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# Theory of Exclusive Nonleptonic $B$ Meson Decays

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**Abstract**

We show that the controversies on the gauge dependence and the infrared singularity emerged in the generalized factorization approach for nonleptonic heavy meson decays within the framework of the operator product expansion can be resolved by perturbative QCD factorization theorem. Gauge invariance of the decay amplitude is maintained under radiative corrections by assuming on-shell external quarks. For on-shell external quarks, infrared poles in radiative corrections have to be extracted using the dimensional regularization. These poles, signifying nonperturbative dynamics of a decay process, are absorbed into bound-state wave functions. Various large logarithms produced in radiative corrections are summed to all orders into the Wilson and Sudakov evolution factors. The remaining finite part gives a hard subamplitude. A decay rate is then factorized into a convolution of the hard subamplitude, the Wilson coefficient, and the Sudakov factor with the bound-state wave functions, all of which are well-defined and gauge invariant.

## I. INTRODUCTION

The effective Hamiltonian is the standard starting point for describing the nonleptonic weak decays of hadrons. Consider the decay $\overline{B}^0 \to D^+\pi^-$ as an example. The relevant effective $\Delta B = 1$ weak Hamiltonian is

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}} V_{cb}V_{ud}^*\Big[c_1(\mu)O_1(\mu) + c_2(\mu)O_2(\mu)\Big], \tag{1}$$

where

$$O_1 = (\bar{c}b)_{V-A}(\bar{d}u)_{V-A}, \qquad O_2 = (\bar{d}b)_{V-A}(\bar{c}u)_{V-A}, \tag{2}$$

with $(\bar{q}_1q_2)_{V\pm A} \equiv \bar{q}_1\gamma_\mu(1\pm\gamma_5)q_2$. In order to ensure the renormalization-scale and -scheme independence for the physical amplitude, the matrix elements of 4-quark operators have to be evaluated in the same renormalization scheme as that for Wilson coefficients and renormalized at the same scale $\mu$.

Although the hadronic matrix element $\langle O(\mu)\rangle$ can be directly calculated in the lattice framework, it is conventionally evaluated under the factorization hypothesis so that $\langle O(\mu)\rangle$ is factorized into the product of two matrix elements of single currents, governed by decay constants and form factors. In spite of its tremendous simplicity, the naive factorization approach encounters two principal difficulties. First, it fails to describe the color-suppressed weak decay modes. For example, the predicted decay rate of $D^0 \to \overline{K}^0\pi^0$ by naive factorization is too small by two orders of magnitude compared to experiment. Second, the hadronic matrix element under factorization is renormalization scale $\mu$ independent as the vector or axial-vector

current is partially conserved. Consequently, the amplitude $c_i(\mu)\langle O\rangle_{\rm fact}$ is not truly physical as the scale dependence of Wilson coefficients does not get compensation from the matrix elements.

A plausible solution to the aforementioned scale problem is to extract the $\mu$ dependence from the matrix element $\langle O(\mu)\rangle$, and combine it with the $\mu$-dependent Wilson coefficients to form $\mu$-independent effective Wilson coefficients. After making a physical amplitude explicitly $\mu$-independent, the factorization hypothesis is applied to the hadronic matrix elements. However, the $\mu$-evolution factor extracted from $\langle O(\mu)\rangle$ depends on an infrared cutoff, which is originally implicit in $\langle O(\mu)\rangle$. Since an off-shell external quark momentum is usually chosen as the infrared cutoff, the $\mu$-evolution factor also contains a gauge dependent term accompanied off-shell external quarks. Therefore, this solution, though removes the scale and scheme dependence of a physical amplitude in the framework of the factorization hypothesis, often introduces the infrared cutoff and gauge dependence.

In this report we shall show that the above controversies can be resolved by perturbative QCD (PQCD) factorization theorem [1]. In this formalism, partons, *i.e.*, external quarks, are assumed to be on shell, and both ultraviolet and infrared divergences in radiative corrections are isolated using the dimensional regularization. Because external quarks are on shell, gauge invariance of the decay amplitude is maintained under radiative corrections. The obtained ultraviolet poles are subtracted in a renormalization scheme, while the infrared poles are absorbed into nonperturbative bound-state wave functions. Various large logarithms produced in radiative corrections are summed to all orders into the Wilson and Sudakov evolution factors. The remaining finite piece is grouped into a hard decay subamplitude. The decay rate is then factorized into the convolution of the hard subamplitude, the Wilson coefficient, and the Sudakov factor with the bound-state wave functions, all of which are well-defined and gauge invariant. The partition of the nonperturbative and perturbative contributions is quite arbitrary. Different partitions correspond to different factorization schemes. However, the decay rate, as the convolution of the above factors, is independent of factorization schemes as it should be.

## II. GAUGE DEPENDENCE AND INFRARED SINGULARITY

The aforementioned scale problem with naive factorization can be circumvented in the following approach. It is postulated that $\langle O(\mu)\rangle$ is related to the tree-level hadronic matrix element via the relation $\langle O(\mu)\rangle = g(\mu)\langle O\rangle_{\rm tree}$ and that $g(\mu)$ is independent of the external hadron states. Then schematically we can write

$$c(\mu)\langle O(\mu)\rangle = c(\mu)g(\mu)\langle O\rangle_{\rm tree} \equiv c^{\rm eff}\langle O\rangle_{\rm tree}. \tag{3}$$

The factorization approximation is applied afterwards to the hadronic matrix element of the operator $O$ at the tree level. Since the tree-level matrix element $\langle O\rangle_{\rm tree}$ is renormalization scheme and scale independent, so are the effective Wilson coefficients $c_i^{\rm eff}$. However, the problem is that we do not know how to carry out first-principles calculations of $\langle O(\mu)\rangle$ and hence $g(\mu)$. It is natural to ask the question: Can $g(\mu)$ be calculated at the quark level in the same way as the Wilson coefficient $c(\mu)$ ? One of the salient features of the operator product expansion (OPE) is that the determination of the short-distance $c(\mu)$ is independent of the choice of external states. Consequently, we can choose quarks as external states in order to extract $c(\mu)$. For simplicity, we consider a single multiplicatively renormalizable 4-quark operator $O$ (say, $O_+$ or $O_-$) and assume massless quarks. The QCD-corrected weak amplitude induced by $O$ in full theory is

$$A_{\rm full} = \left[1 + \frac{\alpha_s}{4\pi}\left(-\frac{\gamma}{2}\,\ln\frac{M_W^2}{-p^2} + a\right)\right]\langle O\rangle_q, \tag{4}$$

where $\gamma$ is an anomalous dimension, $p$ is an off-shell momentum of the external quark lines, which is introduced as an infrared cutoff, and the non-logarithmic constant term $a$ in general depends on the gauge chosen for the gluon propagator. The subscript $q$ in (4) emphasizes the fact that the matrix element is evaluated between external quark states. In effective theory, the renormalized $\langle O(\mu)\rangle_q$ is related to $\langle O\rangle_q$ in full theory via

$$\begin{aligned}\langle O(\mu)\rangle_q &= \left[1 + \frac{\alpha_s}{4\pi}\left(-\frac{\gamma}{2}\,\ln\frac{\mu^2}{-p^2} + r\right)\right]\langle O\rangle_q \\ &\equiv g'(\mu, -p^2, \lambda)\langle O\rangle_q,\end{aligned} \tag{5}$$

where $g'$ indicates the perturbative corrections to the 4-quark operator renormalized at the scale $\mu$. The constant term $r$ is in general renormalization scheme, gauge and external momentum dependent, which has the general expression [2]:

$$r = r^{\rm NDR,HV} + \lambda r^{\lambda}, \tag{6}$$

where NDR and HV stand for the naive dimension regularization and 't Hooft-Veltman renormalization schemes, respectively, and $\lambda$ is a gauge parameter with $\lambda = 0$ corresponding to Landau gauge. Matching the effective theory with full theory, $A_{\rm full} = A_{\rm eff} = c(\mu)\langle O(\mu)\rangle_q$, leads to

$$c(\mu) = 1 + \frac{\alpha_s}{4\pi}\left(-\frac{\gamma}{2}\ln\frac{M_W^2}{\mu^2} + d\right), \tag{7}$$

where $d = a - r$. Evidently, the Wilson coefficient is independent of the infrared cutoff and it is gauge invariant as the gauge dependence is compensated between $a$ and $r$. Of course, $c(\mu)$ is still renormalization scheme and scale dependent.

Since $A_{\rm eff}$ in full theory [Eq. (8)] is $\mu$ and scheme independent, it is obvious that

$$c'^{\rm eff} = c(\mu)g'(\mu, -p^2, \lambda) \tag{8}$$

is also independent of the choice of the scheme and scale. Unfortunately, $c'^{\rm eff}$ is subject to the ambiguities of the infrared cutoff and gauge dependence, which come along with $g'$ extracted from $\langle O(\mu)\rangle_q$. As stressed in [2], the gauge and infrared dependence always appears as long as the matrix elements of operators are calculated between quark states. Therefore, it is unreliable to define the effective Wilson coefficients by applying the existing calculations in the literature. The reason has been implicitly pointed out in [3] that "off-shell renormalized vertices of gauge-invariant operators are in general gauge dependent".

The existing problems associated with the off-shell regularization scheme are as follows:

1. When working with off-shell fermions, there exists the so-called $P$ operator [3] e.g., $\not{p}(1-\gamma_5)\otimes\not{p}(1-\gamma_5)$ which cannot be removed by the equation of motion.

2. The finite terms are external momentum dependent (see Fig. 3 of [4]) and they are obtained in some specific condition. For example, two incoming fermion legs 1,2 and two outgoing legs 3,4 with external momentum $p$ are chosen in Figs. 3a and 3c of [4,5], while legs 1,3 incoming and 2,4 outgoing with $p$ in Fig. 3b.

Hence we cannot avoid the gauge problems if adopting off-shell fermions and the finite parts of $c'^{\rm eff}$ are not well-defined. To circumvent this difficulty, we should work in a physical on-shell scheme and employ the dimensional regularization for infrared divergences. Gauge invariance of the decay amplitude is maintained under radiative corrections, and the infrared poles are absorbed into the hadronic matrix element as stated in the Introduction. Consequently, the effective coefficient $c^{\rm eff} = c(\mu)g(\mu)$ does not suffer from the gauge ambiguity.

## III. PQCD FACTORIZATION THEOREM

The one-loop contributions in full theory are ultraviolet finite because of the current conservation stated above. The existence of the infrared poles simply signifies the nonperturbative dynamics, which demands the inclusion of bound-state effects into the formalism of heavy meson decays. The standard treatment of infrared poles is to absorb them into a universal meson wave function. To absorb the infrared poles associated with the $b$ quark, such as those from the self-energy corrections, it is necessary to introduce a $B$ meson wave function. That is, we must take into account the spectator quark of the $B$ meson in order to develop a complete theory of heavy meson decays.

Accordingly, the decay amplitude $A_{\rm full}$ to one-loop in full theory can be rewritten as

$$\begin{aligned} A_{\rm full} &= 1 + \frac{\alpha_s}{4\pi}\left(\frac{c}{\epsilon_{IF}} + \gamma\ln\frac{M_W}{M_b} + \gamma'\ln\frac{M_b}{\mu_f} + a\right) \\ &= \left[1 + \frac{\alpha_s}{4\pi}\left(\gamma\ln\frac{M_W}{M_b} + \gamma'\ln\frac{M_b}{\mu_f} + a\right)\right]\left(1 + \frac{\alpha_s}{4\pi}\frac{c}{\epsilon_{\rm IR}}\right) + O(\alpha_s^2)\,, \end{aligned} \tag{9}$$

where the factorization scale $\mu_f$ arises from the dimensional regularization of infrared divergences, and the factorization of the infrared pole is performed in the minimal subtraction scheme. The anomalous dimensions of the logarithms $\ln(M_W/M_b)$ and $\ln(M_b/\mu_f)$, $\gamma$ and $\gamma'$, respectively, are different, since the latter involves an extra contribution related to the spectator quark. The factor containing the infrared pole can be formulated as a matrix element of a nonlocal operator, which is the definition of a meson wave function $\phi(\mu_f)$. A wave function, describing the amplitude that a parton carries a fraction of the meson momentum, cannot be derived in perturbation theory. It must be parametrized as a function of parton momentum fraction.

We further factorize the infrared finite part into

$$A_{\rm full} = \left[1+\frac{\alpha_s}{4\pi}\left(\gamma\ln\frac{M_W}{\mu}+a'\right)\right]\left[1+\frac{\alpha_s}{4\pi}\left(\gamma\ln\frac{\mu}{M_b}+\gamma'\ln\frac{M_b}{\mu_f}+a-a'\right)\right] \times\left(1+\frac{\alpha_s}{4\pi}\frac{c}{\epsilon_{\rm IR}}\right)+O(\alpha_s^2)\ . \tag{10}$$

The first factor, characterized by the matching scale $M_W$, is identified as the Wilson coefficient $c(\mu)$ after summing $\ln(M_W/\mu)$ to all orders using renormalization group (RG) equations. The second factor, characterized by the $b$ quark mass $M_b$, is the hard subamplitude which will be denoted by $H(M_b,\mu,\mu_f)$ below. Extending the above procedures to all orders, we obtain the factorization formula for $B$ meson (not $b$ quark) decays,

$$A_{\rm full} = c(\mu)H(M_b,\mu,\mu_f)\phi(\mu_f)\ , \tag{11}$$

which is exactly the three-scale factorization formula for exclusive nonleptonic decays derived in [6]. Note that Eq. (11) in fact denotes a convolution relation, because the momentum fractions should be integrated out.

Compared to Eq. (3), the matrix element $\langle O(\mu)\rangle$ corresponds to

$$\langle O(\mu)\rangle = H(M_b,\mu,\mu_f)\phi(\mu_f)\ . \tag{12}$$

Summing $\ln(\mu/M_b)$ in $H$ to all orders using RG equations, we obtain an evolution factor $g_1(\mu)$, whose behavior from $\mu$ to $M_b$ is governed by the same anomalous dimension as that of $c(\mu)$. Summing $\ln(M_b/\mu_f)$ in $H$ to all orders, we obtain another factor $g_2(\mu_f)$ describing the evolution from $M_b$ to $\mu_f$, whose anomalous dimension differs from that of $c(\mu)$ because of the inclusion of the dynamics associated with spectator quarks. Note that $g_2$ is part of the Sudakov evolution obtained in [6]. Hence, the $\mu$ dependence of $H$ is extracted as

$$H(M_b,\mu,\mu_f) = g_1(\mu)g_2(\mu_f)H(M_b,M_b,M_b)\ . \tag{13}$$

The combination of $c$, $g_1$, and $g_2$ leads to the effective coefficient

$$c^{\rm eff} = c(\mu)g_1(\mu)g_2(\mu_f)\ , \tag{14}$$

which is not only $\mu$ and scheme independent but also gauge invariant. The factor $g(\mu)$ in Eq. (3) can be identified as $g_1(\mu)g_2(\mu_f)$, which describes the evolution down to the factorization scale. However, $g_1(\mu)g_2(\mu_f)$ contains a matching condition at the scale $M_b$ between the Wilson and Sudakov evolutions with different anomalous dimensions. Therefore, there is an ambiguity of the matching condition: the two evolutions can also match at $rM_b$ with $r$ a constant of order unity. Obviously, Eq. (14) is subtler than the naive definition of $c^{\rm eff}$ in Eq. (3).

The matrix element $\langle O\rangle_{\rm tree}$ in Eq. (3) is then identified as

$$\langle O\rangle_{\rm tree} = H_{\rm tree}(M_b,M_b,M_b)\phi(\mu_f)\ , \tag{15}$$

where the hard subamplitude is evaluated to lowest order with one hard gluon exchange, since all large logarithms have been organized by RG equations. We emphasize that the factorization hypothesis for $\langle O\rangle_{\rm tree}$ in the conventional approach is not necessary in the PQCD formalism. The purpose of the factorization hypothesis is to simplify the decay amplitude into products of decay constants and form factors, which are

then parametrized as various models. To have a better fit to experimental data, nonfactorizable contributions, parametrized as $\chi$ [see Eq. (7)], are included. Note that $H_{\rm tree}$ in the PQCD approach includes both factorizable contributions (form factors), when the hard gluon attaches to the two quarks in a meson, and nonfactorizable contributions (octet amplitudes), when the hard gluon attaches to the quarks in different mesons. Therefore, we may compute all possible diagrams for $H_{\rm tree}$ [7] and convolute them with the same meson wave functions $\phi$. That is, we use the single parametrization, *i.e.*, the meson wave functions, for both factorizable and nonfactorizable contributions based on Eq. (15). In this sense the PQCD formalism is more systematic.

At last, we explain how to handle the nonperturbative meson wave functions with the dependence of the factorization scale $\mu_f$. It can be shown that these wave functions are universal for all decay processes involving the same mesons. For example, the $B$ meson wave function for the nonleptonic decays $B \to D^{(*)}\pi(\rho)$ and for the radiative decay $B \to K^*\gamma$ is the same. This universality can be easily understood, since a wave function collects long-distance (infrared) dynamics, which should be insensitive to short-distance dynamics involved in the decay of the $b$ quark into light quarks with large energy release. Based on the universality of wave functions, the application of factorization formulas is as follows [8]. We evaluate the Wilson and Sudakov evolutions down to a factorization scale $\mu_f$ and the hard subamplitude for a decay mode, say, $B \to K^*\gamma$, in perturbation theory. These calculations are simply performed at the quark level with infrared poles dropped (in the minimal subtraction scheme). Adjust the $B$ meson wave function such that the predictions from the relevant factorization formula match the experimental data. At this stage, we determine the $B$ meson wave function defined at the scale $\mu_f$. Then evaluate the Wilson and Sudakov evolutions down to the same scale $\mu_f$ and the hard subamplitude for another decay, say, $B \to D\pi$. Convolute them with the same $B$ meson wave function and make predictions. At this stage, there are no free parameters in the formalism. With the above strategy, the PQCD factorization theorem possesses predictive power.

## IV. CONCLUSION

In this paper we have shown how to construct a gauge invariant and infrared finite theory of exclusive nonleptonic $B$ meson decays based on PQCD factorization theorem. Gauge invariance is maintained under radiative corrections by working in the physical on-shell scheme. The infrared divergences in radiative corrections should be then isolated using the dimensional regularization. The resultant infrared poles are absorbed into the universal meson wave functions, which can be determined once for all from experimental data. The absorption of the poles associated with the $b$ quark requires the inclusion of the spectator quark into the theory. The remaining finite contributions form a hard subamplitude. Applying RG analyses to sum various large logarithms in the above factorization formula, the scale and scheme dependences are removed. Hence, in the PQCD formalism physical quantities are guaranteed to be gauge invariant, infrared finite, scale and scheme independent. By working out the evolution factor $g_1(\mu)$ explicitly, we have constructed gauge invariant, scale and scheme independent effective Wilson coefficients $c(\mu)g_1(\mu)$ at the factorization scale $\mu_f = M_b$. We have shown explicitly that $c^{\rm eff}$ are renormalization scheme and scale independent.


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