Liouville Theory: with a boundary, and without.

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Liouville Theory:

- Noncritical bosonic string (Polyakov, 1981)

$$\vec{X}(\xi)\ ,\quad g_{ab}(\xi) = e^{\varphi(\xi)}\hat{g}_{ab}$$

$$A_{string} = \int \sqrt{g}\,\left(g^{ab}\partial_a\vec{X}\,\partial_b\vec{X} + \lambda\right)d^2\xi$$

Liouville action

$$\mathcal{A}_L = \int \sqrt{\hat{g}}\left(\frac{26-D}{96\pi}(\partial\varphi)^2 + \mu e^{\varphi}\right),\quad c_L = 26-D$$

- 2D Quantum gravity. ~~In conformal gauge~~

"Matter fields" + "Gravity"

In conformal gauge

$$\mathcal{A}_M + \mathcal{A}_L + \mathcal{A}_{Ghost} + \sum_i t_i \int O_i e^{\gamma_i \varphi} d^2\xi$$

$$c_M + c_L + 26 = 0$$

Scaling (Knizhnik, Polyakov, Z. (88), David (88), Distler, Kawai (89))

- Serious illnes if $c_L < 25$ (i.e. $c_M > 1$)

- Agreement with the "matrix models" of 2D Quantum Gravity (Kazakov (85), Kazakov, Kostov, Migdal (85), Brezin, Kazakov (90), Douglas, Shenker (90) Gross, Migdal (90)

Early works on Liouville

Curtright, Thorn (82) D'Hoker, Jackiw (82), Gervais, Neveu (84)

Seiberg (90), Polchinski (91)

Unlike usual "Compact" CFT, in Liouville

- Continous spectrum of primary states
- "Operator-state correspondence" requires new ~~understanding~~ interpretation.

In usual CFT

Local fields $\Psi_i(\xi)$

$|i\rangle \Longleftrightarrow \Psi_i(\xi)$

States $|i\rangle$: $1 = |i\rangle\langle i|$

Liouville is a (simplest) example of "noncompact" CFT

Liouville Theory

Renormalized Action (Distler, Kawai (29))

$$\mathcal{A}_L = \int \sqrt{\hat{g}} \left[\frac{1}{4\pi} (\partial\phi)^2 + \mu e^{2b\phi} + \frac{Q}{4\pi} \hat{R} \phi \right] d^2\xi$$

$$Q = \frac{1}{b} + b \quad , \quad \underline{c_L = 1 + 6Q^2 > 25} \quad \text{(real } b\text{)}$$

μ - "cosmological constant"

Formal Classical limit $b \to 0$

$$\varphi(\xi) = 2b\phi(\xi)$$

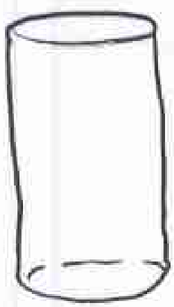
Liouville equation

$$\Delta\varphi = \mu_0 e^{\varphi} \qquad \mu_0 = 2\pi\mu b^2 = -R$$

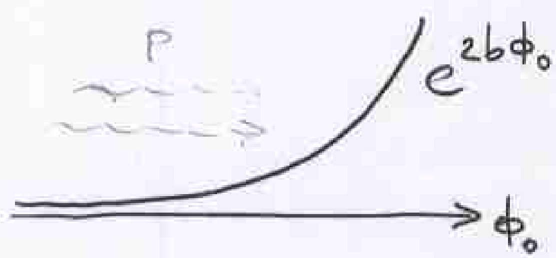
Local primary fields

$$\Phi_\alpha(\xi) = e^{2\alpha\phi(\xi)} \qquad \Delta(\alpha) = \alpha(Q-\alpha)$$

Primary States



$|p\rangle$



Operator-state correspondence

$$|p\rangle \longleftrightarrow \Phi_{Q/2 + ip}$$

Identity operator $I = \Phi_0$ does not correspond to a normalizable state

Operator Product Expansions

$$\Phi_{\alpha_1}\,\Phi_{\alpha_2} = \frac{1}{2}\int dp\;\mathbb{C}(\alpha_1,\alpha_2,Q/2+ip)\left[\Phi_{Q/2+ip} + \text{descendants}\right]$$

$$+ \text{ Discret terms}$$

$$\mathbb{C}(\alpha_1,\alpha_2,\alpha_3) = \langle \Phi_{\alpha_1}\,\Phi_{\alpha_2}\,\Phi_{\alpha_3}\rangle_{\text{sphere}}$$

Explicit form of 2- and 3-point function

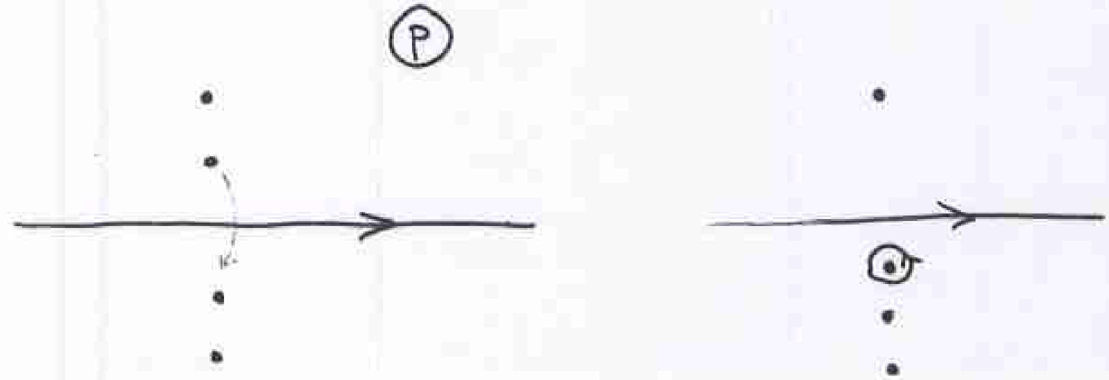
(Conjectured by Dorn, Otto (94) (using proposal of Goulian, Li) (91)

$$\mathbb{C}(\alpha_1,\alpha_2,\alpha_3) = \left(\pi\mu\,\gamma(b^2)\,b^{2-2b^2}\right)^{\frac{(Q-\Sigma\alpha_i)}{b}} \times$$

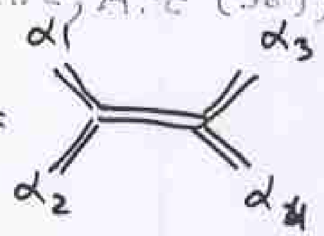
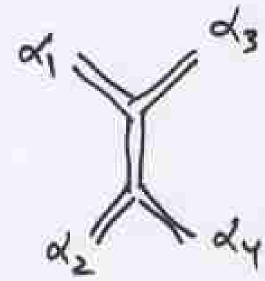
$$\frac{\Upsilon_0\,\Upsilon(2\alpha_1)\,\Upsilon(2\alpha_2)\,\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\,\Upsilon(\alpha_1+\alpha_2-\alpha_3)\,\Upsilon(\alpha_2+\alpha_3-\alpha_1)\,\Upsilon(\alpha_1+\alpha_3-\alpha_2)}$$

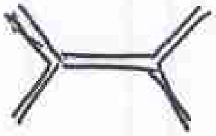
$$\log\Upsilon(x) = \int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2 e^{-2t} - \frac{\operatorname{sh}^2\left(\left(\frac{Q}{2}-x\right)t\right)}{\operatorname{sh}(bt)\,\operatorname{sh}(t/b)}\right]$$

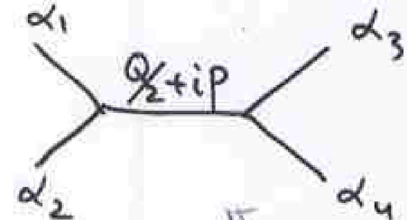
<u>Discret terms</u>



Bootstrap Equation (Al.Z, A.Z (96))

$\langle \Phi_{\alpha_1} \Phi_{\alpha_2} \Phi_{\alpha_3} \Phi_{\alpha_4} \rangle =$  $=$ 

 $= \frac{1}{2} \int dp \, \mathbb{C}(\alpha_1, \alpha_2, Q/2 + ip) \, \mathbb{C}(\alpha_3, \alpha_4, Q/2 - ip) \times$

$\left| \right.$  $\left. \right|^2$ + Discret Terms

"Conformal block"

Properties

- "Duality" $b \leftrightarrow 1/b$.

- Case of "degenerate fields"

$$\alpha = \alpha_{n,m} = (1-m)\frac{b}{2} + (1-n)\frac{1}{2b}$$

Vanishing null-states $\rightarrow$ Discret fusion rules

$$\Phi_{-b/2} : \left(\frac{1}{b^2} L_{-1}^2 + L_{-2}\right) \Phi_{-b/2} = 0$$

$$\Phi_{-b/2} \Phi_{\alpha} = C_+(\alpha)\left[\Phi_{\alpha - b/2} + \cdots\right] + C_-(\alpha)\left[\Phi_{\alpha + b/2}\right]$$

i) Only "discret terms" contribute

$$\mathbb{C} \sim \frac{\epsilon}{(\alpha-\alpha_0)^2 + \epsilon^2}$$

P

ii) "Special structure constants" exactly coinside with their free-field (FFDF) realizations

$$C_+(\alpha) = \mathbb{C}(\alpha, -b/2, Q-\alpha+b/2) =$$
$$\langle e^{2\alpha\phi(0)} e^{-b\phi(1)} e^{2(Q-\alpha+b/2)\varphi(\infty)} \rangle = 1$$

$$C_-(\alpha) = \mathbb{C}(\alpha, -b/2, Q-\alpha-b/2) = \langle e^{2b\phi(x)} e^{2\alpha\phi(0)} e^{-b\phi(1)} e^{2(Q-\alpha-b/2)\phi(\infty)} \rangle =$$
$$-\mu \int \langle e^{2b\phi(x)} e^{2\alpha\phi(0)} e^{-b\phi(1)} e^{2(Q-\alpha-b/2)\phi(\infty)} \rangle$$
$$= -\mu \frac{\pi \gamma(2b\alpha-1-b^2)}{\gamma(-b^2)\gamma(2b\alpha)}, \quad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$$

<u>shift Relations</u> (Teschner (95))

Two-poin function $\langle \Phi_\alpha(z) \Phi_\alpha(0) \rangle = \frac{D(\alpha)}{|z|^{4\Delta_\alpha}}$

$$\langle \Phi_{-b/2}(z) \Phi_\alpha(z_1) \Phi_{\alpha+b/2}(z_2) \rangle = *$$

1. $\Phi_{-b/2} \Phi_\alpha = C_-(\alpha) \Phi_{\alpha+b/2} + C_+(\alpha) \Phi_{\alpha-b/2}$;

$$* = C_-(\alpha) D(\alpha + b/2)$$

2. $\Phi_{-b/2} \Phi_{\alpha+b/2} = C_+(\alpha+b/2) \Phi_\alpha + C_-(\alpha) \Phi_{\alpha+b}$;

$$* = C_+(\alpha+b/2) D(\alpha)$$

$$\cancel{\quad} \; \frac{D(\alpha+b/2)}{D(\alpha)} = \frac{C_+(\alpha+b/2)}{C_-(\alpha)}$$

with $b \to 1/b$

$$D(\alpha) = (\pi\mu\,\gamma(b^2))^{\frac{Q-2\alpha}{b}} \frac{\gamma(2b\alpha - b^2)}{b^2\,\gamma(2-\frac{2\alpha}{b}+\frac{1}{b^2})}$$

$D(\alpha)$, $C(\alpha_1, \alpha_2, \alpha_3)$

Completely determine "bulk" properties of Liouville CFT

Liouville CFT with boundary

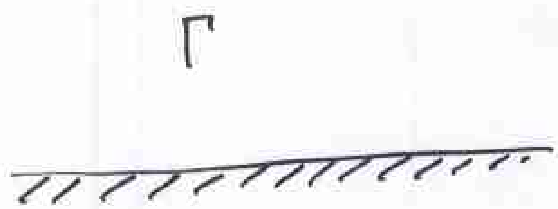
$$\mathcal{A} = \mathcal{A}_{\text{bulk}} + \int_{\partial\Gamma} \hat{g}^{1/4}\left(\frac{QK}{2\pi}\phi + \mu_B e^{b\phi}\right) d\xi$$

"boundary cosmological constant"

$\Gamma = \text{Disk}$,

Flat $\hat{g}$,

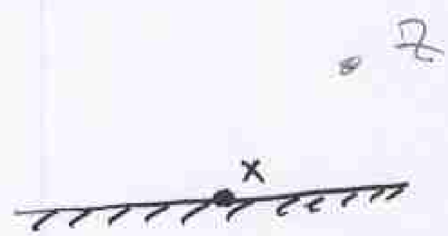
$$\mathcal{A} = \int_{\Gamma}\left(\frac{1}{4\pi}(\partial\phi)^2 + \mu e^{2b\phi}\right) d^2\xi + \int_{\partial\Gamma}\left(\frac{Q\kappa}{2\pi}\phi + \mu_B e^{b\phi}\right) d\xi$$



Two scales, μ_B and $\mu^{1/2}$

$$t = \frac{\mu_B^2}{\mu}$$

- The "bulk" OPE remain the same.
- Boundary primaries



$$B_\beta(x) = e^{\beta\phi(x)}$$

$$\boxed{\Delta(\beta) = \beta(Q-\beta)}$$

Basic CFT data

1. $\langle \Phi_\alpha(z)\rangle = \dfrac{U(\alpha)}{|z-\bar{z}|^{2\Delta_\alpha}}$

2. $\langle \Phi_\alpha(z) B_\beta(x)\rangle = \dfrac{R(\alpha|\beta)}{|z-\bar{z}|^{2\Delta_\alpha-\Delta_\beta}|z-x|^{2\Delta_\beta}}$

$R(\alpha|\beta)$ controls "bulk-boundary OPE"

$$\Phi_\alpha(z) = \int dp\, R(\alpha|Q/2-ip)\left[B_{Q/2+ip} + \dots\right] +$$

Discret terms

Case of "degenerate" Φ_α : only discret terms contribute (with "special structure constants")

Example

$$\Phi_{-b/2} = R(-b/2|Q)[B_0] + R(-b/2|Q+b)[B_{-b}]$$

FFDF representation

$$R(-b/2|Q) = \int_{-\infty}^{\infty} \langle -\mu_B e^{b\phi(x)} e^{-b\phi(i)} e^{2Q\phi(\infty)} \rangle dx$$

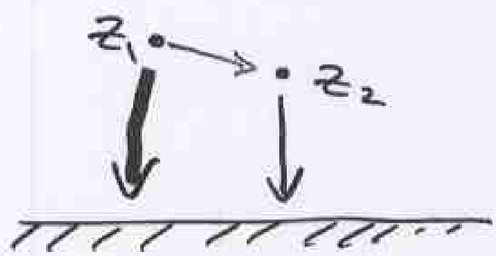
$$= -\frac{2\pi\mu_B\, \Gamma(-1-2b^2)}{\Gamma(-b^2)}$$

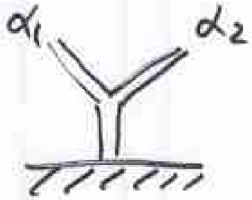
4. Boundary 2- and 3-point functions

$(1 \div 4) \longrightarrow$ All correlation functions

Bootstrap Equations (Cardy (84, 86))

$$\langle \Phi_{\alpha_1}(z_1)\, \Phi_{\alpha_2}(z_2) \rangle =$$





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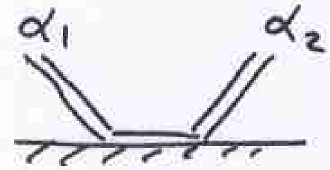
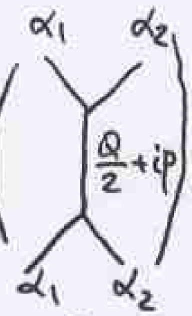


$$\langle \Phi_{\alpha_1}\, \Phi_{\alpha_2} \rangle = \frac{|z_2-\bar z_2|^{2\Delta_1-2\Delta_2}}{|z_1-\bar z_2|^{4\Delta_1}}\, G_{\alpha_1\alpha_2}(\eta)$$

$$\eta = \frac{(z_1-z_2)(\bar z_1-\bar z_2)}{(z_1-\bar z_2)(\bar z_1-z_2)}$$

$(\eta\to 0)$

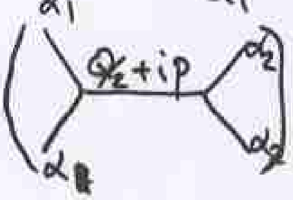
$$G_{\alpha_1\alpha_2} = \int dp\, \mathbb{C}(\alpha_1,\alpha_2,Q/2+ip)\, U(Q/2-ip)\Bigg($$



$\Bigg)$

$(\eta\to 1)$

$$G_{\alpha_1\alpha_2} = \int dp\, R(\alpha_1,\tfrac{Q}{2}+ip)\, R(\alpha_2,\tfrac{Q}{2}-ip)\Bigg($$



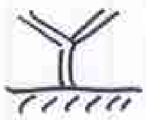
$\Bigg)$

Shift Relations

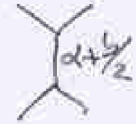
$$\langle \Phi_\alpha(z)\, \Phi_{-b/2}(\zeta) \rangle$$

$$\left(\frac{1}{b^2} L_{-1}^2 + L_{-2}\right) \Phi_{-b/2} = 0$$

1. $\Phi_{-b/2}\, \Phi_\alpha = C_+(\alpha)\left[\Phi_{\alpha - b/2}\right] + C_-(\alpha)\left[\Phi_{\alpha + b/2}\right]$



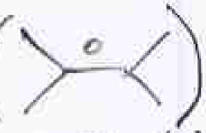
$$= C_+(\alpha)\, U(\alpha - b/2)\, \mathcal{F}_+(\eta) + C_-(\alpha)\, U(\alpha + b/2)\, \mathcal{F}_-(\eta)$$



2. $\Phi_{-b/2} = R(-b/2 \,|\, Q)\left[B_0\right] + R(-b/2 \,|\, Q+b)\left[B_{-b}\right]$

$$R(-b/2 \,|\, Q) = -\frac{2\pi \mu_B\, \Gamma(-1-2b^2)}{\Gamma(-b^2)}$$

$$= R(-b/2 \,|\, Q)\, U(\alpha) \left(\right) +$$



$$R(-b/2 \,|\, Q+b)\, R(\alpha \,|\, -b) \left(\right)$$

$$-\frac{2\pi \mu_B}{\Gamma(-b^2)}\, U(\alpha) = \frac{\Gamma(-b^2 + 2b\alpha)}{\Gamma(-1-2b^2+2\alpha b)}\, U(\alpha - b/2) -$$

$$-\frac{\pi \mu\, \Gamma(-1-b^2+2b\alpha)}{\gamma(-b^2)\, \Gamma(2b\alpha)}\, U(\alpha + b/2)$$

Solution

$$U(Q/2 + ip) = \widetilde{U}(p),$$

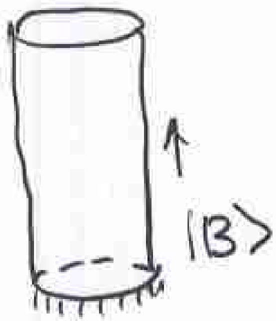
$$\widetilde{U}(p) = \left(\pi\mu\,\gamma(b^2)\right)^{-\frac{ip}{b}} \Gamma(1+2ibp)\,\Gamma\!\left(1+\frac{2ip}{b}\right) \frac{\cos(2\pi s p)}{ip}$$

where

$$\mathrm{ch}\,\pi b s = \frac{\mu_B}{\sqrt{\mu}} \sqrt{\sin \pi b^2}$$

Boundary state

$$\langle \cdots \rangle_B = \langle 0 | \cdots | B \rangle$$ "Boundary state"



In "compact" CFT

$$|B\rangle = \sum_\alpha U_\alpha \, |I_\alpha\rangle$$

"Ishibashi states"

$$|I_\alpha\rangle = \left(1 + \frac{L_{-1}\bar{L}_{-1}}{2\Delta_\alpha} + \cdots\right)|\alpha\rangle$$

In Liouville

$$|B_s\rangle = \frac{1}{2}\int dp\, \widetilde{U}(p)\, |I_p\rangle$$

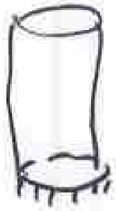
$$|I_p\rangle = \left(1 + \frac{L_{-1}\bar{L}_{-1}}{2\Delta_p} + \cdots\right)|p\rangle \qquad \Delta_p = \frac{Q^2}{4} + p^2$$

$$\widetilde{U}(p) = \langle p | B_s \rangle$$

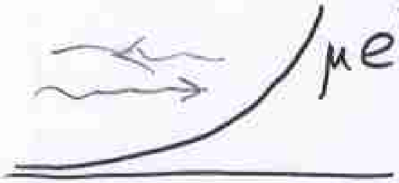
Semiclassical Case

$b \to 0$, $sb = s_0$ finite

Also $p \sim b$



$|B_s\rangle$



$\mu e^{2b\phi}$ $\qquad |p\rangle$

"Minisuperspace approximation"

$$\phi(x) = \phi_0 + \phi_{osc}(x) \; ; \; |p\rangle \to \Psi_p(\phi_0)$$

$$\left(-\frac{1}{2}\frac{d^2}{d\phi_0^2} + 2\pi\mu\, e^{2b\phi_0} - 2p^2\right)\Psi_p(\phi_0) = 0$$

$$\Psi_p(\phi_0) = \frac{2\left(\frac{\pi\mu}{b^2}\right)^{-\frac{ip}{b}}}{\Gamma\left(-\frac{2ip}{b}\right)} K_{\frac{2ip}{b}}\left(2\sqrt{\frac{\pi\mu}{b^2}}\, e^{b\phi_0}\right)$$

$$|B_s\rangle \to \Psi_{B_s}(\phi_0) = e^{-2\pi\mu_B e^{b\phi_0}}$$

$$\int_{-\infty}^{\infty} \Psi_{B_s}(\phi_0)\Psi_p(\phi_0)\, d\phi_0 =$$

$$\left(\frac{\pi\mu}{b^2}\right)^{-\frac{ip}{b}} \Gamma\left(1 + \frac{2ip}{b}\right)\frac{\cos(2\pi p s)}{ip}$$

Boundary length distribution

Boundary length $(ds^2 = e^{2b\phi} dz d\bar{z})$

$$\ell = \int_{\partial\Gamma} e^{b\phi(x)} dx$$

$$U(\alpha) = \int_0^\infty \frac{d\ell}{\ell} W_\alpha(\ell) e^{-\mu_B \ell}$$

$$W_\alpha(\ell) = \frac{2}{b} \left(\pi\mu\gamma(b^2)\right)^{\frac{Q-2\alpha}{2b}} \frac{\Gamma(2\alpha b - b^2)}{\Gamma(1+\frac{1}{b^2}-\frac{2\alpha}{b})} \times$$

$$K_{\frac{Q-2\alpha}{b}}(\varkappa\ell) \sim e^{-\varkappa\ell}$$

$$\boxed{\varkappa = \sqrt{\frac{\mu}{\sin\pi b^2}}}$$

Stability bound $\qquad \mu_B > -\sqrt{\frac{\mu}{\sin\pi b^2}}$

$$\mathrm{ch}\,\pi b s = \mu_B \sqrt{\frac{\sin\pi b^2}{\mu}}$$

I.e. $\quad s$ real $\qquad \left(\mu_B > \sqrt{\frac{\mu}{\sin\pi b^2}}\right)$

$\qquad s = i\sigma \qquad \left(|\mu_B| < \sqrt{\cdots}\right)$

$$\sigma = \pm\frac{1}{b} \iff \mu_B = -\sqrt{\frac{\mu}{\sin\pi b^2}}$$

"No boundary" solution

Classical Liouville

$$\Delta\varphi = \mu_0 e^{\varphi} \; ; \; \mu_0 = -R$$

Basic solution

$$ds^2 = e^{\varphi} dz\, d\bar{z} = \frac{1}{2|R|(\operatorname{Im} z)^2}$$

"Lobachevskiy plane" = "Poincare disc" = "Pseudosphere"
= "Euclidean AdS_2"

<u>Quantum theory</u>

- Bulk OPE
- Boundary state $|B\rangle$

Conformal Invariance →

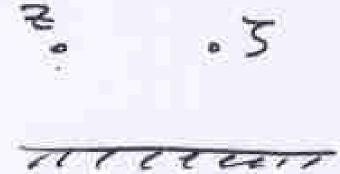
$$|B\rangle = \frac{1}{2}\int dp\, \tilde{V}(p)\, |I_p\rangle$$

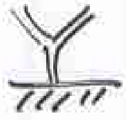
$$\tilde{V}(p) = V\left(\frac{Q}{2} + ip\right)$$

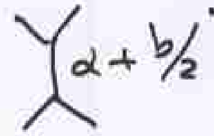
$$\langle \Phi_\alpha(z) \rangle = \frac{V(\alpha)}{|z-\bar{z}|^{2\Delta_\alpha}}$$

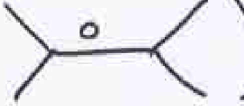
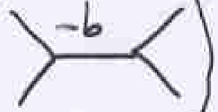
shift Relation analysis

$$\langle \Phi_\alpha(z)\,\Phi_{-b/2}(\xi)\rangle$$



 $= C_+(\alpha)\,V(\alpha - b/2)\,\Big($  $\Big) +$

$C_-(\alpha)\,V(\alpha + b/2)\,\Big($  $\Big)$

$= B_0(\alpha)\,\Big($  $\Big) + B_{-b}(\alpha)\,\Big($  $\Big)$

$$\eta = \frac{(\xi - z)(\bar\xi - \bar z)}{(\xi - \bar z)(\bar\xi - z)} = \mathrm{th}^2\Big(\sqrt{\tfrac{|R|}{8}}\, s(z,\xi)\Big)$$

geodesic distance

\$ Decay of correlations

$$\langle \Phi_{\alpha_1}(z)\,\Phi_{\alpha_2}(\xi)\rangle \xrightarrow{\eta\to 1} \langle \Phi_{\alpha_1}\rangle\langle \Phi_{\alpha_2}\rangle$$

I.e.

$$B_0(\alpha) = V(\alpha)\,V(-b/2)$$

Closed Equation

$$\frac{\Gamma(-b^2)\,V(\alpha)\,V(-b/2)}{\Gamma(-1-2b^2)\,\Gamma(2\alpha b - b^2)} = \frac{V(\alpha - b/2)}{\Gamma(2\alpha b - 2b^2 - 1)} -$$

$$- \frac{\pi\mu\,\Gamma(1+b^2)\,V(\alpha + b/2)}{(2\alpha b - b^2 - 1)\,\Gamma(-b^2)\,\Gamma(2\alpha b)}\ .$$

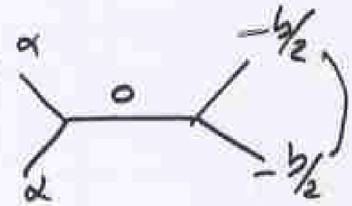
"Basic" solution

$$\tilde{V}(p) = V_{(1,1)}(p) =$$

$$\frac{\left(\pi\mu\gamma(b^2)\right)^{+\frac{ip}{b}}\Gamma(1+b^2)\Gamma(1+\frac{1}{b^2})}{\Gamma(-2pib)\Gamma(-\frac{2ip}{b})\, ip}$$

Properties

1. $B_{-b}(\alpha) = 0$, i.e.

$$\langle \Phi_\alpha \Phi_{-b/2} \rangle = V(\alpha)\, V(-b/2)\, \Bigg($$  $$\Bigg)$$

$|B_{(1,1)}\rangle$ allows no "boundary operators" exept for I

- $|B_{(1,1)}\rangle \sim |B_{i\sigma_+}\rangle - |B_{i\sigma_-}\rangle$

  $\sigma_\pm = \frac{1}{b}(1 \pm b^2)$ (compair to $\sigma_{crit} = \frac{1}{b}$)

- $V_{(1,1)}(\alpha)$ agrees with loop expansion around classical Lobachevskiy plane (up to two loops)

$$\phi = \phi_{cl} + \chi$$

$$e^{-A_{cl}} \int \mathcal{D}\chi\, e^{-A(\chi)}$$

General solution $(n, m > 0)$

$$\tilde{V}_{(n,m)}(p) = \frac{sh\left(\frac{\pi m p}{b}\right)}{sh\left(\frac{\pi p}{b}\right)} \cdot \frac{sh(\pi n b p)}{sh(\pi b p)} \tilde{V}_{(1,1)}(p)$$

$$\downarrow |B_{(n,m)}\rangle \longleftrightarrow \psi_{(n,m)}$$

$$\Delta_{(n,m)} = \frac{Q^2}{4} - \frac{\left(\frac{m}{b} + nb\right)^2}{4}$$

~~Ever~~ $|B_{(n,m)}\rangle$ admits "degenerate" boundary operators $B_{(p,q)}$
~~satyz degenera~~

$$\psi_{(n,m)} \psi_{(n,m)} = \bigoplus_{p,q} \psi_{(p,q)}$$

- $|B_{(1,m)}\rangle$ have "classical limit" $b \to 0$ which agrees with loop p.t. which agrees with two loops (but not two loops) to one loop (but not two loops)

- $\langle \Phi_{\alpha_1} \Phi_{\alpha_2} \rangle_{B_{n,m}} \to \langle \Phi_{\alpha_1} \rangle \langle \Phi_{\alpha_2} \rangle$ + growing term

Is there any meaningful interpretation to such "phases" in 2D quantum gravity?