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An uncertainty relation of space-time — non-commutative geometry ? *

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Abstract

We discuss several aspects of the previously proposed space-time uncertainty relation $l_P^2 T \lesssim \delta V$, where T and δV denote an observation time and a spatial volume uncertainty, respectively. The derivation of the uncertainty relation is extended to a thought experiment with a photon instead of a stationary clock. We show that this uncertainty relation qualitatively reproduces the life time of a black hole derived from Hawking radiation. We discuss some of the interesting features of the entropy formula $S = \sqrt{EV}/l_P$, which was previously derived from the uncertainty relation. One of them is that the entropy of a spatial region surrounded by a surface should be bounded by its area, which is a weak statement of the holographic principle of 't Hooft and Susskind. We discuss the Lorentz transformation. Lastly we comment on some possible algebraic realizations of the uncertainty relation.

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1 Introduction

Despite the various excellent attempts, the cosmological constant problem remains as one of the major mysteries in physics[1]. A large cosmological constant seems to be an inevitable conclusion from the following two widely believed physical assumptions. One is that the low energy dynamics of gravity is described by the classical Einstein's general relativity, and the other is that, under some scale, our world including gravity is described by some quantum field theory. These assumptions predict a cosmological constant of at least the order of the scale below which these assumptions are valid. This scale might not be lower than GeV.

One of the possible resolutions of this apparent inconsistency would be to change the notion of the space-time itself. Of course, this would be a hopeless direction without any guidance. Since we do not presently have any experimental facts on this issue, one of the best things we can do would be to consider a thought experiment.

In our previous paper[2], we derived a measurement bound of metric, $l_P^2 T \lesssim \delta V$, through a thought experiment with a stationary clock, simply based on quantum mechanics and classical gravity[3]. Here T denotes an observation time, and δV denotes an uncertainty of volume. The precise measurement of the metric is a doubtful notion, since we need a high energy probe to minimize the quantum uncertainty, but, on the other hand, the high energy probe will disturb unavoidably the metric to be observed. This difficulty does not appear in gauge theories, since the fields do not couple directly with energy. Thus the measurement bound of metric would be regarded as an uncertainty of the space-time.

This space-time uncertainty relation can be extended to a thought experiment with a photon instead of a stationary clock. Let us consider a photon with a momentum p in a direction x. We assume that the wave function of this photon is localized with the size δx in the x-direction and δr in the transverse directions. Then the quantum has a momentum uncertainty $\delta p_r \sim 1/\delta r$ in the transverse directions. Hence it has a velocity uncertainty of order $\delta p_r/p$ in the transverse directions. (Note that there is no uncertainty of the velocity in the x-direction in the first order estimation.) Now we impose the condition that the size of the wave function in the transverse directions remain in the order of δr during the observation time T;

$$\frac{\delta p_r}{p} T \lesssim \delta r. \tag{1}$$

As in the case of a stationary clock, the gravitational field generated by the energy of the photon will disturb the observation. The condition that the disturbance remain within the order of the size of the wave function would be

$$l_P^2 p \lesssim \delta x. \tag{2}$$

Deleting the photon momentum p with (1) and (2), we obtain a space-time uncertainty relation

$$l_P^2 T \lesssim \delta V. \tag{3}$$

Since it is impossible to perform an observation more accurately than (3), it would be natural to propose that the fundamental degrees of freedom (FDF) of the space-time exists one per the space-time region $l_P^2 T \sim \delta V$.

2 Black hole life time, entropy bound and holographic principle

An interesting direct application of the uncertainty relation is the prediction of the black hole life time. The standard analysis through Hawking radiation shows that a black hole with mass M will evaporate in time $l_P^4 M^3$. The discussion with the uncertainty relation, which was originally done by Barrow [4], is as follows. Since the Schwarzschild radius of the black hole is $l_P^2 M$, the volume associated with it would be $V_{BH} = (l_P^2 M)^3$. According to the proposition in the previous section, the maximum time during which there is at least one FDF in this black hole is estimated as $V_{BH}/l_P^2 = l_P^4 M^3$. Since there can be no FDF for a time period longer than this, the black hole should evaporate within this time period, which agrees with the standard estimation.

An indirect application of the uncertainty relation is the maximum number of the FDFs in a region. Using a standard uncertainty relation of quantum mechanics, $\delta E > 1/T$, the uncertainty relation (3) can be rewritten in the form

$$\delta E \delta V \gtrsim l_P^2. \tag{4}$$

As discussed in [2], the uncertainty relation (4) yields the number of the FDFs in a region with a volume V and an energy E to be suppressed by

$$N \lesssim \frac{\sqrt{EV}}{l_P}. \tag{5}$$

This suppression of the number of FDFs can be smoothly related to the holographic principle [5] as follows. Let us consider a physical system with a total energy E and a size R. Since this system is a black hole or not, the Schwarzschild bound $l_P^2 E \le R$ holds. Then

$$\frac{\sqrt{EV}}{l_P} \lesssim \frac{R^2}{l_P^2} \sim \frac{A}{l_P^2}, \tag{6}$$

where A denotes the area of a sphere with a size R. Thus the number of the FDFs in the region is always suppressed by the area of a surface surrounding it. This can be understood as a weak version of the holographic principle [5].

If we assume that there are a finite number of excitations associated to each FDF, the entropy contained in a region is suppressed by a quantity proportional to the number of FDFs in it. Hence

$$S \lesssim \frac{\sqrt{EV}}{l_P}. \tag{7}$$

For a black hole, $l_P^2 E \sim R$ holds. If we assume that a black hole is a saturated case of (7), then this agrees qualitatively with the Bekenstein-Hawking entropy formula.

3 Thermodynamics

In this section, we shall show a peculiarity of the maximum entropy $S_{max} = \sqrt{EV}/l_P$. Let us consider a general case $S = E^\alpha V^\beta$, and discuss its consistency. We assume that α and β are real numbers, and we put $l_P = 1$ for simplicity. If we impose that the entropy should be an extensive quantity, we obtain

$$\alpha + \beta = 1, \tag{8}$$

since an energy and a volume are extensive.

Another condition can be obtained from the velocity of the "sound" in a perfect fluid with the entropy formula $E^\alpha V^\beta$. The first law of thermodynamics, $dE = TdS - PdV$, implies that the pressure is proportional to the energy density: $P = \frac{\beta}{\alpha}\frac{E}{V}$. The square root of the proportional constant is the velocity of the sound in the perfect fluid. Hence, if we impose that the sound velocity should not be larger than the speed of light, we obtain

$$\beta \le \alpha. \tag{9}$$

The last condition is given by the consistency with the Bekenstein-Hawking entropy formula. Let us consider a system with an energy E and a size R, and assume that the system almost saturates the Schwarzschild bound:

$$E < R < E + \Delta E, \tag{10}$$

where ΔE is a small amount of energy. When ΔE is added to the system, the system forms a black hole. ¿From the second law of thermodynamics, the entropy should not decrease in this process. Hence

$$S_{BH}(E + \Delta E) = (E + \Delta E)^2 \geq E^\alpha V^\beta + \Delta S, \tag{11}$$

where ΔS denotes the entropy associated to the small energy ΔE. Now we take a limit $\Delta E \to 0$, then $\Delta S \to 0$, and

$$E^2 \geq E^{\alpha+3\beta}. \tag{12}$$

Since we are considering a macroscopic energy $E > 1$, (12) yields

$$\alpha + 3\beta \leq 2. \tag{13}$$

When we solve (8) and (9), we obtain

$$\alpha \geq \frac{1}{2}. \tag{14}$$

This result can be also derived from (8) and (13). The entropy formula $S_{max} = \sqrt{EV}/l_P$ is the case saturating (14). Thus the sound velocity is the speed of light, and, as discussed in the previous section, the entropy formula agrees with the Bekenstein-Hawking formula in case $E = R$.

Now let us consider a mixed system of a fluid with the entropy formula $S_{max} = \sqrt{E_1 V_1}/l_P$ and one with $E_2^\alpha V_2^{(1-\alpha)}$ ($\alpha > 1/2$), which is consistent with (14). We assume that these two fluids can be converted to each other, and that the total energy $E = E_1 + E_2$ and the total volume $V = V_1 + V_2$ are conserved. The system becomes thermodynamically stable when the temperatures and the pressures agree with each other. However one finds that there is no solution for general E and V, and all the energy and volume are absorbed into the fluid with the entropy formula S_{max}.

4 Lorentz transformation and string theory

Under a Lorentz transformation, a spatial length in the direction of a boost is contracted, while a time interval get longer with the inverse proportion of this Lorentz contraction. Thus the uncertainty relation (3) is not invariant under the Lorentz transformation. There is also another aspect which shows the violation of the Lorentz transformation. As discussed in [2], we need an observation time longer than the spatial length to detect: $T > \delta l$. Substituting this and $\delta V \sim (\delta l)^3$ into the uncertainty relation (3), we obtain

$$\delta l \geq l_P. \tag{15}$$

This implies the Planck length is the minimal length detectable. The existence of such a minimal length is not consistent with the Lorentz contraction. One may obtain a smaller scale by a Lorentz transformation. Hence the Lorentz transformation needs to become fuzzy in a scale near where the uncertainty relation (3) is saturated.

The violation of the Lorentz transformation in the uncertainty relation (3) can be traced to the equation $p < \delta x$ in the derivation of the first section (For the case with a stationary clock, this is $m < \delta x$, where m is the clock mass.). In this equation, the left-hand side and the right-hand side transform differently. Since the derivation of the uncertainty relation is merely based on quantum mechanics and classical gravity, the usage of the inequality $p < \delta x$ ($m < \delta x$) may be questioned. However, in fact, this inequality is not just derived from these elementary theories, but is derived from an intuitive argument that the uncertainty caused by gravitation should be negligible over the scale δx. To derive this inequality more rigorously, we need a theory containing quantum gravity, and presently it is impossible. But we have a good suggestion from string theory. The longitudinal size of a string is proportional to its longitudinal momentum, and so an abnormal behavior under a Lorentz boost is pointed out [6]. Thus there would be an interesting possibility that the Lorentz invariance does not hold in its primitive sense in quantum gravity [7].

5 Summary and discussions

In this paper, we see that the uncertainty relation (3) represents qualitatively very well the quantum aspects of space-time. Considering the fact that the uncertainty relation has a very simple form and it is derived from a simple thought experiment based on quantum mechanics and classical gravity, these successes are quite remarkable. The uncertainty relation seems to catch the essential properties of the quantum space-time.

Now let us see whether our uncertainty relation can be derived from the dissipation relation of a matter, $E = \sqrt{p^2 + m^2}$, and the existence of a minimum length, $\delta x, \delta y, \delta z > l_P$. We find

$$\delta E \delta x \sim \frac{p}{E} \delta p \delta x > \frac{p}{E} < 1. \tag{16}$$

Hence $\delta E \delta x \delta y \delta z$ can be larger or smaller than l_P^2, and eq. (4) cannot be derived[1] . The situation does not change when we assume $\delta y \delta z > l_P^2$, which is motivated from the holographic principle. Moreover our uncertainty relation is different from the stringy ones listed in the followings:

[1] I would like to thank A. Hashimoto and H. Suzuki for the discussions on this point.

- $\delta t \delta x \gtrsim {l_s}^2$ [8],
- $\delta x \delta p \gtrsim 1 + {l_s}^2 \delta p^2$ [9],
- $\delta t \delta x_1 \cdots \delta x_p \gtrsim g_s l_s^{p+1}$ [10].

Thus our uncertainty relation is new.

There are some possibilities of the algebraic realization of this uncertainty relation[2] . Let us consider the 2+1 dimensional case. The uncertainty relation (3) is $l_P T < \delta x \delta y$. Thus we may propose

$$2 l_P T = i[x, y]. \tag{17}$$

By the standard procedure in quantum mechanics, this leads to

$$l_P \langle T \rangle \leq \sqrt{\langle \delta x^2 \rangle \langle \delta y^2 \rangle}, \tag{18}$$

which could be understood as a more rigorous representation of the uncertainty relation. In four dimensions, new ingredients seems to be needed. We may put the uncertainty relation into the form

$$l_P^2 T = N(x, y, z), \tag{19}$$

where $N(x, y, z)$ is a generalization of the usual commutation. The idea by Nambu [11] and the quantum version [12] may be helpful for its realization.

Finally, in the derivation of the uncertainty relation, we have ignored interactions other than gravity. As was discussed in [13], if we take into account other interactions, the derivation breaks down. This is also expected in the discussion of black hole life time in section 2, because a BPS black hole is stable and the discussion should not be applicable for this case. Hence our uncertainty relation is at most just for a pure gravitational system. How the uncertainty relation can be generalized to a system with other interactions is an open question.

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