Review on Attractors (I)

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Abstract

This is a small review of attractor mechanism and its implication on the number theory.

1 Attractor mechanism of six-dimensional strings

In this section we shall give a review of the attractor mechanism in supergravity. As an example we shall consider the chiral $N = 2$ supergravity in six dimensions[1]. The black string solutions obey an "attractor equation" exhibited below.

The chiral $N = 2$ supergravity has 16 supercharges. There are two possible supermultiplets in this theory. Above all there must be a gravity multiplet[2],

$$e_{\mu}^{\underline{\mu}}, \psi_{\mu a}, B^{I}_{(L)\mu\nu} (I = 1, \ldots, 5),$$

and we can incorporate an arbitrary number of tensor multiplets

$$B_{(R)\mu\nu}, \chi_a \quad \text{and 5 scalars.}$$

Here the gravitinos are left-handed, while the tensorinos are right-handed[3]:

$$\gamma^7 \psi_{\mu a} = \psi_{\mu a} \ , \ \gamma^7 \chi_a = -\chi_a.$$

The sets of two-form potentials $B_{(L,R)\mu\nu}$ satisfy a certain self-duality condition given below.

Classically the number of tensor multiplets is arbitrary, but anomaly cancellation requires the number to be 21. In this case the theory is regarded as the $K3$ compactification of ten-dimensional type IIB supergravity. We shall hereafter focus on this case.

It is convenient to formulate the theory in a way in which the symmetry

$$SO(5, 21)_{\text{global}} / SO(5) \times SO(21)_{\text{local}}$$

is manifest. To do this, let us introduce a $(5 + 21)$-dimensional matrix V. The local $SO(5) \times SO(21)$ acts on the left of V, while the global $SO(5, 21)$ acts on the right. This matrix represents a point on the coset space $SO(5, 21)/SO(5) \times SO(21)$. The dimension of this space is equal to the number of scalar fields in this theory.

We denote the components of V as follows:

$$V = \begin{pmatrix} u^i_I & v^i_M \\ w^m_I & x^m_M \end{pmatrix} \ , \ V^T \eta V = \eta \ , \quad \eta = \text{diag}((+)^5, (-)^{21}) \tag{1}$$

[1] We largely follow the conventions of [1]

[2] We denote the curved(flat) indices without(with) an underline

[3] γ^7 is defined as $\gamma^{\underline{\alpha\beta\gamma\delta\epsilon\tau}} = \varepsilon^{\underline{\alpha\beta\gamma\delta\epsilon\tau}} \gamma^7$

$$\partial_\mu V \cdot V^{-1} = \begin{pmatrix} Q_{\mu\ j}^{\ i} & P_{\mu\ n}^{\ i} \\ (P_\mu^T)^m_{\ j} & S_{\mu\ n}^{\ m} \end{pmatrix} \tag{2}$$

The equation of motion imposes the self-duality condition on the set of three-form field strengths in the following way. Define

$$\begin{pmatrix} K^i_{(L)\mu\nu\rho} \\ K^m_{(R)\mu\nu\rho} \end{pmatrix} = V \begin{pmatrix} 3\partial_{[\mu} B^{\ i}_{(L)\nu\rho]} \\ 3\partial_{[\mu} B^{\ m}_{(R)\nu\rho]} \end{pmatrix},$$

then they satisfy the following conditions:

$$(*K_{(L)})^i_{\mu\nu\rho} = \frac{1}{6}\varepsilon_{\mu\nu\rho\lambda\sigma\tau} K^{i,\lambda\sigma\tau}_{(L)} = K^i_{(L)\mu\nu\rho}\ , \quad (*K_{(R)})^m_{\mu\nu\rho} = -K^m_{(R)\mu\nu\rho}.$$

The SUSY transformation law of fermions are given by

$$\delta\psi_{\mu a} = D_\mu \varepsilon_a + \frac{1}{4} K_{(L)\mu\nu\rho}^{\ \ i} \gamma^{\nu\rho} (\Gamma^i)_a^{\ b} \varepsilon_b \tag{3}$$

$$\delta\chi_a^m = \frac{1}{2}\gamma^\mu P_\mu^{im} (\Gamma^i)_a^b \varepsilon_b - \frac{1}{12}\gamma^{\mu\nu\rho} K_{(R)\mu\nu\rho}^{\ \ m} \varepsilon_a \tag{4}$$

Here $i = 1, \ldots, 5$ and $m = 1, \ldots, 21$, respectively. Indices a, b are $SO(5) \simeq USp(4)$ spinors. Γ^i is the hermitian matrix obeying $SO(5)$ Clifford algebra.

We shall now turn to find the black string solutions of this theory. We shall limit ourselves to supersymmetric ones, therefore all we have to do is to find the backgrounds on which the above SUSY transformation of fermions vanishes for certain nonzero ε.

We take the following ansatz:

$$\begin{aligned} -ds^2 &= -A^2 dt^2 + A^2 d\sigma^2 + B^2(dr^2 + r^2 d\Omega^2) \\ & \quad d\Omega^2 = d\theta^2 + \cos^2\theta\phi^2 + \sin^2\theta d\xi^2 \end{aligned} \tag{5}$$

For later convenience, we introduce the following basis of tangent frame:

$$\begin{array}{llllllll} \varpi^{\underline{t}} &= Adt &,& \varpi^{\underline{r}} &= Bdr &,& \varpi^{\underline{\phi}} &= Br\cos\theta d\phi \\ \varpi^{\underline{\sigma}} &= Ad\sigma &,& \varpi^{\underline{\theta}} &= Brd\theta &,& \varpi^{\underline{\xi}} &= Br\sin\theta d\xi \end{array} \tag{6}$$

We assume that the string worldsheet is stretching along the $t\sigma$-plane. Then the three-form field strengths have only the following components:

$$\varpi^{\underline{t\sigma r}} = A^2 B dt d\sigma dr\ , \quad \varpi^{\underline{\theta\phi\xi}} = B^3 r^3 \sin\theta\cos\theta d\theta d\phi d\xi \tag{7}$$

We assume that the three-form field strengths take the following form:

$$\begin{aligned} K^i_{(L)} \equiv \frac{1}{6} K_{(L)\underline{\alpha\beta\gamma}} \varpi^{\underline{\alpha\beta\gamma}} &= k^i_{(L)}(\varpi^{\underline{t\sigma r}} + \varpi^{\underline{\theta\phi\xi}}) \\ K^m_{(R)} \equiv \frac{1}{6} K_{(R)\underline{\alpha\beta\gamma}} \varpi^{\underline{\alpha\beta\gamma}} &= k^m_{(R)}(\varpi^{\underline{t\sigma r}} - \varpi^{\underline{\theta\phi\xi}}) \end{aligned} \tag{8}$$

The equations are easily solved in a gauge $Q_{\mu\ j}^{\ i} = 0$. They yield the following conditions:

$$AB = const = 1 \tag{9}$$

$$P_u^{im} = -\frac{k^i_{(L)} k^m_{(R)}}{|k_{(L)}|} \tag{10}$$

$$\partial_u k^i_{(L)} = -|k_{(R)}|^2 |k_{(L)}|^{-1} k^i_{(L)} \tag{11}$$

$$\partial_u k^m_{(R)} = -|k_{(L)}| k^m_{(R)} \qquad (u = r^{-2}) \tag{12}$$

Hence both of the left- and right-moving components of k reduce their lengths as they approach to $u = \infty (r = 0)$ keeping their orientations fixed. At $u = \infty$ k has only the left-moving components. This is the attractor mechanism.

2 Attractor $K3$'s and AdS_3/CFT_2 correspondence

The black string solutions of the previous section are particularly interesting, as they have the $AdS_3 \times S^3$ near-horizon geometry. Therefore the theory of gravity or the string theory on such backgrounds are expected to be dual to certain two-dimensional CFT's defined on the boundary of AdS_3. This correspondence has been extensively studied in relation to the physics of certain black holes in five dimensions. Now we are expecting that the boundary CFT is the superconformal sigma model on the moduli space of instantons on $K3$. In some cases the moduli space is given by the Hilbert scheme of n-points on $K3$, where n is given later.

Among various attempts to verify the duality, Dijkgraaf proved the correspondence between the moduli spaces of the two theories[3]. In this section we shall give a brief review of his work.

His statement is:

(i) The moduli space of sigma models on "attractor $K3$'s" with charge vector v is the same as the moduli space of HyperKähler(HK) structures on the Hilbert scheme of n-points on $K3$. Here n is given by $n = \frac{1}{2}v^2 + 1$.
(ii) The moduli space of string theory on "attractor $K3$'s" with charge vector v is the same as the moduli space of sigma models on the Hilbert scheme of n-points on $K3$.

Here we should know what the "attractor $K3$'s" are. They are defined in the following way. Consider the $K3$ compactification of IIB supergravity. The black string solutions in this theory were discussed in the previous section. Generically they have the r-dependent scalar fields. As the scalars are parameterizing certain geometrical structure of the internal $K3$ (as well as other informations), we imagine that these solutions have the internal $K3$ whose structure varies as a function of r. However, due to the attractor mechanism, the values of scalars right at $r = 0$ is (partially) fixed by the charges of the black string. As a consequence, the internal $K3$ at $r = 0$ must satisfy the attractor condition and cannot be arbitrary. We call these $K3$'s as the attractor $K3$'s.

2.1 Statement (i)

We first give an explanation of the first statement. We shall begin with the analysis of the sigma models on an attractor $K3$.

As is well-known, the moduli space of sigma models on $K3$ is given by

$$\frac{O(3,19)}{O(3)\times O(19)} \times \mathbf{R}^{3,19} \times \mathbf{R}_+ \simeq \frac{O(4,20)}{O(4)\times O(20)} \tag{13}$$

where the three factors in the LHS correspond to the HK structures, B-fields and the volume. We can interpret the RHS as the space of possible embeddings of a positive 4-plane W in $\mathbf{R}^{4,20}$. In fact there is a simple and easy way to do this. Using the quaternionic notation we can write W as

$$W = \exp(B + \mathbf{i}\omega_1 + \mathbf{j}\omega_2 + \mathbf{k}\omega_3), \tag{14}$$

or, in the usual notation, W is spanned by

$$(0, \vec{\omega}, B\wedge\vec{\omega}) \ , \quad (1, B, \frac{1}{2}B^2 - V). \tag{15}$$

Here we use the normalization $\omega_a \wedge \omega_b = \frac{2}{3}\delta_{ab}V$. The above four vectors are indeed positive with respect to the following quadratic form

$$v\cdot v = \int_{K3} (v^2\wedge v^2 - 2v^0\wedge v^4). \tag{16}$$

Here v^i means the projection of v to the i-th cohomology.

An element in the moduli space of sigma models on $K3$ is represented by a $(4+20)$-dimensional matrix $\hat{V}$. It acts on a $(4+20)$-dimensional vector space which we identify with $H^*(K3, \mathbf{R})$. It contains an even self-dual lattice $\Gamma^{4,20} \simeq H^*(K3, \mathbf{Z})$. This lattice is spanned by the D-brane charge vectors v.

Obviously we can embed the above matrix and the charge vector into the problem of the previous section. Namely we are freezing NS-brane charges and RR-scalars, thereby reducing the dimension of the charge lattice by $(1+1)$. Then the attractor condition puts a constraint on the value of the moduli at $r=0$. It simply requires the charge vector v at $r=0$ is purely left-moving, i.e.

$$\eta \hat{V} v = \hat{V} v.$$

Consequently, the moduli space of sigma models on an attractor $K3$ is the space of $\hat{V}$ satisfying the above condition. This condition fixes one positive direction, so that the moduli space is given by

$$\frac{O(3,20)}{O(3) \times O(20)}. \tag{17}$$

This is understood as the space of embeddings of a 3-plane in the orthogonal complement of v in $\mathbf{R}^{4,20}$.

Now we would like to compare the above moduli space with the moduli space of HK structure on the Hilbert scheme of n-points of $K3$. The latter moduli space naturally appears if we consider the effective field theory on the worldvolume of the bound state of D-branes forming a black string solution. The effective theory is given by the gauge theory defined on the D5-brane worldvolume, $K3 \times \Sigma$. The existence of various D-branes of lower dimensions is realized by the nontrivial excitation of the worldvolume gauge field. The charge vector v is given in the gauge theory as follows:

$$v = \mathrm{Tr} \exp\left(\frac{iF}{2\pi}\right) \wedge Td(K3)^{\frac{1}{2}} \tag{18}$$

The worldvolume gauge theory is described by the following action:

$$S = \int_{\Sigma \times K3} \frac{1}{g_s} \mathrm{Tr} \mathcal{F} \wedge * \mathcal{F} + \mathcal{C} e^B \wedge v(F) \ , \quad \mathcal{F} = F - 2\pi i B \tag{19}$$

We assume that the size of $K3$ is much smaller than that of Σ, so that the effective theory reduces to the two-dimensional theory defined on Σ. It is given by the sigma model on the moduli space of instantons on $K3$ with charge vector v.

Let us review here the basic property of the moduli space $\mathcal{M}$ of instantons on $K3$. First of all, $\mathcal{M}$ has a HK structure, as it is viewed as an (infinite dimensional) HK quotient $\mu^{-1}(0)/\mathcal{G}$, where $\mu = F \wedge \vec{\omega}$ and $\mathcal{G}$ is the local gauge transformations. Considering in this way we can see that the complex structure of $\mathcal{M}$ is determined by that of $K3$.

In some cases $\mathcal{M}$ is identical with the Hilbert scheme of n-points on $K3$, where $n = \frac{1}{2}v^2 + 1$. Although this holds only under certain conditions on v, we shall hereafter ignore the complexity and limit ourselves to the cases where the identity holds. Then we obtain the cohomology group of $\mathcal{M}$ from that of the symmetric product of $K3$'s. In particular, $H^2(\mathcal{M})$ is spanned by the symmetrization of elements of $H^2(K3)$ and an exceptional $(1,1)$-form which is dual to the $\mathbf{Z_2}$-fixed locus of the symmetric product.

By Torelli's theorem and the Yau's theorem the moduli space of HK structures of $\mathcal{M}$ (with fixed volume) is given by the coset

$$\frac{O(3,20)}{O(3) \times O(20)}$$

As a consequence we come to the statement (i).

Furthermore we can find a map from $H^*(K3)$ to $H^*(\mathcal{M})$. The map is defined in terms of the second term in (19) (with B_{NS} set to zero):

$$\mu(C) = \int_{K3} C \wedge v(\hat{F})$$

This map is called as the Mukai map. It transforms a vector in $v^{\perp}$ into $H^2(\mathcal{M})$. In fact, we can see the following relations:

$$\begin{aligned} H^{2,0}(\mathcal{M}) &\simeq H^{2,0}(K3) \\ H^{0,2}(\mathcal{M}) &\simeq H^{0,2}(K3) \\ H^{1,1}(\mathcal{M}) &\simeq (H^{0,0}(K3) \oplus H^{1,1}(K3) \oplus H^{2,2}(K3)) \cap v^{\perp} \end{aligned}$$

2.2 Statement (*ii*)

The moduli space of IIB string theory on $K3$ is given by

$$\frac{O(4,20)}{O(4)\times O(20)} \times \mathbf{R}^{4,20} \times \mathbf{R}_+ \simeq \frac{O(5,21)}{O(5)\times O(21)} \tag{20}$$

The second and the third factors in the LHS come from the RR-scalars and the dilaton, respectively. This coset is understood as the embeddings of a positive 5-plane in $\mathbf{R}^{5,21}$. This $(5+21)$-dimensional vector space contains a lattice $\Gamma^{5,21}$ which is identified as the charge lattice of strings in six dimensions. We denote a general charge vector in $\Gamma^{5,21}$ as

$$v = (Q_{D5}, \vec{Q}_{D3}, -Q_{D1}, Q_{NS5}, -Q_{NS1})^T.$$

The 5-plane embedded in $\mathbf{R}^{5,21}$ is spanned by the following vectors:

$$(\exp b, 0, \mathcal{C}\exp b), \quad (\mathcal{C}, 1, 1/g_s) \tag{21}$$

By a similar argument the moduli space of string theory on an attractive $K3$ is given by

$$\frac{O(4,21)}{O(4)\times O(21)}$$

The moduli space of sigma model on $\mathcal{M}$ is also given by

$$\frac{O(3,20)}{O(3)\times O(20)} \times \mathbf{R}^{3,20} \times \mathbf{R}_+ \simeq \frac{O(4,21)}{O(4)\times O(21)},$$

where the three factors in the LHS correspond to the HK structure, B-field and the volume. Thus we come to the statement (*ii*). Note that the RR fields in the string theory are mapped by $\mu(C)$ to the B-field on $\mathcal{M}$.

2.3 An example: Bound states of $D1$ and $D5$-branes

Let us give you one simple example. Consider a black string that is made of $D1$ and $D5$-branes on the top of each other. This is the most familiar example first analyzed by Strominger and Vafa in the context of the physics of certain BPS black holes in five dimensions.

We repeat below the expression for the five vectors spanning a positive 5-plane in $\mathbf{R}^{5,21}$:

$$\begin{array}{lllllll} (& 1 & , B & , \frac{1}{2}B^2 - V & , 0 & , -G + B\wedge\tilde{B} + \theta V - \frac{1}{2}\theta B\wedge B &) \\ (& 0 & , \vec{\omega} & , B\wedge\vec{\omega} & , 0 & , (\tilde{B}-\theta B)\wedge\vec{\omega} &) \\ (& \theta & , \tilde{B} & , G & , 1 & , g_s^{-1} &) \end{array} \tag{22}$$

The charge vector of the black string is given by

$$v = (Q_5, 0, -Q_1, 0, 0) \tag{23}$$

If we require the charge vector to lie along the positive 5-plane, the moduli must satisfy the following conditions:

$$\begin{array}{l} B \text{ is a linear combination of } \vec{\omega} \quad (B \text{ is self-dual}). \\ \frac{1}{2}B\wedge B + V = \frac{Q_5}{Q_1} \\ G = \theta\cdot\frac{Q_5}{Q_1} \end{array} \tag{24}$$

We can easily see that the above conditions reduce the number of independent moduli by 21.

3 Attractor mechanism of four-dimensional black holes

In four-dimensional supergravity theories there are similar attractor mechanisms. Among other cases, the attractor mechanism works when we try to find a supersymmetric black hole solution in $\mathcal{N}=2$ (or higher) supergravity. Let us see how the attractor condition can be expressed, focusing ourselves mainly in the case of $\mathcal{N}=2$ supergravity. The attractor mechanism in this setting has a close relation with the arithmetic, as was conjectured by Moore[4].

The CY3 compactification of type IIB string theory yields a string theory with $\mathcal{N}=2$ SUSY in four dimensions. There are $h^{2,1}$ abelian vectormultiplets and $h^{1,1}$ hypermultiplets in the effective field theory, where $h^{p,q}$ is the Hodge number of the CY manifold. In the following we would like to obtain a charged BPS black hole solution of this theory. In doing this we shall neglect the degrees of freedom of hypermultiplets, and consider the remaining system of abelian vectormultiplets coupled to gravity.

Dyonic black hole solution

We shall work in a $\mathcal{N}=2$ supergravity in four dimensions that is obtained by CY3 compactification of IIB theory. To obtain the dyonic black hole solution we simply freeze the hypermultiplet degree of freedom. Then the relevant field contents are

$$\begin{array}{ll} \text{gravity multiplet:} & (g_{\mu\nu}, \psi^A_\mu, A^0_\mu) \\ \text{vectormultiplet:} & (z^i, \lambda^{Ai}, A^i_\mu)\ , \quad i=1,\ldots,h^{2,1} \end{array} \tag{25}$$

The fully generic action and SUSY transformation law are found in, say, [2]. We quote some of the formulas in it.

We can regard the black holes in this theory as a D3-brane wrapping on a certain three-cycle of the internal CY3 which we denote by X. Let us denote the charge of the black hole as $\gamma \in H^3(X,\mathbf{Z})$. Then the attractor condition is given by

$$\Pi^{2,1}(\gamma)=0$$

or, equivalently,

$$\gamma=\gamma^{3,0}+\gamma^{0,3}.$$

This condition has the following meaning. We have a Hodge decomposition of $H^3(X,\mathbf{R})$ once we choose a specific complex structure or the holomorphic $(3,0)$-form Ω. The polarization of Ω in $H^3(X,\mathbf{R})$ varies as a function of the spacetime, in the present case it is a function of r. Hence it follows that although γ is constant, its Hodge decomposition is not. The attractor mechanism fixes (partially) the value of the moduli at $r=0$ so that the Hodge decomposition at $r=0$ satisfies $\gamma=\gamma^{3,0}+\gamma^{0,3}$.

3.1 Attractor mechanism of $D=4, \mathcal{N}=4$ supergravity

We consider the $K3\times T^2$ compactification of type IIB string. Hereafter we shall denote the $K3$ surface as S. At low energies the theory is described by $\mathcal{N}=4$ supergravity in four dimensions. The moduli space is given by:

$$\mathcal{M}=\frac{SO(6,22)}{SO(6)\times SO(22)}\times\frac{SL(2,\mathbf{R})}{SO(2)} \tag{26}$$

The first factor is the moduli space of $\mathcal{N}=2$ vector and hypermultiplets, whereas the second is that of a tensormultiplet, which appears when we decompose the $\mathcal{N}=4$ gravity multiplet into multiplets of $\mathcal{N}=2$ SUSY.

There are $(6+22)$ electric and $(6+22)$ magnetic charges, p and q. A point in $\mathcal{M}$ determines the decomposition of a vector $v\in\Gamma^{6,22}$ as follows:

$$v=v^{6,0}\oplus v^{0,22}=v_L+v_R \tag{27}$$

The attractor condition is given by:

$$p_R=q_R=0 \tag{28}$$

This condition defines the attractor moduli space, a 88-dimensional subvariety in $\mathcal{M}$. It is given by

$$\mathcal{V}(p,q) \simeq \frac{SO(4,22)}{SO(4)\times SO(22)}. \tag{29}$$

where we simply omit the tensormultiplet moduli. As is obvious the attractor equation puts no constraint on the tensormultiplet moduli.

Geometrical meaning

Let us discuss the geometrical meaning of the attractor condition. To do this, we use U-duality to take $p,q \in H^3(S;\mathbf{Z})$. The attractor condition becomes

$$\bar{Z}e^{\frac{K}{2}}\Omega^{3,0} = \frac{p^2}{2\sqrt{-D}}(q-\bar{\tau}p)\wedge dz \tag{30}$$

together with the conditions fixing the value of NSNS and RR tensors. Here D is a quantity called as the discriminant, defined as follows:

$$D = -4\det Q_{p,q} = (p\cdot q)^2 - p^2q^2\ , \quad Q_{p,q} \equiv \frac{1}{2}\begin{pmatrix} p^2 & -p\cdot q \\ -p\cdot q & q^2 \end{pmatrix} \tag{31}$$

The above condition has the same meaning as the one for CY compactification, namely, the cohomology element defined by the charge has only $(3,0)$ and $(0,3)$-components at the attractor point. This condition determines $\Omega^{2,0}$ on $K3$ as a function of p,q and τ, the moduli of T^2. Note that τ is not arbitrary, since

$$\int_S \Omega^{0,2}\wedge\Omega^{0,2} = 0 \Rightarrow p^2\tau^2 - 2p\cdot q\tau + q^2 = 0$$

$$\tau = \frac{p\cdot q + \sqrt{D_{p,q}}}{p^2}. \tag{32}$$

Thus p,q determine τ completely and $\Omega^{2,0}$ of S up to an overall complex constant.

The condition that $\Omega^{2,0} = C(q-\bar{\tau}p)$ has a remarkable consequence. Recalling that the Neron-Severi (NS) lattice is the kernel of the period map ($\gamma \to \int_\gamma \Omega$), we find that the attractor $K3$ has the maximal NS lattice of rank 20. Therefore the attractor $K3$'s are "maximally algebraic".

We can say more about the attractor variety $X_{p,q}$, using the Shioda-Inose theorem. Skipping the detailed math arguments and quoting only the final results, the attractor variety

$$X_{p,q} = S_{2Q_{p,q}} \times E_{\tau(p,q)}.$$

is closely related to a product of three elliptic curves:

$$X_{p,q} \stackrel{2:1}{\to} Km(E_{\tau_{p,q}}\times E_{\tau'_{p,q}})\times E_{\tau_{p,q}}.$$

Here $\tau_{p,q},\tau'_{p,q}$ are given by

$$\tau_{p,q} = \frac{p\cdot q+\sqrt{D}}{p^2}\ , \quad \tau'_{p,q} = \frac{-p\cdot q+\sqrt{D}}{2}$$

Moore conjectured that all the attractor varieties are arithmetic. Roughly speaking, we call a manifold arithmetic if it is defined on a number field which is "smaller" than $\mathbf{R}$. For example we may as well imagine that such manifolds are defined by polynomials defined on a number field which is "smaller" than $\mathbf{R}$.

As we have seen, the attractor varieties of $\mathcal{N}=4$ compactification is closely related to the products of elliptic curves with the moduli τ valued in quadratic imaginary fields, namely, τ is a solution of the equation

$$a\tau^2 + b\tau + c = 0\ , \quad a,b,c\in\mathbf{Z}\ , \quad D = b^2-4ac < 0.$$

As is shown below, these elliptic curves are arithmetic.

First we quote a theorem.

Theorem. Suppose τ is a solution of $a\tau^2 + b\tau + c = 0$ with g.c.d$(a, b, c) = 1$. Then $j(\tau)$ is an algebraic integer of degree $h(D)$. In other words $j(\tau)$ is a solution of a certain monic polynomial equation of degree $h(D)$ with integral coefficients. Here $h(D)$ is the class number, the number of $SL(2, R)$ inequivalent quadric forms of discriminant D.

Since any elliptic curve with moduli τ can be expressed by the Weierstrass polynomial,

$$y^2 = 4x^3 - c(x+1)\ , \quad c = \frac{27j(\tau)}{j(\tau) - (12)^3}$$

we find that any elliptic curve of the above property is indeed arithmetic, defined on a certain algebraic extension of $\mathbf{Q}$ containing $j(\tau)$.

The attractor variety of the $\mathcal{N} = 4$ supergravity is made from the above special elliptic curves by tensoring, taking a quotient and the resolution of the singularities. Therefore they are also arithmetic, if the procedure of tensoring, taking a quotient and so on preserves the arithmetic property. Of course the precise argument requires a more careful investigation, see the original paper of Moore. Thus we conclude that the attractor varieties of $\mathcal{N} = 4$ are arithmetic.

As for the $\mathcal{N} = 2$ case or the case of CY3 compactification, much less can be said. Moore has made the following attractor conjecture: Attractor CY3 with charge γ is defined over a number field $K(\gamma)$, and the quantity such as the period vector or the special coordinates associated to the attractor points are valued in a field $\hat{K}(\gamma)$ that is related to $K(\gamma)$.

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