# Plan

I. Introduction (I tell what I'm gonna tell)

II. Main part (I tell)

III. Conclusion (I tell what I've just told)

# More about Ising Field Theory correlation functions

P. Fonseca, S. Lukyanov, A Z (unpublished)

"Result": New derivation of well known result – <u>nonlinear integrable differential equations for the Ising spin correlation functions</u> – plus some generalizations.

Method: Use of <u>Noncommutative Local Integrals of Motion</u>

Infinite set of
$\sqrt{}$ <u>NLIM</u>

- Conformal Field Theories

e.g. $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m}$

- Free field Theories

$$I_n = \int \phi(x)\, P_n(x, \tfrac{\partial}{\partial x})\, \phi(x)\, dx$$

# Ising Field Theory

- Exact Solution of the Ising Model (Onsager, 44) at $H=0 \Rightarrow$ Free Fermions

- Scaling limit $(T \to T_c,\ R_c \to \infty)$

  $\Downarrow$

  Ising Field Theory = Free Majorana Fermions

- Spin correlation functions

  $\langle \sigma(x_1) \ldots \sigma(x_N) \rangle$

Can be obtained through solutions of certain integrable nonlinear equations $\left\{\begin{array}{l}\text{Barouh, McCoy, Tracy, Wu;} \\ \text{Sato, Miwa, Jimbo}\end{array}\right.$

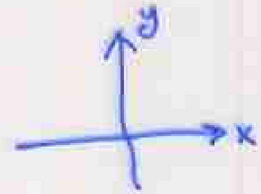
E.g: Two-point functions

$\sigma(x)$ - "spin field"

$\mu(x)$ - "dual spinfield"

Complex coordinates

$z = x + iy\ ,\ \bar{z} = x - iy$




$$G(z,\bar{z}) = \langle \sigma(z,\bar{z})\,\sigma(0,0)\rangle$$

$$\tilde{G}(z,\bar{z}) = \langle \mu(z,\bar{z})\,\mu(0,0)\rangle$$

Equations ("Bilinear form")

$$\partial G\,\partial G - G\,\partial^2 G = \partial\tilde{G}\,\partial\tilde{G} - \tilde{G}\,\partial^2\tilde{G} \qquad (H1)$$

$$\tilde{G}\,\partial\bar{\partial}\tilde{G} - \partial\tilde{G}\,\bar{\partial}\tilde{G} + G\,\partial\bar{\partial}G - \partial G\,\bar{\partial}G = 0 \qquad (H2)$$

$$G\,\partial\bar{\partial}\tilde{G} + \tilde{G}\,\partial\bar{\partial}G - \partial\tilde{G}\,\bar{\partial}G - \bar{\partial}\tilde{G}\,\partial G = \left(\frac{m}{2}\right)^2 G\tilde{G} \qquad (H3)$$

$\partial = \frac{\partial}{\partial z}\ ,\ \bar{\partial} = \frac{\partial}{\partial \bar{z}}\ ,\ m = R_c^{-1} \sim (T_c - T)$

Introduce $\chi$, $\varphi$

$$G_+ = G + \widetilde{G} = e^{\frac{\chi}{2}} e^{\frac{\varphi}{2}}$$

$$G_- = G - \widetilde{G} = e^{\frac{\chi}{2}} e^{-\varphi/2}$$

$(H1 \div 3)$

$$\hookrightarrow \begin{cases} \partial\bar{\partial}\varphi = \frac{m^2}{8} \operatorname{sh}\varphi \\ \partial\bar{\partial}\chi = \frac{m^2}{8}(1 - \operatorname{ch}\varphi) \\ \partial^2\chi + \partial\varphi\partial\varphi = 0 \end{cases}$$

Derivations:

- <u>Barouh, McCoy, Tracy, Wu</u>: Using determinant representation of the correlation functions

- <u>Sato, Miwa, Jimbo</u>: Isomonodromic deformation Theory

# Free Fermions in IFT

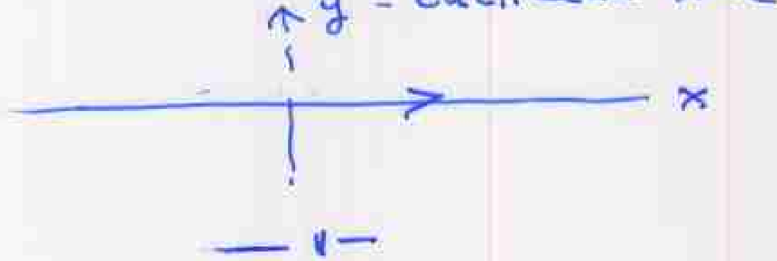
(Euclidean) Majorana fermi field $(\psi, \overline{\psi})$

$$\begin{cases} \bar\partial \psi = i \frac{m}{2} \overline{\psi} \\ \partial \overline{\psi} = -i \frac{m}{2} \psi \end{cases}$$

Energy and momentum

$$\underline{P} = \frac{E-p}{2} = -\frac{1}{4\pi} \int \psi \partial \psi \, dz + i \frac{m}{2} \overline{\psi} \psi \, d\bar z$$

$$\overline{P} = \frac{E+p}{2} = -\frac{1}{4\pi} \int \overline{\psi} \bar\partial \overline{\psi} \, d\bar z + i \frac{m}{2} \overline{\psi} \psi \, dz$$




Spin fields $\sigma(x), \mu(x)$ are nonlocal expressions (quadratic exponentials) in terms of $\psi, \overline{\psi}$

(Relatively) simple locality properties

$\psi(x)\sigma(0)$


$\psi(x)\sigma(0) \to -\psi(x)\sigma(0)$

# Operator product expansions

$$\psi(\vec{x})\,\sigma(0) \longrightarrow \frac{e^{i\frac{\pi}{4}}}{\sqrt{2z}}\,\mu(0)$$

$$\psi(\vec{x})\,\mu(0) \rightarrow \frac{e^{-i\pi/4}}{\sqrt{2z}}\,\sigma(0)$$

# Doubled Ising Model

Two copies

\#1 $(\psi_1, \bar{\psi}_1)$, $\sigma_1$, $\mu_1$, ..

\#2 $(\psi_2, \bar{\psi}_2)$, $\sigma_2$, $\mu_2$, ...

The "doubled" system has infinitely many noncommutative local IM

(single Ising also does)

1) Energy and Momentum

$$(P_1, \bar{P}_1) = \left(\frac{E_1 - p_1}{2}, \frac{E_1 + p_1}{2}\right)$$

$$(P_2, \bar{P}_2) = \left(\frac{E_2 - p_2}{2}, \frac{E_2 + p_2}{2}\right)$$

$(P, \bar{P}) = (P_1 + P_2, \bar{P}_1 + \bar{P}_2)$ - total EM

But

$$X = P_1 - P_2$$

$$\bar{X} = \bar{P}_1 - \bar{P}_2$$

plays interesting role

2) Rotation of $\psi_1, \psi_2$ (U(1) charge)

$$H = \frac{1}{2\pi}\int\left(\psi_1\psi_2\,dz - \bar{\psi}_1\bar{\psi}_2\,d\bar{z}\right)$$

3) Two additional quadratic IM

$$Y = -\frac{1}{2\pi}\int\left(\psi_1\partial\psi_2\,dz + i\frac{m}{2}\bar{\psi}_1\psi_2\,d\bar{z}\right)$$

$$\bar{Y} = \frac{1}{2\pi}\int\left(\bar{\psi}_1\bar{\partial}\bar{\psi}_2\,d\bar{z} - i\frac{m}{2}\psi_1\bar{\psi}_2\,dz\right)$$

Commutation relations

$$\begin{cases} [X,H] = 2Y, & [\bar{X},H] = 2\bar{Y} \\ [Y,H] = -2X, & [\bar{Y},H] = -2\bar{X} \\ [X,\bar{Y}] = [\bar{X},Y] = \frac{m^2}{2}H & \end{cases}$$

and $\Big\{ \quad [X,\bar{X}] = [Y,\bar{Y}] = 0$

$$\underbrace{\text{Affine Lie Algebra}}_{\text{Infinite-dimensional}} \quad \widehat{SL(2)} \text{ (of level 0)}$$

$$[X,Y] = 2H_2, \quad [\bar{X},\bar{Y}] = 2\bar{H}_2$$

$$[X, Y] = 2H_2 \,, \quad [\bar{X}, \bar{Y}] = 2\bar{H}_2$$

$$H_2 = \frac{1}{2\pi}\int \partial\psi_1 \partial\psi_2 \, dz + \left(\frac{m}{2}\right)^2 \psi_1 \psi_2 \, d\bar{z}$$

$$\bar{H}_2 = \frac{1}{2\pi}\int \bar{\partial}\bar{\psi}_1 \bar{\partial}\bar{\psi}_2 \, d\bar{z} + \left(\frac{m}{2}\right)^2 \bar{\psi}_1 \bar{\psi}_2 \, dz$$

— " —

Statement: Nonlinear diff. equations for spin corr. functions follow from this symmetry.

Commutators with the spin fields

$$\{X_i\} = \{X, \bar{X}, H, Y, \bar{Y}\}$$

$$[X_i, \text{Field}(x)] = \text{another local field}$$

$$\text{Field}(x) = \{\sigma_1(x)\sigma_2(x), \ \sigma_1(x)\mu_2(x), \ \text{etc}\}$$

$$\psi_1(x)\psi_2(x)\,\sigma_1(0)\mu_2(0)$$

Commutators

1. $X = P_1 - P_2$

$$i\,[X, (\sigma_1\sigma_2)] = \partial\sigma_1\,\sigma_2 - \sigma_1\,\partial\sigma_2$$

etc

2. $i\,[H, (\sigma_1\sigma_2)] = i\,[H, (\mu_1\mu_2)] = 0$ ;

$i\,[H, (\sigma_1\mu_2)] = \mu_1\sigma_2$ ;

$i\,[H, (\mu_1\sigma_2)] = -\sigma_1\mu_2$ ;

3. $i\,[Y, (\sigma_1\sigma_2)] = i\,(\partial\mu_1\,\mu_2 - \mu_1\,\partial\mu_2)$ ;

$i\,[Y, (\mu_1\mu_2)] = i\,(\partial\sigma_1\,\sigma_2 - \sigma_1\,\partial\sigma_2)$ ;

$i\,[Y, (\sigma_1\mu_2)] = \partial\mu_1\,\sigma_2 - \mu_1\,\partial\sigma_2$

$i\,[Y, (\mu_1\sigma_2)] = -\partial\sigma_1\,\mu_2 + \sigma_1\,\partial\mu_2$

Also useful

$$i\,[H_2, (\sigma_1\sigma_2)] = i\,(2\,\partial\mu_1\,\partial\mu_2 - \partial^2\mu_1\,\mu_2 - \mu_1\,\partial^2\mu_2)$$

$$i\,[H_2, (\mu_1\mu_2)] = i\,(2\,\partial\sigma_1\,\partial\sigma_2 - \partial^2\sigma_1\,\sigma_2 - \sigma_1\,\partial^2\sigma_2)$$

$X_i|0\rangle = 0 \Rightarrow$ Identities ~~for~~ among the CF

(*) $\langle 0| \underbrace{[H_2, \sigma_1(x)\sigma_2(x)\mu_1(0)\mu_2(0)]}_{\mathrm{I} + \mathrm{II}} |0\rangle = 0$

$\langle \mathrm{I} \rangle = \langle (2\partial\mu_1\partial\mu_2 - \partial^2\mu_1\,\mu_2 - \mu_1\partial^2\mu_2)_{(x)} (\mu_1\mu_2)_{(0)} \rangle$

$= 2\langle \partial\mu_1(x)\mu_1(0)\rangle\langle\partial\mu_2(x)\mu_2(0)\rangle -$
$\langle\partial^2\mu_1(x)\mu_1(0)\rangle\langle\mu_2(x)\mu_2(0)\rangle -$
$\langle\mu_1(x)\mu_1(0)\rangle\langle\partial^2\mu_2(x)\mu_2(0)\rangle$

$\langle \mathrm{II} \rangle$ = same with $\sigma$'s

(*) $\partial\tilde{G}\partial\tilde{G} - \tilde{G}\partial^2\tilde{G} = \partial G\partial G - G\partial^2 G$ **(H1)**

$\tilde{G}(x) = \langle\mu(x)\mu(0)\rangle$ $\quad x = (z, \bar{z})$
$G(x) = \langle\sigma(x)\sigma(0)\rangle$

(**) $\langle 0| \underbrace{[Y, [\bar{X}, \sigma_1(x)\sigma_2(x)]\mu_1(0)\mu_2(0)]}_{\mathrm{I}' + \mathrm{II}'} |0\rangle = 0$

$[Y, [\bar{X}, (\sigma_1\sigma_2)]] = [\bar{X}, [Y, (\sigma_1\sigma_2)]] + [\underbrace{[Y, \bar{X}]}_{= -\frac{m^2}{2}H}, (\sigma_1\sigma_2)]$

**(H2)**

(***) $\langle 0 | [\underline{Y}, [\overline{X}, \sigma_1(x)\mu_2(x)]\mu_1(0)\sigma_2(0)] | 0 \rangle = 0$

$\hookrightarrow$ (H3)

## Symetry identities $\rightarrow$ (H 1 $\div$ 3)

Simple generalizations:

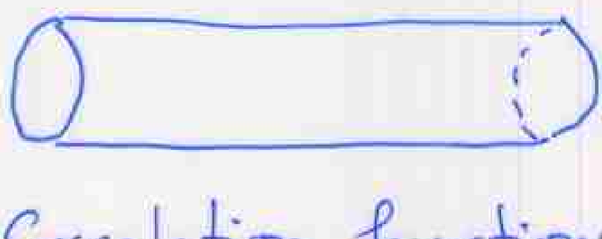
1. $\langle 0 | \ldots | 0 \rangle \rightarrow tr(\ldots e^{-\beta E})$

$$E = E_1 + E_2$$

As $[X_i, E] = 0$,

$$\underline{tr([X_i, \ldots] e^{-\beta E}) = 0}$$

All the equations hold for finite $\beta$



$\beta$

2. Correlation functions on a torus
3. —— " —— in the presence of a boundary

4. Matrix elements $\langle \Psi_1 | \sigma(x) \sigma(0) | \Psi_2 \rangle$ between excited states

Excited states: Free fermions

One particle state $|A(\theta)\rangle$

$$E = m\,\mathrm{ch}\,\theta,\ p = m\,\mathrm{sh}\,\theta$$

$$\frac{E \pm p}{2} = \frac{m}{2} e^{\pm\theta}$$

$$F(x) = \langle 0|\sigma(x)\mu(0)|A(\theta)\rangle$$

$$\tilde{F}(x) = \langle 0|\mu(x)\sigma(0)|A(\theta)\rangle$$

In "doubled" system $|A_1(\theta)\rangle$, $|A_2(\theta)\rangle$.

$$\langle 0|[X_i, \sigma_1(x)\sigma_2(x)\mu_1(0)\mu_2(0)]|A_1(\theta)\rangle \neq 0$$

one needs $X_i|A_{1,2}(\theta)\rangle$.

$$X|A_{1,2}(\theta)\rangle = \pm\frac{m}{2}e^{\theta}|A_{1,2}(\theta)\rangle;$$

$$H|A_{1,2}(\theta)\rangle = \pm|A_{2,1}(\theta)\rangle;$$

$$Y|A_{1,2}(\theta)\rangle = \frac{m}{2}e^{\theta}|A_{2,1}(\theta)\rangle;$$

Differential equations for $F, \tilde{F}$

$$\Psi_1(x) = e^{-\frac{\chi(x)}{2}}(F(x) - i\tilde{F}(x))$$

$$\Psi_2(x) = e^{-\frac{\chi(x)}{2}}(F(x) + i\tilde{F}(x))$$

$$G \pm \tilde{G} = e^{\chi/2}e^{\pm\varphi/2}$$

$$\partial \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\partial\varphi & -\lambda e^{-\varphi} \\ -\lambda e^{\varphi} & -\frac{1}{2}\partial\varphi \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

$$\bar\partial \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\bar\partial\varphi & -\bar\lambda e^{\varphi} \\ -\bar\lambda e^{-\varphi} & \frac{1}{2}\bar\partial\varphi \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

$$\lambda = -i\frac{m}{4}e^{\theta}, \quad \bar\lambda = i\frac{m}{4}e^{-\theta}$$

Zero-curvature representation for

$$\partial\bar\partial\varphi = \frac{m^2}{8}\,\mathrm{sh}\,\varphi$$

i.e.

$\Psi_1, \Psi_2$ are (essentialy) $\langle 0|\sigma\mu|A\rangle$

~~Multiparticle~~ Two-particle elements

$$\cancel{G(\lambda_1,\lambda_2|x) = \langle 0|\sigma(x)\sigma(0)|A(\theta_1)A(\theta_2)}$$

$$G(x|\lambda_1,\lambda_2) = \langle A(\theta_1)|\sigma(x)\sigma(0)|A(\theta_2)\rangle$$

$$\tilde G(x|\lambda_1,\lambda_2) = \langle A(\theta_1)|\mu(x)\mu(0)|A(\theta_2)\rangle$$

$$\lambda_1 = -i\frac{m}{4}e^{\theta_1}, \quad \lambda_2 = -\frac{im}{4}e^{\theta_2}$$

$$(\lambda_2-\lambda_1)\,G_+(\lambda_1,\lambda_2) = e^{(\lambda_2-\lambda_1)z+(\bar\lambda_2-\bar\lambda_1)\bar z}\, e^{\frac{\chi+\varphi}{2}} \times (\text{times})$$

$$\left[\lambda_1 \Psi_2(\lambda_1)\Psi_1(\lambda_2) - \lambda_2 \Psi_1(\lambda_1)\Psi_2(\lambda_1)\right].$$

$$(\lambda_2-\lambda_1)\,G_-(\lambda_1,\lambda_2) = e^{(\lambda_2-\lambda_1)z+(\bar\lambda_2-\lambda_1)\bar z}\quad e^{\frac{\varphi+\chi}{2}}$$

$$\left[-\lambda_1 \Psi_1(\lambda_1)\Psi_2(\lambda_2) + \lambda_2 \Psi_2(\lambda_1)\Psi_1(\lambda_2)\right].$$

where

$$G_\pm(\lambda_1,\lambda_2) = G(x|\lambda_1,\lambda_2) \pm \tilde G(x|\lambda_1,\lambda_2)$$

## Conclusion

- Yet another derivation of known result (diff. equations for Ising spin correlation functions)
- Extension to finite-size and boundary
- Multi-particle matrix elements: Useful in development of h-perturbation theory in the presence of magnetic field