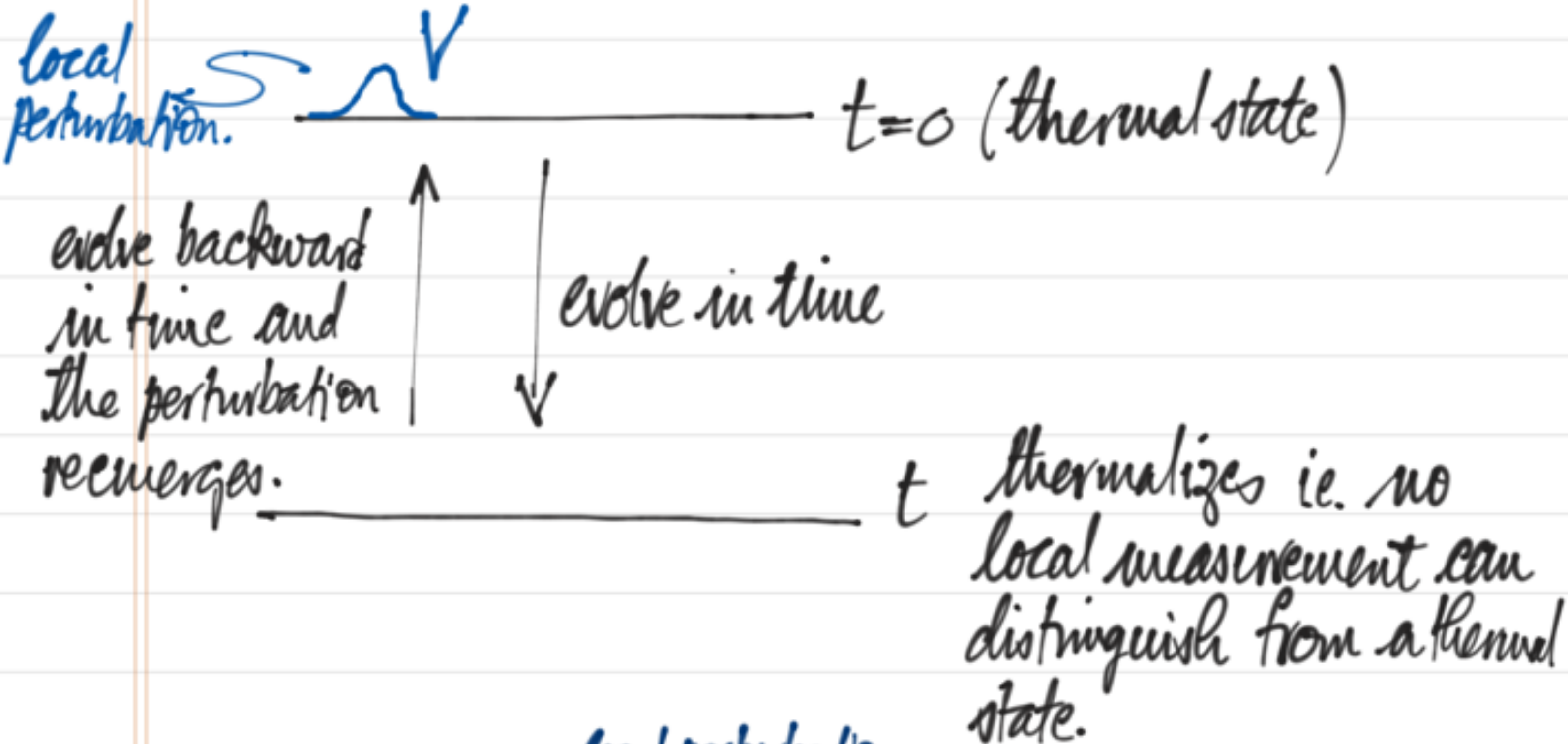
* To get an idea, we can numerically study the coefficients and plot the result:

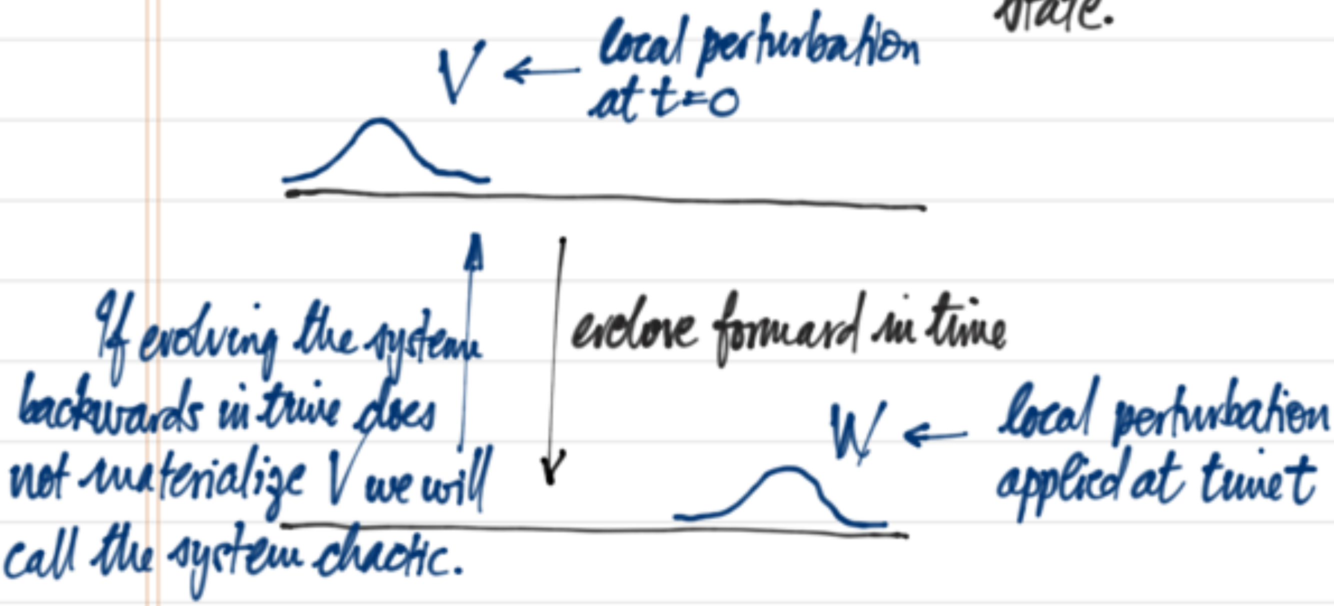



As time increases, the operator gets more and more weight in longer Pauli strings.

* How do we understand this behaviour?

* Let's prepare a thermal state at $t=0$




* How to characterize this?

* The statement here is that the perturbation by $W(t)$ interfered with the measurement of $V(0)$.

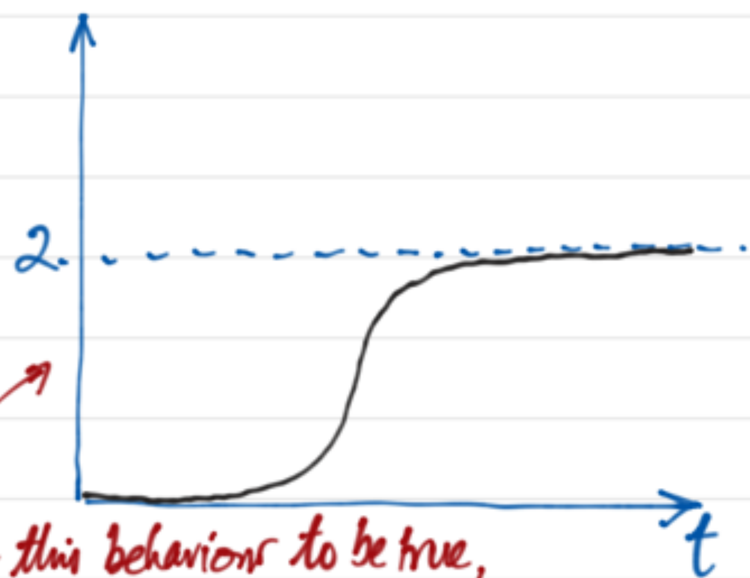
* A useful measure of the interference is the commutator

$$\langle -[V(0), W(t)]^2 \rangle$$

to make the e.v. +.

Hermitian operators

thermal expectation value.




taking, for concreteness, $V(0) = Z_1$ and $W(t) = X_5$ say.

for this behavior to be true,
we need the operator corresponding to $W(t)$ to maximally not commute with $V(0)$ $\Rightarrow W(t) = e^{iHt} W e^{-iHt} \sim \Sigma\, \sigma\sigma\sigma\sigma \ldots$

# LECTURE 2 SOME TOOLS

* Having spent some time introducing the SYK model of disordered (fermionic) quantum mechanics, we would now like to ask:

How can we extend this model to higher dimensions, say 1+1?

* There are many approaches to extend SYK to higher dimensions:

→ SYK chains [Gu, Xi, Stanford]

→ Nonstandard Kinetic terms [Turiaci-Verlinde]

→ Random Thirring model [Berkooz et al.]

But this is tricky because:

→ In $D=1$, a canonical fermion has

$[\psi]=0$ so the 4-fermi interaction is relevant and in the IR the theory flows to a strongly interacting SYK phase

* In $D=2$, $[\psi]=\frac{1}{2} \Rightarrow$ a 4-fermi interaction is marginal while $q(>4)$-fermi interactions are irrelevant.
$\Rightarrow$ Hard to get SYK-like physics

* What about bosons? Since $[\phi]=0 \Rightarrow$ random $q$-boson interactions are all relevant!

* So, we might consider a model with action

$$I=\int d^2x\left[\frac{1}{2}(\nabla\phi_i)^2+J_{i_1\cdots i_q}\phi_{i_1}\cdots\phi_{i_q}\right]$$

But the potential has -ve directions and the theory is unstable!

* Or, we might consider something like

$$I = \int d^2x \left[ \frac{1}{2} (\nabla \phi_i)^2 + \sum_a \left( C^a_{i_1 \cdots i_q} \phi_{i_1} \cdots \phi_{i_q} \right)^2 \right]$$

random tensor

This model is well-defined BUT does not seem to flow to an SYK-like fixed point in the IR.

* The goldilocks model will turn out to be a random supersymmetric CFT

$$I = \int d^2x \, d^2\theta \left[ \frac{1}{2} D_{\bar\theta} \Phi_i D_\theta \Phi_i + i C_{i_1 \cdots i_q} \Phi_{i_1} \cdots \Phi_{i_q} \right]$$

with $\mathcal{N}=1$ supersymmetry in 1+1 dim.

* We will expand much more on this new model but first we are going to need some tools.

## 2.1. Superconformal Symmetry in 1 Dimension.

* In this section we will summarize some salient features of superconformal symmetry in $D=1$ in a way that will facilitate our extention to $D=2$ in a straightforward way.

* The conformal group in 1 spacetime dimension is $SL(2,\mathbb{R})$. This extends to the 1-dimensional superconformal group $Osp(1/2)$ with the addition of two fermionic generators so that the algebra of $osp(1/2)$ is defined by

$$[L_m, L_n] = (m-n) L_{m+n} \qquad m,n = \pm 1, 0$$

$$\{G_r, G_s\} = 2 L_{r+s} \qquad r,s = \pm \tfrac{1}{2}.$$

$$[L_m, G_{\pm 1/2}] = \left(\frac{m}{2} \mp \frac{1}{2}\right) G_{m \pm 1/2}$$

* the algebra is realized by acting on a 1-dim superspace of functions $\Phi(t,\theta)$ with the differential operators

$$L_0 = -t\partial_t - \tfrac{1}{2}\theta\partial_\theta - \Delta$$

conformal dim of $\Phi$.

$$L_{-1} = -\partial_t$$

$$L_{+1} = -t^2\partial_t - t\theta\,\partial_\theta - 2\Delta t$$

$$G_{-1/2} = \partial_\theta - \theta\partial_t$$

$$G_{+1/2} = t\partial_\theta - t\theta\partial_t - 2\Delta\theta$$

* The quadratic Casimir of $osp(1|2)$

$$C = L_0^2 - \tfrac{1}{2}\{L_{-1}, L_{+1}\} - \tfrac{1}{4}[G_{-1/2}, G_{+1/2}]$$

has a 1-particle realization of

$$C = \Delta^2 - \tfrac{1}{2}\Delta$$ ← c.f. $\Delta^2 - \Delta$ for just bosons.

* In what follows, we will need the Casimir $C_{12}$ for a 2-particle system with coords. $(t_1, \theta_1)$ and $(t_2, \theta_2)$. Assuming the same conformal dimension $\Delta = 0$;

$$\begin{cases} C_{12}\Phi(t_1,\theta_1,t_2,\theta_2) = -\left(t_{12} - \frac{1}{4}\theta_1\theta_2\right)^2 \partial_1\partial_2\Phi \\ +\frac{1}{2}\left(t_{12} - \theta_1\theta_2\right)\partial_{\theta_1}\partial_{\theta_2}\Phi + t_{12}\left(\theta_2 - \frac{1}{2}\theta_1\right)\partial_{\theta_2}\partial_1\Phi \\ -t_{12}\left(\theta_1 - \frac{1}{2}\theta_2\right)\partial_{\theta_1}\partial_2\Phi \end{cases}$$

$\equiv t_1 - t_2$

for more general $\Delta$, conjugate by $(t_{12} - \theta_1\theta_2)^{2\Delta}$

* Much of what we will study will derive from the normalized 4-point function of the theory. For a 4-particle system with coordinates $(t_1, \theta_1) \ldots (t_4, \theta_4)$ the 4-point function must be:

→ a superconformally invariant $f^{n}$ of the

$(t_i, \theta_i)$ and

$\rightarrow$ Grassman even

* These two facts together imply that it will only depend on the Grassman even invariants

$$\chi \equiv \frac{\langle 1,2\rangle\langle 3,4\rangle}{\langle 1,3\rangle\langle 2,4\rangle} - \zeta$$

This is the superanalog of the cross ratio of 4 points on the real line.

$$\zeta \equiv \frac{\langle 1,2\rangle\langle 3,4\rangle + \langle 2,3\rangle\langle 1,4\rangle + \langle 3,1\rangle\langle 2,4\rangle}{\langle 1,3\rangle\langle 2,4\rangle}$$

this is nilpotent since $\zeta^2 = 0$ which can be seen by using the Osp(1|2) to fix 3 bosonic and 2 fermionic coords so that $\zeta$ is a bilinear in the remaining 2 $\theta$'s.

* The 4-point functions that we seek will be eigenfunctions of the 2-particle Osp(1|2) Casimir. so we now need only to understand how

$C_{12}$ acts on functions of the form

$$\Phi(\chi,\zeta) = F(\chi) + \zeta G(\chi)$$

* We can make things easier by using the $OSP(1|2)$ symmetry to fix $t_3 = 1$, $t_4 = \infty$ and $\theta_3 = \theta_4 = 0$

$$\Rightarrow \chi = \frac{t_1 - t_2}{t_1 - 1} \quad \text{and} \quad \zeta = \frac{\theta_1\theta_2}{1 - t_1}$$

* Then the action of $C_{12}$ on $\Phi(\chi,\zeta)$ can be written

$$\begin{pmatrix} \chi^2(1-\chi)\partial_\chi^2 - \chi^2\partial_\chi & \chi/2 \\ \frac{\chi}{2}(1-\chi)\partial_\chi^2 - \frac{\chi}{2}\partial_\chi & \chi^2(1-\chi)\partial_\chi^2 + \chi(2-3\chi)\partial_\chi - \chi + \frac{1}{2} \end{pmatrix} \begin{pmatrix} F(\chi) \\ G(\chi) \end{pmatrix}$$

$$\equiv D \begin{pmatrix} F(\chi) \\ G(\chi) \end{pmatrix}$$

* The operator $D$ is not Hermitian with respect to any positive-definite inner product so let's talk about measures for a little.

* In the bosonic case á la Maldacena-Stanford the 2-particle Casimir

$$\mathcal{C}_{12} F(t_1,t_2) = -t_{12}^2\, \partial^2_{t_1 t_2} F(t_1,t_2)$$

is Hermitian with respect to the inner product

$$\langle F|G\rangle = \int \frac{dt_1 dt_2}{t_{12}^2} F^* G. = \int_0^2 \frac{d\chi}{\chi^2} F^* G.$$

the restriction of the integral domain anticipates the $\chi \to \chi/\chi-1$ symmetry of the SYK model.

* A natural extent of this inner product to superspace is

$$\langle A|B\rangle = \int \frac{dt_1 dt_2 d\theta_2 d\theta_1}{\langle 1,2\rangle} AB$$

our fermion measure satisfies $\int d\theta_2 d\theta_1\, \theta_1\theta_2 = 1$

* This inner product:
  1) Is not positive definite and hence defined as a bilinear inner product instead of Hermitian.

2) Reduces to the bosonic case if $A$ and $B$ is $\theta$-independent and we integrate out over the $\theta_1$ and $\theta_2$ variables.

* $C_{12}$ is Hermitian with respect to this inner product in the sense that

$$\langle A(t_1,\theta_1,t_2,\theta_2) \,|\, C_{12}\, B(t_1,\theta_1,t_2,\theta_2)\rangle = \langle C_{12} A | B \rangle$$

* In order to write the inner product in terms of $OSp(1|2)$ invariants we:

1) Introduce additional variables $t_3, \theta_3, t_4$ and $\theta_4$
2) Restrict to functions $A$ and $B$ of $OSp(1|2)$ invariant
3) Gauge fix the $OSp(1|2)$ action by setting $t_1 = 0,\ t_3 = 1,\ t_4 = \infty$ and $\theta_3 = \theta_4 = 0$

* Then our inner product is, quite simply

$$\langle A | B \rangle = -\int \frac{d\chi\, d\zeta}{\chi + \zeta}\, A(\chi,\zeta)\, B(\chi,\zeta)$$

* One point to note is that the SYK model has

a symmetry under exchange of particles $1 \leftrightarrow 2$ (or $3 \leftrightarrow 4$). In terms of the $OSp(1|2)$ invariants this is a symmetry under

$$\chi \to \frac{\chi}{\chi-1} \quad \text{and} \quad \zeta \to \frac{\zeta}{\chi-1}$$

and for functions with this symmetry we can restrict the integration range from $0 \to 2$.

* A function $\Phi(\chi,\zeta) = F(\chi) + \zeta G(\chi)$ is invariant under particle exchange symmetry if $G(\chi) \to (\chi-1)G$. Combined with the transforms of the derivatives as

$$\partial_\chi \to -(\chi-1)^2 \partial_\chi \quad \text{and} \quad \partial_\chi^2 \to (\chi-1)^4 \partial_\chi^2 + 2(\chi-1)^3 \partial_\chi$$

it follows that $D$ is also invariant under the symmetry.

* Now we can return to the super-Casimir eigenvalue problem

$$D\begin{pmatrix}F\\G\end{pmatrix} = h\left(h-\tfrac{1}{2}\right)\begin{pmatrix}F\\G\end{pmatrix}$$

This is a system of $2^{nd}$ order differential eq$^{\underline{ns}}$

this is the Casimir for two particles coupled to a dim. h primary.

* To solve the eigenvalue problem we set $G(x) = \frac{h}{x}F(x)$ so that

$$x^2(1-x)\frac{d^2F}{dx^2} - x^2\frac{dF}{dx} - h(h-1)F = 0$$

this is a hypergeometric equation and condition for F to be an eigenf$^{\underline{n}}$ of the $SL(2,\mathbb{R})$ Casimir.

* General solution:

$$F(x) = c_1 x^h\, {}_2F_1(h,h,2h;x) + c_2 x^{1-h}\, {}_2F_1(1-h,1-h,2-2h;x)$$

integration constants.

* Is this consistent? The eigenvalue equation is a $2^{\underline{nd}}$ order differential equation for 2 functions

and so we would expect $\underline{\underline{4}}$ linearly independent solutions. Here's how the counting works:

→ Since the eigenvalue problem is invariant under $h \to \frac{1}{2} - h$ we could have solved it also with the independent ansatz $G(x) = \frac{(\frac{1}{2}-h)}{x} F(x)$ (replacing $h \to \frac{1}{2} - h$ everywhere)

→ The two choices of ansatz $+ c_1 + c_2 = 4$ linearly indep. solutions.

* We now need to select a subset to form a complete basis for the space of $f^{\underline{ns}}$ subject to appropriate boundary conditions. In order to do this, we will use the shadow formalism.

## 2.2. THE SHADOW FORMALISM

* The shadow representation allows us to construct the 4-pt. $f^{\underline{n}}$ in a CFT for a specified value

of the conformal Casimir without solving the Casimir eigenvalue equation directly but instead giving an integral representation of the solution.

* We want to compute:

$$\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\,\mathcal{O}'(x_3)\,\mathcal{O}'(x_4)\rangle$$ ← connected 4-point f$^{\underline{n}}$.

dimension $\Delta$ conformal primaries in $D$-dimensions.

* Strategy: → Think of $\mathcal{O}$ and $\mathcal{O}'$ as living in 2 decoupled $D$-dimensional CFTs.

→ $V \in CFT_1$ with $\dim(V) = h$ and $V' \in CFT_2$ with $\dim(V) = D-h$.

* In $CFT_1 \otimes CFT_2$ the connected 4-point f$^n$ $= 0$ but if we perturb by $\epsilon \int d^D y\, V(y)\, V'(y)$ then

$$\langle O(x_1) O(x_2) O'(x_3) O'(x_4) \rangle$$

$$= \epsilon \int d^D y \langle O(x_1) O(x_2) V(y) \rangle \langle V'(y) O'(x_3) O'(x_4) \rangle$$

This is a single-valued, conformally invariant eigenf$^{n}$ of the 2-particle Casimir $C_{12}$.

* This will allow us to construct basis f$^{\underline{ns}}$ in which to expand the full 4-point f$^{\underline{n}}$ in the SYK model.

* Let's see how this works for the SYK model in 1-D. Since there is no reason not to, we will consider the model with $q$-fold couplings where the fermionic primaries (in the large N limit) have dimension $\Delta = 1/q$. and

$$\langle \psi_i(t) \psi_{i'}(t') \rangle = \delta_{ii'} \frac{\mathrm{sgn}(t-t')}{|t-t'|^{2\Delta}}$$

normalized

disorder averaged 2-pt. f$^{\underline{n}}$.

* Usually we would want to compute the normalized 4-pt. function

$$W(t_1,t_2,t_3,t_4) = \frac{\langle \psi_i(t_1)\psi_i(t_2)\psi_j(t_3)\psi_j(t_4)\rangle}{\langle \psi_i(t_1)\psi_i(t_2)\rangle\langle \psi_j(t_3)\psi_j(t_4)\rangle}$$

This is conformally invariant but not convenient in disordered models where we want to average over we want to:

1) average over disorder
2) average over labels
3) remove the contribution from the identity in the 12 channel.
4) have a well-defined large N limit.

* To that end, whenever we refer directly to SYK models (and variants) we will refer to

$$F(t_1,t_2,t_3,t_4) \equiv N \frac{\langle \psi_i(t_1)\psi_i(t_2)\psi_j(t_3)\psi_j(t_4)\rangle'}{\langle \psi_i(t_1)\psi_i(t_2)\rangle'\langle \psi_j(t_3)\psi_j(t_4)\rangle'}$$

* In $D=1$

$$\langle \psi_i(t_1)\psi_i(t_2)V(t_3)\rangle = \frac{\mathrm{sgn}(t_1-t_2)}{|t_1-t_2|^{2\Delta-h}|t_1-y|^h|t_2-y|^h}$$

dim $\Delta$ fermionic primary

dim. $h$ bosonic primary.

with a similar 3-pt. $f^{\underline{n}}$ for the shadow CFT.

* Inserting into the shadow representation gives a contribution to F of

$$\Psi_h(t_1,t_2,t_3,t_4)$$
$$=\frac{1}{2}\int dy\,\frac{|t_1-t_2|^h\,|t_3-t_4|^{1-h}}{|t_1-y|^h|t_2-y|^h|t_3-y|^{1-h}|t_4-y|^{1-h}}$$

* Notes:

1) The integral converges for $0<\mathrm{Re}\,h<1$ but can be analytically cont. to a meromorphic

function on the whole complex h-plane.

2) Conformal invariance implies that $\Psi$ is a $f^{\underline{n}}$ only of $\chi$.

$$\Psi(\chi) = \frac{1}{2}\int_{-\infty}^{\infty} dy \, \frac{|\chi|^h}{|y|^h |\chi-y|^h |1-y|^{1-h}}$$

* Actually, from this representation we can deduce most of the properties of the Casimir eigenvectors. Since these will have analogs in all the models we are interested in (1D, 2D, SUSY etc) it is worth spending some time elaborating:

1) Changing variables from $y \to y/y-1$ this is a $GL(2,\mathbb{R})$ transform.

$$\Rightarrow \Psi_h(\chi) = \Psi_h\left(\chi/\chi-1\right)$$

reflecting symmetry under the exchange $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$.

2) With $(t_1, t_2, t_3, t_4) = (0, \chi, 1, \infty)$ the change of variables $y \to \chi/y$

$$\Rightarrow \Psi_h(\chi) = \Psi_{1-h}(\chi)$$

reflecting the fact that $\Psi_h$ is an eigenf$^{\underline{n}}$ of $C_{12}$ with eigenvalue

$$h(1-h)$$

which is invariant under $h \to 1-h$.

* In order to understand the OPE of the 4-pt function we need to know the behaviour of $\Psi_h$ near $\chi \gtrsim 0$. Again, this behaviour can be read off from the integral representation as

$$\Psi_h(\chi) \sim \begin{cases} \chi^h \dfrac{\tan \pi h}{2 \tan(\pi h/2)} \dfrac{\Gamma^2(h)}{\Gamma(2h)} & 0 < \mathrm{Re}(h) < 1/2 \\[2ex] \chi^{1-h} \dfrac{\tan \pi(1-h)}{2 \tan(\frac{\pi}{2}(1-h))} \dfrac{\Gamma^2(1-h)}{\Gamma(2(1-h))} & 1/2 < \mathrm{Re}\, h < 1 \end{cases}$$

* Note that the eigenvalue equation

$$C_{12}\Psi_h = h(1-h)\Psi_h$$

is just the hypergeometric equation with solution

$$\Psi_h = \tilde{A}(h)F_h(x) + \tilde{B}(h)F_{1-h}(x) \quad \text{on } 0<x<1$$

where:

$$\tilde{A}(h) = \frac{\tan \pi h}{2\tan(\pi h/2)} \frac{\Gamma^2(h)}{\Gamma(2h)}$$

$$\tilde{B}(h) = \tilde{A}(1-h)$$

and $F_h(x) \equiv x^h {}_2F_1(h,h,2h;x)$ is the usual $SL(2,\mathbb{R})$ conformal block.

* Again, this expansion can be directly established from the integral representation without any reference to the differential equation itself.

* Before moving on to extending the shadow construction to higher dimensions, I want to say something about the integrals of the representation that we will see a lot of. Since such integrals were first popularized in the high energy literature by Kawai-Lewellyn and Tye in the study of relations between open and closed string amplitudes, we might as well call them KLT integrals.

* We start with the dimensional analysis observation that

2-component vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\int \frac{d^2\vec{x}}{|\vec{x}|^{2a}} e^{i\vec{p}\cdot\vec{x}} \sim |\vec{p}|^{2a-2} \qquad \vec{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$= c(a)\, |\vec{p}|^{2a-2}$$

This coeff. can be found by doing a Gaussian integral on both sides and found to be

$$c(a) = \frac{\pi}{2^{2a-2}} \frac{\Gamma(1-a)}{\Gamma(a)}$$

* Now let's change from Cartesian $\to$ Complex coordinates $x \equiv x_1 + i x_2$ and $p = p_1 + i p_2$ and, for example,

$|\vec{x}|^2 = x\bar{x}$ and $\vec{p}\cdot\vec{x} = \frac{1}{2}(p\bar{x} + \bar{p}x)$. Then, taking derivatives of the integral with respect to $p$ repeatedly

$$\int \frac{d^2x}{|x|^{2a}} \left(\frac{i}{2}\bar{x}\right)^n e^{ip\cdot x} = C(a,n)\, p^{a-1-n}\, \bar{p}^{a-1}$$

and $C(a,n) \equiv \dfrac{\pi}{2^{2a-2}} \dfrac{\Gamma(1-a)}{\Gamma(a-n)}$

* Finally, solving for $p$ and relabeling we get a Fourier representation

$$x^{b+n}\bar{x}^{b} = \frac{(i/2)^n}{C(b+1+n,n)} \int \frac{d^2p\, e^{i\vec{p}\cdot\vec{x}}}{p^{b+1}\,\bar{p}^{b+1+n}}$$

* Now we can evaluate the desired KLT integral

$$\int d^2x\, x^{a+n}\bar{x}^{a}(1-x)^{b+m}(1-\bar{x})^{b}$$

we get this by applying the transform to each pair and then integrate over x.

$$= \pi \frac{\Gamma(1+a)\Gamma(1+b)\Gamma(-1-a-b-m-n)}{\Gamma(2+a+b)\Gamma(-b-m)\Gamma(-a-n)}$$