Vortex condensation and black hole in the matrix model of 2-d string theory

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1. 2-d string theory, Sine-Liouville theory and 2-d black hole: V. Fateev, A. & Al. Zamolodchikov duality
2. Twisted periodic matrix quantum mechanics and vortices on planar graphs.
3. String partition function as a τ-function of Toda hierarchy
4. Scaling solutions of 2-d string theory for various genera from Toda equation.
5. „Black hole" limit
6. Correlators of vorticities
7. Comparison with dilaton gravity
8. Discussion

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A Matrix Model for the 2d Black Hole

I. Kostov, hep-th/0101011
D. Kutasov
V.K.

Two-dimensional bosonic string theory:

$$S = \frac{1}{4\pi}\int_{\Sigma} d^2\sigma \left[G_{\mu\nu}(X)\nabla_a X^\mu \nabla^a X^\nu + T(X) + \hat{R}^{(2)}\Phi(X) \right]$$

where $X^1 = x(\sigma_1,\sigma_2)$, $X^2 = \varphi(\sigma_1,\sigma_2)$,

has two classical solutions:

① Flat space: $G_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Phi = 2(\varphi_0 - \varphi)$, $T = 2(\varphi_0-\varphi)\, e^{-2(\varphi-\varphi_0)}$

Well described by the „old" Matrix (singlet) Quantum Mechanics (A. Migdal & V.K. 1988)

$$\mathcal{L} = \mathrm{tr}\left[\dot{M}^2 + V(M)\right]$$

World sheets ⟷ Planar Graphs

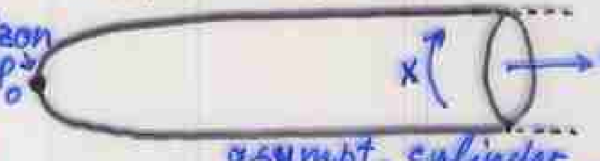
(F. David, V.K. 1985)

② „Cigar" space: $ds^2 = g(\varphi)\, d\varphi^2 + \frac{1}{g(\varphi)}\, dx^2$

$\Phi = 2(\varphi_0 - \varphi)$, $T=0$, $g(\varphi) = 1 - \exp\left[2(\varphi_0-\varphi)\frac{1}{R}\right]$

(G. Mandal et al., S. Elitzur et al. 1991)

$\boxed{\text{What is the „holographic" Matrix Model description?}}$



Our Model: condensation of Kosterlitz-Thouless-Berezinski vortices on the world sheet (described by non-singlet states in MQM)

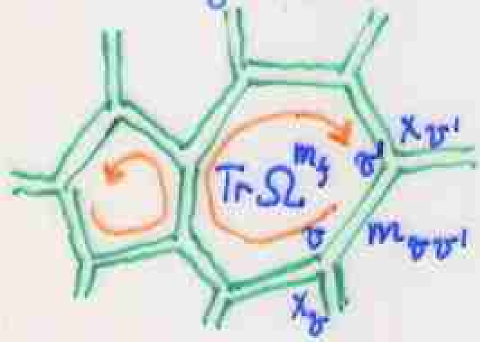
Matrix Quantum Mechanics and Vortices on planar graphs

$$Z_N[\Omega] = \int \mathcal{D}^{N^2} M(x) \exp -\mathrm{Tr}\int_0^{2\pi R} dx \left(\dot{M}_x^2 + M^2 + \frac{g}{\sqrt{N}} M^3\right)$$

$M(2\pi R) = \Omega^+ M(0)\, \Omega$ - twisted periodic boundary condition

$\Omega \in SU(N)$

Planar graphs as worldsheets in periodic 1D space



x_v - coord. of vertex v

$m_{vv'} \in \mathbb{Z}$ - winding number of propagator $\langle v v'$

$m_f = \sum_{\langle vv'\rangle \in f} m_{vv'}$ vorticity through the face f

Periodic propagator

$$x_v \underset{\Omega^{+m}}{\overset{\Omega^m}{\rightleftarrows}} x_{v'} = \sum_{m \in \mathbb{Z}} \exp\left[-|x_v - x_{v'} + 2\pi R\, m_{vv'}|\right] \cdot \Omega^m \otimes \Omega$$

String partition function

$$F_N(\Omega) = \sum_{h=0}^{\infty} N^{2-2h} \sum_K g^K \sum_{\text{graphs}} \sum_m \prod_{\text{faces}} \frac{\mathrm{tr}}{N} \Omega^{m_f} \times$$

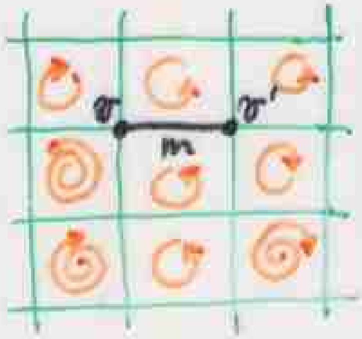
$$\times \int \prod_{\text{vertices}} dx_v \exp\left[-\sum_{\langle v v'\rangle} |x_v - x_{v'} + 2\pi R\, m_{vv'}\right.$$

{D. Gross '90
I. Klebanov

{D. Boulatov
V. K. '91

P. Zinn-Justin

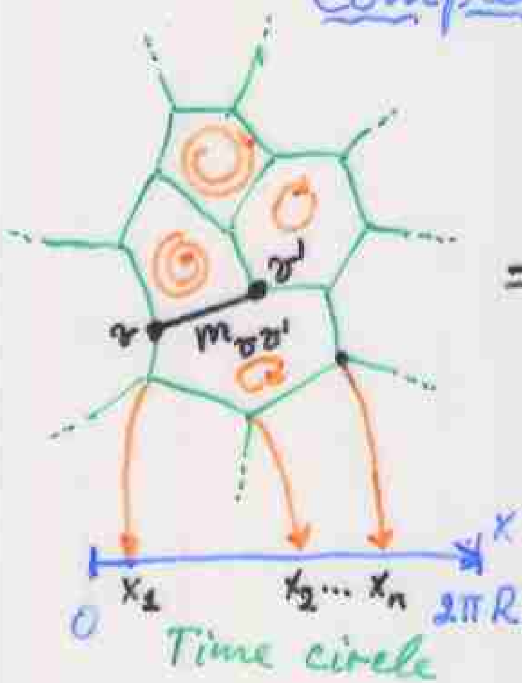
Comparison with Villain model of Berezinski-Kosterlitz-Thouless vortices



$$Z^{(R)}_{BKT} = \sum_{\{m\in\mathbb{Z}\}} \int_0^{2\pi R} \prod_{v} dx_v \prod_{\langle vv'\rangle} e^{-(x_v - x_{v'} + 2\pi R m_{vv'})^2}$$

vertices   edges

Compare with:



$$Z^{(R)}_n =$$

$$= \sum_{\substack{\text{Graphs}\\ G_n}} \sum_{\{m\in\mathbb{Z}\}} \int \prod_{v\in G_n} dx_v \prod_{\langle vv'\rangle\in G_n} e^{-|x_v - x_{v'} + 2\pi R m_{vv'}|}$$

$$m_f = \sum_{\langle vv'\rangle\in f} m_{vv'} \to$$ vorticity throug face f

face

New couplings: fugacities of vortices

$$Z_N[t_{\pm 1}, t_{\pm 2}, \dots] = \int [d\Omega]_{SU(N)} \exp\Big[\sum_{n\neq 0} t_n \,\mathrm{tr}\,\Omega^n\Big] Z_N(\Omega)$$

At $N \gg 1$: $(\mathrm{tr}\,\Omega^n)$ behave as independent gaussian random quantities

Douglas '94

$$\int [d\Omega]\, e^{\sum_n t_n \mathrm{tr}\,\Omega^n} = e^{\sum_n n\, t_n t_{-n}} + O(e^{-N})$$

Vorticity n_f through the face f has a weight t_{n_f}

Comment: there are also correlations between different faces: e.t.c.

They should be important for the correct combinatorics of graphs, as the results for singlet sector show

$\sim 1/N^2$

Grand canonical partition function

$$Z(\mu, t) \equiv e^{F(\mu,t)} = \sum_{N=1}^{\infty} e^{2\pi R \mu \cdot N} Z_N[t_{\pm 1}, t_{\pm 2}, \dots]$$

Expansion in irreducible representations

$$Z_N^{(R)}(\Omega) = \sum_r \underbrace{\chi_r(\Omega)}_{\substack{\text{character}\\ \text{in irrep } r}} \underbrace{\mathrm{Tr}_r\, e^{-\beta \hat{H}_r}}_{\substack{\text{Partition function}\\ \text{in irrep } r}} \qquad (\beta = 2\pi R)$$

Singlet

$$\Psi_{sing}(M) = \prod_{i>j}(x_i - x_j)\cdot \underbrace{\varphi(x_1,\dots,x_N)}_{\text{antisymmetric}}$$

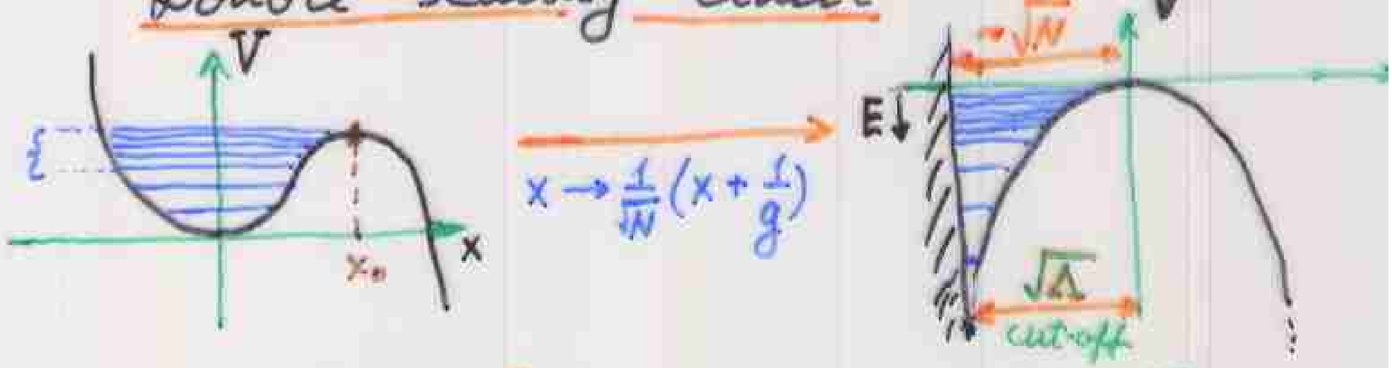
$$\sum_{k=1}^{N}\left(-\frac{1}{2N}\frac{\partial^2}{\partial x_k^2} + \frac{N}{2}x_k^2 - N\frac{g}{3}x_k^3\right)\varphi(x_1,\dots,x_N) = E\varphi(x_1,\dots$$

non-interacting fermions E. Brezin et al. '78

$$\varphi(x_1,\dots x_N) \sim \det_{i,j}\varphi_{n_i}(x_j)$$ - Slater det.

$$\left[-\frac{1}{2N}\frac{\partial^2}{\partial x^2} + N\left(\frac{x^2}{2}-\frac{g}{3}x^3\right)\right]\varphi_n(x) = E_n\varphi_n(x)$$

Double scaling limit



$$\frac{1}{2}\partial_x^2\varphi + \frac{x^2}{2}\varphi + \cancel{\frac{g x^3}{3\sqrt{N}}\varphi} = E\varphi$$ - parabolic cylinder eq.

$$\rho_{sing}(E) = \frac{1}{\pi}\mathrm{Re}\,\Psi\left(iE+\frac{1}{2}\right) + \frac{1}{\pi}\log\Lambda = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{k+1/2}{E^2+\left(k+\frac{1}{2}\right)^2}$$

Density of states

Grand canonical partiton function ($\frac{1}{T}=\beta=2\pi$

$$\log Z_{sing}(\mu) \equiv F_{sing}(\mu) = \int_{-\infty}^{\infty} dE\,\rho_{sing}(E)\log\left(1+e^{\beta(-\mu+E)}\right)$$

$$= -\frac{R}{2}\mu^2\log\frac{\mu}{\Lambda} - \frac{R+R^{-1}}{24}\log\mu + \sum_{h=2}^{\infty}\frac{f_h(R)}{\mu^{2h-2}} + O(e^{-2\pi\mu})$$

μ - chemical potential conjugated to N

D. Gross & I. Klebanov

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Matrix Model of the two dimensional black hole

SOMETIMES THE CIGAR IS JUST A CIGAR ...
S. FREU

'00 { D. Kutasov, I. Kostov, V. K

Euclidean 2D black hole
$$\begin{cases} ds^2 = \frac{k}{2}\left[dr^2 + 2\tanh^2\frac{r}{2}\, d\theta^2\right] \\ \varphi - \varphi_0 = -2\log\cosh\frac{r}{2} \end{cases}$$

A. Zamolodchikov, Al. Zamolodchikov, V. Fateev } conjecture

$R = 3/2$

2D bosonic string perturbed by Sine-Gordon term ("Sine-Liouville")

$$\mathcal{L}_{c=1} = \frac{1}{4\pi}\left[(\partial x)^2 + (\partial\varphi)^2 + Q\hat{R}\varphi + \lambda e^{(R-2)\varphi}\cos R(x_L - x_R) + \mu\varphi e^{-2\varphi}\right]$$

$Q = 2$

'90 { D. Gross, I. Klebanov

'91 { D. Boulatov, V. K

'88 A. Migdal, V. K.

Matrix model of $c=1$ string with vortices

$$S = \int_0^{2\pi R} dx\ \mathrm{tr}\left[(M_x')^2 - M^2\right]$$

$$M(2\pi R) = \Omega^+ M(0)\,\Omega$$

$M_{ij} = M_{ji}^*$

$\Omega \in SU(N)$

'00 { D. Kutasov, I. Kostov, V. K.

Toda Chain hierarchy: $\partial_{\lambda_+}\partial_{\lambda_-} F_\mu = \exp\left[2F_\mu - F_{\mu+i} - F_{\mu-i}\right]$

Free energy $F_\mu(\lambda) = \tau$-function: $\exp F_\mu(\lambda) = \sum_{N=1}^{\infty} e^{2\pi R N\mu} Z_N(\lambda)$

$$Z_N(\lambda) = \int [d\Omega]_{SU(N)}\, e^{\lambda\,\mathrm{tr}(\Omega^+ + \Omega)}\, Z_N(\Omega)$$

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Partition function as Toda τ-function

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hep-th/010101

In terms of twist angles:

$$\Omega = \mathrm{diag}(z_1, z_2, \dots, z_N)$$

the partition function looks as solitonic τ-function of Toda hierarchy (after taking the gaussian matrix integral)

$$Z[\mu + il, t] = e^{\sum_n n t_n t_{-n}} \tau_l[t] =$$

$$= \sum_{N=1}^{\infty} \frac{e^{2\pi R(\mu+il)N}}{N!} \oint \prod_{k=1}^{N} \frac{dz_k}{2\pi i} \frac{e^{\sum_n \lambda_n z_k^n}}{(q^{1/2} - q^{-1/2})^N} \prod_{m\neq j} \frac{z_m - z_j}{q^{1/2} z_m - q^{-1/2} z_j}$$

$$\boxed{\lambda_n = (q^{n/2} - q^{-n/2}) t_n} \qquad q = e^{i2\pi R}$$

Fredholm determinant representation

$$F[\mu, \lambda] = \log Z[\mu, t] = \log \mathrm{Det}\left[\mathbb{1} + e^{2\pi R\mu} \hat{\mathcal{K}}\right]$$

$$\hat{\mathcal{K}} f(z) = \oint \frac{dz'}{2\pi i} \frac{\exp\left[\sum_n \lambda_n (z^n + z'^n)\right]}{q^{1/2} z - q^{-1/2} z'} f(z')$$

Earlier discussions of Toda structure in 2-d string theory: R. Dijkgraaf, G. Moore, R. Plesser '92
T. Eguchi, H. Kanno '94
K. Takasaki '94

They should be related to Toda structure of our model

Toda equation for free energy.

$$\lambda_{\pm 1} = \lambda e^{\pm i\alpha} , \quad \lambda_{\pm 2} = \lambda_{\pm 3} = \ldots = 0$$

$t_{\pm n} = \pm \lambda_n$

$$\frac{\partial}{\partial \lambda_{+1}} \frac{\partial}{\partial \lambda_{-1}} F(\mu, \lambda_\pm) + e^{2F(\mu,\lambda_\pm) - F(\mu+i,\lambda_\pm) - F(\mu-i,\lambda_\pm)} = 1$$

Charge symmetry: $\lambda_n \to e^{in\alpha} \lambda_n$

$$\frac{1}{4} \lambda^{-1} \partial_\lambda \lambda \partial_\lambda F(\mu,\lambda) + e^{2F(\mu,\lambda) - F(\mu+i,\lambda) - F(\mu-i,\lambda)} = 1$$

Boundary condition (free fermions on a circle)

$$\partial_\mu^2 F(\mu,0) = -\frac{1}{4} \int_{\Lambda^{-1}}^{\infty} \frac{ds \cdot \sin \mu s}{\sinh \frac{s}{2} \sinh \frac{s}{2R}} =$$

$$\underset{\mu \to \infty}{\simeq} - R \log \mu/\Lambda - \frac{R + R^{-1}}{24} \cancel{\log \mu/\Lambda} + \sum_{h=2}^{\infty} \frac{f_h(R)}{\mu^{2h-2}} + O(e^{-2\pi\mu}) + O(e^{-2\pi R \mu})$$

T-duality: $\begin{cases} R \longleftrightarrow \frac{1}{R} \\ \mu \longrightarrow R\mu \end{cases}$

Zero mode of Toda eq.

$$F \to F + \sum_n (A_n + B_n \log \lambda) e^{-2\pi n \mu}$$

non-perturbative terms

(11)

Topological expansion (in $1/\mu \sim 1/N$)

Scaling ansatz: $\mu \to \infty, \ \lambda \to \infty, \ y = \frac{\mu}{\lambda^{\frac{2}{2-R}}}$ - fix.

$$F(\mu,\lambda) = -\frac{R}{2}\mu^2 \log \mu/\Lambda + \mu^2 \tilde{F}_0(y) -$$

$h=0$

$$- \frac{R+R^{-1}}{24} \log \mu/\Lambda + \tilde{F}_1(y) +$$

$h=1$

$$+ \sum_{h=2}^{\infty} \frac{1}{\mu^{2h-2}} F_h(y)$$

$h \geq 2$

Toda equation

$$\frac{1}{4}\lambda^{-1}\partial_\lambda \lambda \partial_\lambda F + \exp -\underbrace{\left(2\sin^2\left[\frac{1}{2}\partial_\mu\right]\right)}_{\text{expand in } \partial_\mu^{2n} F} F = 0$$

Sphere

$$\partial_\mu^2 F_0 \simeq -R \log \lambda^{\frac{2}{2-R}} + X_0(y)$$

$$\frac{1}{4}\lambda^{-1}\partial_\lambda \lambda \partial_\lambda F + e^{-\partial_\mu^2 F} = 0 \implies (y\partial_y)^2 X_0 = \partial_y^2 e^{X_0}$$

Solution $\boxed{y = e^{R^{-1}X_0} - e^{(1-R^{-1})X_0}}$

$$F_0(\lambda,\mu) = -\frac{R}{2}\mu^2 \log \mu + R\mu^2 \sum_{n=1}^{\infty} \left(\frac{\lambda^2}{\mu^{2-R}}\right)^n \frac{\Gamma(n(2-R)-2)}{n!\,\Gamma(n(1-R)+1)}$$

G. Moore '90

self-dual radius

Kosterlitz-Thouless radius

$1 < R < 2$: analytical continuation to $\mu \ll \lambda^{\frac{2}{2-R}}$ ("black hole" limit):

$$\boxed{F_0(\lambda,\mu) = -\frac{(2-R)^2}{R-1}\lambda^{\frac{4}{2-R}}} + \frac{R\mu^2}{2-R}\sum_{n=0}^{\infty}\left(\frac{\mu}{\lambda^{\frac{2}{2-R}}}\right)^{n-1} \frac{\Gamma\left(\frac{n-1}{2-R}\right)}{(n+1)!\,\Gamma\left(\frac{n-1}{2-R}-n+2\right)}$$

$$-\frac{R}{2}\mu^2 \log \lambda^{\frac{4}{2-R}}$$

$\mu\to\infty$: Sphere partition function $F_0(\mu,\lambda)$

$$\begin{cases} \frac{1}{4}\lambda^{-1}\partial_\lambda\lambda\partial_\lambda F_0 + e^{-\partial_\mu^2 F_0} = 0 & \text{- Toda equation} \\ F_0(\mu,0) = -\frac{R}{2}\mu^2\log\mu & \text{- boundary condition} \end{cases}$$

λ^2 expansion (in # vortices)

$$F_0(\mu,\lambda) \simeq -\frac{R}{2}\mu^2\log\mu - \lambda^2\mu^R + O(\lambda^4\mu^{2R-2})$$

$$\simeq \mu^2\left(-\frac{R}{2}\log\mu - \left[\frac{\lambda}{\mu^{\frac{2-R}{2}}}\right]^2 + O\left(\left[\frac{\lambda}{\mu^{\frac{2-R}{2}}}\right]^4\right)\right)$$

Natural scaling ansatz:

$$\partial_\mu^2 F_0 = -R\log\mu + f_0\left(\frac{\lambda^2}{\mu^{2-R}}\right) =$$

$$= -R\log\lambda^{\frac{2}{2-R}} + X_0(y) \qquad \boxed{y = \frac{\mu}{\lambda^{\frac{2}{2-R}}}}$$

From the Liouville theory of 2-d string:

$$\lambda^2\left\langle \int d^2z\, e^{(R-2)\varphi}\cos(Rx)\cdot\int d^2z'\, e^{(R-2)\varphi'}\cos(Rx')\right\rangle_{\varphi,x} \sim \lambda^2\mu^R$$

Correct scaling!

μ-expansion ("black hole" limit)

$$\boxed{F_0(\lambda,\mu) \simeq -\frac{(2-R)^2}{R-1}\lambda^{\frac{4}{2-R}}} - \frac{(2-R)^3}{R(R-1)}\mu\cdot\lambda^{\frac{2}{2-R}} + O(\mu^2)$$

Toda eq. gives a universal non-zero free energy in the black hole limit!

(12)

Partition functions for higher genera

Torus: $h=1$

$$\boxed{F_1(\lambda,\mu) = \frac{R+R^{-1}}{24}\left[-\log\lambda^{\frac{2}{2-R}}\right.} + R^{-1}X_0(y)\Big] -$$

$$-\frac{1}{24}\log\left[1-(R-1)\exp\left(\left(\tfrac{2}{R}-1\right)X_0(y)\right)\right]$$

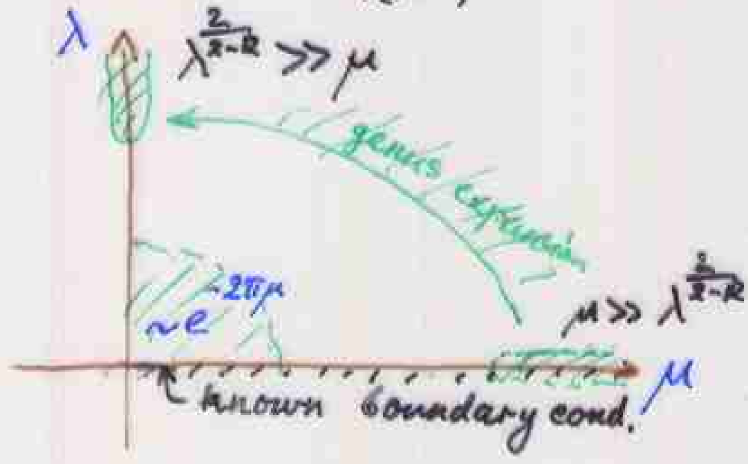
Any genus h:

$$\left(y\partial_y+2h-2\right)^2 f_h - e^{-X_0(y)}\,\partial_y^2 f_h = \Phi(f_0, f_1, \dots, f_{h-1})$$

II-nd order linear ODE with polynomial coeff.

„Black hole" limit: $y=\frac{\mu}{\lambda^{\frac{2}{2-R}}}\longrightarrow 0$

$$F(\lambda,0) = -\frac{(2-R)^2}{4(R-1)}\lambda^{\frac{4}{2-R}} - \frac{R+R^{-1}}{48}\log\lambda^{\frac{4}{2-R}} + \sum_{h=2}^{\infty}\frac{f_h(0)}{\lambda^{\frac{4-4h}{2-R}}}$$



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Correlators of vortices on the sphere

S. Alexandrov
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hep-th/0104094

$$\tilde{X}_{i_1 \dots i_n} = \frac{\partial^n}{\partial t_{i_1} \dots \partial t_{i_n}} \log \tau_0 \Big|_{t_{\pm 2} = t_{\pm 3} = \dots = 0}$$

$\mu = t_0, \ \lambda = t_{\pm 1}$ - fixed

Generating functions of 2-point and 1-point correlators

$$\Phi_{\pm}(a, b) = 2 \sum_{m,n=0}^{\infty} \frac{a^m b^n}{m \cdot n} \tilde{X}_{\pm m, n}$$

$$h(a) = \Phi(a, 0) = \sum_{n=0}^{\infty} \frac{a^n}{n} \tilde{X}_{0,n}$$

Analyzing Hirota equations in dispersionless (sphere) limit we found:

$$\Phi_+(a,b) = \log \left[\frac{4ab}{(a-b)^2} \, sh^2\left(\frac{1}{2}\left(h(a) - h(b) + \log \frac{a}{b}\right)\right)\right]$$

$$\Phi_-(a,b) = 2 \log \left(1 - A ab \, e^{h(a) + h(b)} \right)$$

Knowing these correlators and the free energy $F(\mu, \lambda)$ (or $X_0(y)$) we get:

$$e^{R^{-1} h} - z e^h = 1$$

Need to compare with Sine-Liouville calc
V. Fateev (2001)
T. Fukuda, K. Hosomichi } (2001)

with: $z = a \, \lambda^{\frac{R}{2-R}} e^{-\frac{R-1}{R} X_0(y)}$

correct scaling

$$\tilde{X}_n = \frac{\Gamma(nR+1)(2-R)n \ \lambda^{\frac{nR+2}{2-R}}}{(n+1)! \, \Gamma(n(R-1)+2)\,(R-1)^{n/2}} \left[\frac{n+1}{n(2-R)} e^{-\frac{n(R-1)+1}{R} X_0} - \frac{n(R-1)+1}{n(2-R)} e^{-(n+1)\frac{R-1}{R} X_0} \right]$$

(14) <u>Conformal</u> <u>field theory</u>: (described by matrix quantum mechanics)

compactified bosonic field $X(z_1, z_2)$ coupled to 2-d gravity (Liouville field $\varphi(z_1, z_2)$)

$$\mathcal{L}_{String} = \frac{1}{4\pi}\Big[(\partial X)^2 + (\partial\varphi)^2 - 4\hat{R}\varphi + \mu\varphi e^{-2\varphi} + \sum_{n\neq 0} t_n \underbrace{e^{(|n|R-2)\varphi}\cdot e^{inR(X_L - X_R)}}_{\substack{\text{operator of vortex, charge} = n \\ \text{conf. dimension } (1,1)}}\Big]$$

Central charge $\underline{c = 26}$

Target space dimensions (scaling t_n with μ) can be read off from zero mode $\varphi \to \varphi + \varphi_0$

$$t_n \sim \mu^{1 - \frac{|n|R}{2}} \quad , \quad g_s \sim 1/\mu \ (\text{string coupling})$$

<u>Sine-Liouville $\Longleftrightarrow$ Black hole duality</u>

V. Fateev
Al. & A. Zamolodchikov

$$\mathcal{L}_{SL} = \frac{1}{4\pi}\Big[(\partial X)^2 + (\partial\varphi)^2 - 2Q\hat{R}\varphi + \lambda e^{b\varphi} \underbrace{\cos R(X_L - X_R)}_{\text{charges } \pm 1}\Big]$$

$$c = 2 + Q^{-2}, \quad b = -1/Q, \quad R^2 = 2 + Q^{-2}$$

$\Updownarrow$ duality

$$S_{BH} = \frac{k}{2\pi}\,\mathrm{tr}\Big[\int d^2z\, \nabla g^{-1}\nabla g + \int d^2z\, dy\, \nabla g \wedge \nabla g \wedge \nabla g\Big]$$

$$c = \frac{3k}{k-2} - 1 \quad , \quad g \in \frac{SL(2,C)}{SU(2)\times U(1)}$$

level k WZNW model

At $\underline{c = 26}$, $k = \frac{9}{4}$, $\underset{\mu=0}{\underline{R = 3/2}}$: same as our $\mathcal{L}_{string}$.

Matrix model of 2-d black hole!

V. K., I. Kostov, D. Kutasov — hep-th/0101011

Hirota identity for Toda τ-function

Free fermion representation:

$$\psi(z) = \sum_{r\in\mathbb{Z}} \psi_r z^r , \quad \psi^*(z) = \sum_{r\in\mathbb{Z}} \psi^*_r z^{-r-1}$$

$$[\psi_r, \psi^*_s]_+ = \delta_{rs}$$

Hamiltonians: $H_n = \sum_{r\in\mathbb{Z}} :\psi^*_{r-n}\psi_r:$

Vacuum of charge ℓ: $\psi_r|\ell\rangle = \psi^*_r|\ell\rangle = 0 \quad (r<\ell)$

$$\langle \ell|\psi^*_{-r} = \langle\ell|\psi_r = 0$$

Correlator: $\langle\ell|\psi(z)\,\psi^*(z')|\ell\rangle = \dfrac{(z'/z)^\ell}{z-z'}$

$GL(\infty)$ group element:

$$g = \exp\left[e^{2\pi R\mu}\oint\frac{dz}{2\pi}\,\psi(q^{1/2}z)\,\psi^*(q^{-1/2}z)\right]$$

2-d string partition function as a τ-function: $\tau_\ell[t] \equiv \exp F[\mu - i\ell, t]$

$$\tau_\ell[t] = \Big\langle \ell\Big|\exp\Big(\sum_{n>0}t_nH_n\Big)\cdot g\cdot\exp\Big(\sum_{n<0}t_nH_n\Big)\Big|\ell\Big\rangle$$

Hirota identity — Schur polynomials

$$\sum_{j=0}^{\infty} p_{j+i}(-2y_+)\,p_j(\tilde D_+)\exp\Big(\sum_{k\neq 0}y_kD_k\Big)\tau_{i+\ell+1}[t]\cdot\tau_\ell[t] =$$

$$= \sum_{j=0}^{\infty} p_{j-i}(-2y_-)\,p_j(\tilde D_-)\exp\Big(\sum_{k\neq 0}y_kD_k\Big)\tau_{i+\ell}[t]\cdot\tau_{\ell+1}[t]$$

$y_\pm = (y_{\pm1}, y_{\pm2},\dots)$; $\tilde D_\pm = (D_{\pm1}, D_{\pm2},\dots)$

$D_n f[t]\cdot g[t] = \partial_x f(t_n+x)\,g(t_n-x)\big|_{x=0}$ – Hirota derivative

Toda equation for $F(\mu, \lambda = t_{\pm 1})$ is given by coefficient of y_{-1} in Hirota identity:

$$\tau_\ell \partial_1 \partial_{-1} \tau_\ell - \partial_1 \tau_\ell \partial_{-1} \tau_\ell + \tau_{\ell+1} \tau_{\ell-1} = 0$$

To calculate correlators we need higher Hirota relations.

Coefficient of $y_n \cdot y_m$ $(n, m \geq 0)$ gives: $\;i=0$, $D_0 \equiv 1$

$$\left[2 p_{m+n}(\tilde{D}_+) - p_n(\tilde{D}_+) D_m - p_m(\tilde{D}_+) D_n\right] \tau_{s+1} \cdot \tau_s = 0$$

Multiplying by $a^n \cdot b^m$ and summing over m, n:

$$\Big[\frac{2}{a-b}\Big(a e^{\sum_{k=1}^{\infty} a^k \tilde{D}_k} - b e^{\sum_{k=1}^{\infty} b^k \tilde{D}_k}\Big) -$$

$$- \sum_{m=0}^{\infty} b^m D_m \exp\Big\{\sum_{k=1}^{\infty} a^k \tilde{D}_k\Big\} - \sum_{m=0}^{\infty} a^m D_m \exp\Big\{\sum_{k=1}^{\infty} b^k \tilde{D}_k\Big\}\Big] = 0$$

<u>Notations</u>: $\quad \tau_s[t] = \exp F[\mu - is, t]$

$$\Phi_+(a, b) = \sum_{n,m=0}^{\infty} a^n b^m \tilde{X}_{n,m}$$ – generating function of 2-point correlators

where $\tilde{X}_{n,m} = \frac{2}{m \cdot n} \partial_{t_n} \partial_{t_m} F$; $\tilde{X}_{0,n} = \frac{i}{n} \partial_\mu \partial_{t_n} F$

<u>Spherical</u> (dis<u>persionless</u>) limit:

only 1-st and 2-nd derivatives of $F_0[\mu, t]$ survive.

Using the formula $f(D) e^F \cdot e^G = e^F \cdot e^G f(D + \partial F \cdot 1 - 1 \cdot \partial G)$

we get the PDF:

$$a e^{-\frac{1}{2}\Phi(a,a)} \partial_a \Phi(a,b) + b e^{-\frac{1}{2}\Phi(b,b)} \partial_b \Phi(a,b) = \frac{a+b}{a-b}\Big[e^{-\frac{1}{2}\Phi(b,b)} - e^{-\frac{1}{2}\Phi(a,a)}$$

Analiticity of $\Phi_+(a,b)$ at $a = b = 0$ leads to the unique solution in terms of one point correlator $h(a) = \Phi_+(a, 0)$

One point correlator of vorticity

$$h(a) = \Phi_+(a,0) = \sum_{n=0}^{\infty} \frac{a^n}{n} \partial_\mu \partial_{t_n} F$$

can be determined from:

i) explicite 2-point correlator

$$\Phi_+(a,b) = \log\left[\frac{4ab}{(a-b)^2} sh^2\left(\frac{1}{2}\left(h(a) - h(b) + \log\frac{a}{b}\right)\right)\right]$$

ii) known dependence of free energy on μ, λ $(= \pm t_{\pm 1})$

iii) definition of $\Phi_+(a,b)$, which gives the equation:

$$\partial_\mu \partial_b \Phi(a,b)\big|_{b=0} = \partial_\lambda h(a) + \partial_\lambda \partial_\mu F$$

or $$\partial_\mu \frac{2}{a}\left(1 - e^{-h(a)}\right) = \partial_\lambda h(a) + \partial_\lambda \partial_\mu F$$

One can also write a similar equation starting from $\Phi_-(a,b)$.

The solution is:

$$\boxed{e^{R^{-1}h} - z\, e^{h} = 1}$$

where $z = a \cdot \lambda^{\frac{R}{2-R}} \cdot \frac{e^{(R^{-1}-1)X_0}}{\sqrt{R-1}}$,

and $X_0\left(\frac{\mu}{\lambda^{\frac{2}{2-R}}}\right)$ is related to the free energy and satisfies: $y = e^{-R^{-1}X_0} - e^{-(R^{-1}-1)X_0}$

Comparison with the low energy effective action of σ-model

A. Tseytlin, V.K. '01 hep-th/0104138

$$S_{eff} = \int_{bulk} dx\, d\varphi \sqrt{G}\, e^{-2\Phi}\Big[-\frac{1}{2}\big(R + 4D^2\Phi - 4(\partial\Phi)^2\big) + \frac{4}{\alpha'}\Big] +$$

$$+\ 2\int_{Boundary} dx\, \partial_n\big(\sqrt{G}\, e^{-2\Phi}\big)$$

bulk

Boundary ($\varphi \to \infty$)

$\Lambda \sim e^{-2\varphi}$ - IR cutoff (UV on the world sheet)

On the black hole solution only the boundary term contributes:

$$\mathcal{F} = S_{eff}^{(Bound)}(G_{BH}, \Phi_{BH}) =$$

$$= -\frac{4\pi}{1-p} e^{-2\varphi_0}\Big[1 + \sqrt{p^2 + 4(1-p)\, e^{-4(\varphi-\varphi_0)}}\Big]$$

$$\approx -\frac{8\pi}{\sqrt{1-p}}\Lambda^2 - \frac{4\pi}{1-p} e^{-2\varphi_0} + O(\Lambda^{-2})$$

$\boxed{p = \frac{2}{R} = \frac{2}{3}}$ $\boxed{R=3}$

Compare with the matrix model result

$$\boxed{F(\lambda, 0) \sim -\lambda^{\frac{4}{2-R}}}$$

right scaling!

$$g_s^{-2} \sim \lambda^{\frac{4}{2-R}} \sim e^{-2\varphi_0}$$

Non-zero free energy of the black hole!

Contradicts Gibbons & Perry '92

Comments and Problems

1. Toda hierarchy and commuting flows

$$\mathcal{L} = \frac{1}{4\pi}\left[(\partial X)^2 + (\partial\varphi)^2 + 2\hat{R}\varphi + \mu\varphi e^{\varphi} + \sum_n t_n e^{(|n|R-2)\varphi + inR(X_L - X_R)}\right.$$

$$\updownarrow$$

$$Z(\mu, t_1, t_2, t_3, \dots) = \int d\Omega\, e^{\operatorname{tr}\sum_{n\in\mathbb{Z}} t_n \Omega^n} Z(\mu, \Omega)$$

$t_1, t_2, \dots$ – Toda „times"

Correlators of winding modes can be calculated by Hirota equations

Momentum modes ?

2. Minkowski continuation: by use of eigenvalue hamiltonians in fixed irreps r:

$$\hat{H}_r = P_r \sum_{k=1}^{N}\left[-\frac{1}{2}\frac{\partial^2}{\partial x_k^2} - \frac{1}{2}x_k^2\right] + \frac{1}{2}\sum_{i\neq j}\frac{t^r_{ij}\, t^r_{ij}}{(x_i - x_j)^2}$$

$$Z_N(R,\lambda) = \sum_r g_r(\lambda)\, \operatorname{Tr}_r e^{-2\pi R \hat{H}}$$

Direct calculation of the entropy of states of the black hole ?

Black hole formation ?

3. Das-Jevicki-Polchinski collective field action in presence of vortices and the target space picture ?

4. More evidence for the black hole physics from MQM ?

What is special at $R = \frac{3}{4}R_{KT}$ there?