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Komaba Lectures on Noncommutative Solitons and D-Branes

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Abstract

These lectures provide an introduction to noncommutative geometry and its origins in quantum mechanics and to the construction of solitons in noncommutative field theory. These ideas are applied to the construction of D-branes as solitons of the tachyon field in noncommutative open string theory. A brief discussion is given of the K-theory classification of D-brane charge in terms of the K-theory of operator algebras. Based on lectures presented at the Komaba 2000 workshop, Nov. 14-16 2000.

2/01

0. Introduction . . . 20
1. Lecture 1 . . . 20
1.1. Motivation . . . 20
1.2. Noncommutative Geometry . . . 21
1.3. The Weyl Transform . . . 22
1.4. Noncommutative Geometry in Quantum Mechanics . . . 28
1.5. Noncommutative Field Theory and Solitons . . . 28
1.6. The Solution Generating Technique . . . 31
2. Lecture 2 . . . 33
2.1. Derivatives and Gauge Fields in Noncommutative Field Theory . . . 33
2.2. Noncommutative Solitons at Finite θ . . . 36
2.3. Vortices and Their Analogs in Commutative Field Theory . . . 37
2.4. The Noncommutative ABS Construction . . . 39
3. Lecture 3 . . . 41
3.1. String Theory and Noncommutative Geometry . . . 41
3.2. Bosonic D-branes as Noncommutative Solitons . . . 43
3.3. Type II D-branes as Noncommutative Solitons . . . 47
4. Lecture 4 . . . 49
4.1. Noncommutative Solitons and K-theory . . . 49
4.2. D-brane Charge and the Index of Toeplitz Operators . . . 52
4.3. BDF and all that . . . 54

0. Introduction

These notes are based on lectures given at the Komaba workshop in November of 2000. The notes follow the lectures fairly closely, although I have tried to correct minor errors made in the lectures and to expand on some points which may have caused confusion. In addition, I have divided the third lecture into two lectures and have included more material on the relation between D-branes and the K-theory of operator algebras than I had time to present in the original lectures.

The first lecture discusses how noncommutative geometry arises in quantum mechanics, formulates simple noncommutative field theories, and constructs soliton solutions of these theories. In the second lecture derivatives and gauge fields are introduced and the soliton solutions are generalized in various ways. The third lecture summarizes how noncommutative field theory arises in string theory, discusses unstable D-brane systems and their tachyon excitations, and shows that the noncommutative solitons constructed in the previous lecture lead to configurations of the tachyon field that have the properties of D-branes. The final lecture discusses more mathematical material (although at a fairly low-brow level) which shows how viewing D-branes as noncommutative solitons naturally fits into the rich theory of operator algebras and their K-theory.

Many aspects of noncommutative field theory and tachyon condensation in string theory are not covered here. More mathematically sophisticated reviews of solitons in noncommutative gauge theory which cover many topics not discussed here can found in [1,2]. These references also provide more complete references to the mathematical literature. The star product and its historical origin in quantum mechanics is reviewed in [3]. The lectures by Berkovits [4], Taylor [5], and Zwiebach [6] at this meeting review much of the recent progress in tachyon condensation using cubic string field theory [7] and its supersymmetric generalization [8]. Another approach to tachyon condensation not discussed at this meeting is based on boundary conformal field theory and the closely related techniques of background independent string field theory [9,10]. See [11,12,13,14,15,16,17,18,19,20,21] for treatments using this approach.

1. Lecture 1

1.1. Motivation

In these lectures I will be discussing the application of noncommutative geometry and field theory to the problem of open string tachyon condensation, specifically to the problem of constructing D-branes as solitons in the tachyon field. There are various reasons why I think this topic is of some interest. First, noncommutative geometry and field theory have an interesting structure which parallels the structure of string theory. Soliton solutions in field theory and string theory are always interesting and often shed light on the non-perturbative and strong coupling behavior of the theory, thus these solutions should be investigated in noncommutative field theories. That noncommutative field theory arises in a limit of string theory makes this pursuit even more compelling. Second, open string tachyon condensation is one example of a class of problems that we need to understand better in string theory, namely identifying the vacuum and its excitations in the absence of supersymmetry. In addition, if we are able to obtain a clear understanding of how closed string physics emerges after condensation of open string modes, it is likely to teach us new things about the connection between open and closed strings. Finally, the connection between tachyon condensation and noncommutative geometry is in its own right quite striking and suggests that noncommutative geometry plays a more fundamental role in string theory than I would have previously suspected .

1.2. Noncommutative Geometry

Noncommutative geometry is based on the following idea [22]. The structure of an ordinary commutative manifold M can be captured algebraically by the algebra $A = C^\infty(M)$ of smooth functions on M, $f : M \to \mathbf{C}$ with the product in the algebra being the commutative multiplication of functions. Although it is hard to see what noncommutative geometry should be directly, noncommutative algebras are quite familiar. Thus one tries to deform the commutative algebra A to a noncommutative algebra with a product $*_\hbar$ such that

$$f *_\hbar g = fg + \hbar P(f, g) + \cdots \tag{1.1}$$

where P is a bilinear map $P : A \times A \to A$. As will be explained below, the $\hbar$ notation is supposed to suggest an analogy to quantum mechanics. As $\hbar \to 0$ the noncommutative algebra approaches the commutative algebra defined by the ordinary product of functions. Given such a noncommutative algebra, the noncommutative geometry is then defined in terms of noncommutative generalizations of the algebraic constructs corresponding to various elements in geometry. For example, points in the commutative theory can be defined in terms of ideals since functions vanishing at a point form an ideal in the algebra A.

We will not pursue this very general level of discussion as there is to my knowledge no general definition of noncommutative geometry which starts with an arbitrary manifold

M [23]. An example which has been much studied in the literature occurs when M is taken to be the phase space of a classical system (that is M is a Poisson manifold) and $P(f,g) = \{f,g\}_{P.B.}$ is the Poisson bracket [24]. I will specialize even further and take $M = \mathbf{R}^{2n}$ with a star product defined by

$$f * g(x) = e^{\frac{i}{2}\theta^{ij}\partial_i\partial'_j} f(x)g(x')|_{x^i=x'^i} \tag{1.2}$$

where $\theta^{ij} = -\theta^{ji}$ is a non-degenerate constant antisymmetric matrix (thus θ defines a symplectic form and the skew eigenvalues of θ are analogous to the deformation parameter $\hbar$).

Before explaining the origin of the star product, first note the following. For small θ^{ij} we have

$$f * g(x) = f(x)g(x) + \frac{i}{4}\theta^{ij}(\partial_i f\partial_j g - \partial_j f\partial_i g) + \cdots \tag{1.3}$$

so that the order θ term is indeed the Poisson bracket with symplectic form θ^{ij}. Equation (1.2) thus implies the star bracket

$$x^i * x^j - x^j * x^i \equiv [x^i, x^j] = i\theta^{ij}. \tag{1.4}$$

Above we have taken f, g to be complex functions. Later we will also want to consider real functions, $f: M \to \mathbf{R}$. If f, g are real functions then $f * g$ is not in general real since $(f * g)^{\text{c.c}} = (g * f)$ (we denote complex conjugation by c.c). However $f * f$ is real if f is, and we will use this later to construct an action for a real scalar field on a noncommutative space.

The star product takes a simple form in momentum space. Let

$$\tilde{f}(\vec{\tau}) = \int d^{2n}x e^{i\vec{\tau}\cdot\vec{x}} f(\vec{x}) \tag{1.5}$$

be the Fourier transform of f and $\tilde{g}$ the Fourier transform of g. Then one easily finds that the Fourier transform of $f * g$ is

$$\widetilde{f * g}(\vec{\tau}) = \frac{1}{(2\pi)^{2n}} \int d^{2n}\tau' e^{\frac{i}{2}\theta^{ij}\tau_i\tau'_j} \tilde{f}(\tfrac{1}{2}\vec{\tau} + \vec{\tau}')\tilde{g}(\tfrac{1}{2}\vec{\tau} - \vec{\tau}'). \tag{1.6}$$

1.3. The Weyl Transform

To explain the origin of the star product in quantum mechanics it will be useful to first introduce the Weyl transform. For simplicity I will take $M = \mathbf{R}^2$ and relabel the coordinates as $x_1 = q, x_2 = p$ to conform to standard notation in quantum mechanics. Then $\theta^{12} = \hbar$ although for simplicity I will usually set $\hbar = 1$.

In quantum mechanics the coordinate and momentum operators $\hat{q}, \hat{p}$ obey the Heisenberg commutation relation,

$$[\hat{q}, \hat{p}] \equiv \hat{q}\hat{p} - \hat{p}\hat{q} = i. \tag{1.7}$$

Weyl [25] proposed that we regard this as the Lie algebra of a group (now termed the Weyl-Heisenberg group) with elements

$$U(\tau, \sigma) = \exp[-i(\tau\hat{q} + \sigma\hat{p})]. \tag{1.8}$$

If we denote the adjoint operator by $\overline{U}$ then we have

$$\begin{aligned} U(\tau, \sigma)\hat{q}\,\overline{U}(\tau, \sigma) &= \hat{q} - \sigma, \\ U(\tau, \sigma)\hat{p}\,\overline{U}(\tau, \sigma) &= \hat{p} + \tau. \end{aligned} \tag{1.9}$$

Thus U acts as translations in phase space. However, using the Campbell-Baker-Hausdorff formula we find

$$U(\tau_1, \sigma_1)U(\tau_2, \sigma_2) = e^{-i(\tau_1\sigma_2 - \sigma_1\tau_2)/2}U(\tau_1 + \tau_2, \sigma_1 + \sigma_2). \tag{1.10}$$

Thus we have a representation up to a phase, or a projective representation of the Abelian group of translations in phase space. We can as usual view this as a true representation of a larger group with elements $U'(\alpha, p, q) = e^{i\alpha}U(p, q)$. We will see that the phase factor responsible for the projective representation is at the heart of noncommutative geometry.

Weyl further suggested that we should view operators $\hat{O}(\hat{q}, \hat{p})$ as sums of group elements or more formally as elements of the group algebra [1]

$$\hat{O}_f(\hat{q}, \hat{p}) = \frac{1}{(2\pi)^2} \int d\sigma d\tau U(\tau, \sigma)\tilde{f}(\tau, \sigma). \tag{1.11}$$

If we take $\tilde{f}(\tau, \sigma)$ to be the Fourier transform of a function on phase space,

$$\tilde{f}(\tau, \sigma) = \int dq dp e^{i(\tau q + \sigma p)} f(q, p), \tag{1.12}$$

[1] A group algebra is the algebra formed by linear combinations of group elements with the bilinear product in the algebra given by group multiplication.

then in the limit $\hbar \to 0$ so that $[\hat{q},\hat{p}] \to 0$, $\hat{O}_f$ is simply f. Furthermore, the formula (1.11) gives an ordering prescription (called Weyl ordering) for constructing an operator from a classical function on phase space:

$$\hat{O}_f(\hat{q},\hat{p}) = \frac{1}{(2\pi)^2}\int d\sigma d\tau dq dp e^{-i\tau(\hat{q}-q)-i\sigma(\hat{p}-p)} f(q,p). \tag{1.13}$$

We can also invert the map given by (1.11), $f(q,p) \to \hat{O}_f(\hat{q},\hat{p})$, to associate functions to operators. It is conventional to break the symmetry between coordinates and momenta by evaluating matrix elements of operators between coordinate eigenstates $|q\rangle$. Using conventions with

$$\langle q'|q\rangle = \delta(q-q'), \qquad \langle p'|p\rangle = 2\pi\delta(p-p'), \tag{1.14}$$

we can project out $\tilde{f}$ using

$$\mathrm{Tr}_{\mathcal{H}} U(\tau,\sigma)\overline{U}(\tau',\sigma') = 2\pi\delta(\tau-\tau')\delta(\sigma-\sigma'), \tag{1.15}$$

and then perform the Fourier transform to find

$$f(q,p) = \int d\sigma' e^{-ip\sigma'}\langle q+\sigma'/2|\hat{O}_f(\hat{q},\hat{p})|q-\sigma'/2\rangle. \tag{1.16}$$

We will refer to the formulae (1.11) and (1.16) relating functions on phase space to operators on Hilbert space as the Weyl transform and will call functions and operators so related Weyl transforms of each other (in more mathematical treatments [26] $f(q,p)$ is called the Weyl symbol of $\hat{O}_f$ but we will ignore the distinction between functions and symbols here).

If $\hat{O} = |\psi\rangle\langle\psi|$ is the projection operator onto a state $|\psi\rangle$ then the phase space function corresponding to $\hat{O}$ is

$$f_\psi(q,p) = \int d\sigma' e^{-ip\sigma'}\langle q+\sigma'/2|\psi\rangle\langle\psi|q-\sigma'/2\rangle = \int d\sigma' e^{-ip\sigma'}\psi(q+\sigma'/2)\psi^*(q-\sigma'/2). \tag{1.17}$$

This function is usually called the Wigner distribution function of the state ψ. The Wigner distribution function and more generally the Weyl transform (1.16) have many applications, particularly in the analysis of semi-classical physics [27].

One useful identity relates the integral of a phase space function to the trace of its Weyl transform:

$$\begin{aligned}\int dq dp f(q,p) &= \int dq dp d\sigma' e^{-ip\sigma'}\langle q+\sigma'/2|\hat{O}_f|q-\sigma'/2\rangle \\ &= \int dq d\sigma' 2\pi\delta(\sigma')\langle q+\sigma'/2|\hat{O}_f|q-\sigma'/2\rangle \\ &= 2\pi\int dq\langle q|\hat{O}_f|q\rangle = 2\pi\mathrm{Tr}_{\mathcal{H}}\hat{O}\end{aligned} \tag{1.18}$$

It will be useful to distinguish certain classes of operators and to understand their representation via the Weyl transform. Recall that an operator O is said to be bounded if there is a number k such that $||Ox|| \leq k||x||$ for all x in a Hilbert space $\mathcal{H}$ (which we always take to be separable). The set of all bounded operators forms a C^* algebra denoted $\mathcal{B}(\mathcal{H})$. The compact operators form a subset of bounded operators consisting of the norm completion of finite rank operators. There are various equivalent definitions of compact operators. For example, an operator K is compact if, for every bounded sequence (x_n) in $\mathcal{H}$, the sequence (Kx_n) contains a convergent subsequence, or equivalently, if the image under K of the unit ball in Hilbert space is compact.

If O is bounded and K compact, then clearly KO and OK are compact, so the compact operators form an ideal in the algebra of bounded operators on $\mathcal{H}$. In fact, the vector space of compact operators, $\mathcal{K}(\mathcal{H})$, is the only norm closed and two-sided ideal in $\mathcal{B}(\mathcal{H})$ and thus plays a distinguished role. In the relation between commutative and noncommutative geometry $\mathcal{B}(\mathcal{H})$ should be thought of as analogous to $C_b(X)$, the set of all bounded continuous functions on a space X, while $\mathcal{K}(\mathcal{H})$ should be thought of as the analog of $C_0(X)$, the C^* algebra of continuous functions on X which vanish at infinity. This interpretation is supported by the following relation between operators and their Weyl transforms.

Let $\mathcal{S}^m$ be the set of functions on phase space defined by the condition

$$\mathcal{S}^m = \{f(q,p)|\ |\partial_q^\alpha \partial_p^\beta f| \leq C_{\alpha\beta}(1+q^2+p^2)^{(m-\alpha-\beta)/2}\} \tag{1.19}$$

for constants $C_{\alpha\beta}$. Then one can show that f being in $\mathcal{S}^m$ with $m \leq 0$ implies that $\hat{O}_f \in \mathcal{B}(\mathcal{H})$ while $m < 0$ implies that $\hat{O}_f \in \mathcal{K}(\mathcal{H})$. Thus we see that the Weyl transforms of bounded operators are bounded functions and the Weyl transforms of compact operators vanish at infinity in $\mathbf{R}^2$. These definitions have obvious generalizations to $\mathbf{R}^{2n}$.

In what follows it will be useful to keep two examples in mind. Let $|n\rangle$, with n a non-negative integer be an orthonormal basis for a separable Hilbert space $\mathcal{H}$. A rank k projection operator such as

$$P_k = |0\rangle\langle 0| + |1\rangle\langle 1| + \cdots + |k-1\rangle\langle k-1| \tag{1.20}$$

is an important example of a compact operator. An example of a bounded operator which will appear later is the shift operator

$$S = \sum_{n=0}^{\infty} |n+1\rangle\langle n| \tag{1.21}$$

Although many of the calculations to be presented later can be done without making an explicit choice of basis for Hilbert space, it can be useful in developing physical intuition to choose a basis and to work out the Wigner functions (and their generalizations) using this basis. Thus we consider a basis of the separable Hilbert space $\mathcal{H} = L^2(R)$ consisting of the simple harmonic oscillator eigenstates $|n\rangle$, $n = 0, 1, 2, \cdots\infty$. We write an arbitrary operator on $\mathcal{H}$ as

$$\hat{O} = \sum_{n,m=0}^{\infty} O_{n,m}|n\rangle\langle m|. \tag{1.22}$$

Let $f_{nm}(q,p)$ be the Weyl transform of the operator $|n\rangle\langle m|$:

$$f_{nm}(q,p) = \int dye^{-ipy}\langle q + \frac{y}{2}|n\rangle\langle m|q - \frac{y}{2}\rangle. \tag{1.23}$$

These functions can most easily be computed by introducing the generating function

$$G(\lambda, \overline{\lambda}, q, p) = \sum_{n,m} \frac{\lambda^n}{\sqrt{n!}}\frac{\overline{\lambda}^m}{\sqrt{m!}} f_{nm}(q,p). \tag{1.24}$$

I will follow the treatment in [28] which also contains references to the original literature. Define $\lambda = -(\sigma + i\tau)/\sqrt{2}$, $a = (\hat{q} + i\hat{p})/\sqrt{2}$, and note that the coherent state

$$\begin{aligned} |\lambda\rangle =& U(\tau, -\sigma)|0\rangle = e^{\lambda\overline{a} - \lambda^* a}|0\rangle \\ =& e^{-|\lambda|^2/2}e^{\lambda\overline{a}}|0\rangle \end{aligned} \tag{1.25}$$

obeys

$$|\lambda\rangle\langle\lambda| = U|0\rangle\langle 0|\overline{U} = e^{-|\lambda|^2}\sum_{n,m}\frac{\lambda^n}{\sqrt{n!}}\frac{\overline{\lambda}^m}{\sqrt{m!}}|n\rangle\langle m|. \tag{1.26}$$

We thus have

$$G(\lambda, \overline{\lambda}, q, p) = e^{|\lambda|^2}\int dye^{-ipy}\langle q + y/2|U|0\rangle\langle 0|\overline{U}|q - y/2\rangle. \tag{1.27}$$

Since $\overline{U}|q - y/2\rangle = e^{i\sigma\tau/2 + i\tau(q-y/2)}|q - y/2 + \sigma\rangle$, and $\langle q|0\rangle = \psi_0(q) = \pi^{-1/4}e^{-q^2/2}$, the integral is an elementary Gaussian integral and gives

$$G(\lambda, \overline{\lambda}, q, p) = 2e^{|\lambda|^2}e^{-(q+\sigma)^2 - (p+\tau)^2}. \tag{1.28}$$

Note in particular that $f_{00} = 2e^{-q^2-p^2}$.

An explicit formula for the other f_{nm} can be derived in terms of the Laguerre polynomials,

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\left(\frac{d}{dx}\right)^n (e^{-x}x^{\alpha+n}). \tag{1.29}$$

Introduce polar coordinates in phase space so that $q + ip = re^{i\phi}$. Then a little simple algebra allows us to write the generating function as

$$\begin{aligned} G =& 2e^{-r^2} e^{\bar\lambda\sqrt{2}re^{i\phi}-\bar\lambda\lambda}\ e^{\lambda\sqrt{2}re^{-i\phi}} \\ =& 2e^{-r^2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\bar\lambda\sqrt{2}re^{i\phi}\right)^n\left(1-\frac{\lambda}{\sqrt{2}r}e^{-i\phi}\right)^n e^{\lambda\sqrt{2}re^{-i\phi}}. \end{aligned} \tag{1.30}$$

Setting $y = 2r^2$ and $k = -y^{-1/2}\lambda e^{-i\phi}$ we have

$$G = 2e^{-y/2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\bar\lambda\sqrt{y}e^{i\phi}\right)^n (1+k)^n e^{-yk}. \tag{1.31}$$

The payoff for these manipulations is that we can now use the identity for Laguerre functions

$$\sum_{m=0}^{\infty} L_m^{n-m}(y)k^m = e^{-yk}(1+k)^n, \tag{1.32}$$

to write

$$G = 2e^{-y/2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n!}\left(\bar\lambda\sqrt{y}e^{i\phi}\right)^n L_m^{n-m}(y)\left(-y^{-1/2}\lambda e^{-i\phi}\right)^m. \tag{1.33}$$

Reading off the powers of $\bar\lambda$ and λ then gives the final formula for f_{nm}:

$$f_{nm}(r,\phi) = 2e^{-r^2}\sqrt{\frac{n!}{m!}}(-1)^n(2r^2)^{\frac{m-n}{2}}e^{i\phi(m-n)}L_n^{m-n}(2r^2). \tag{1.34}$$

1.4. Noncommutative Geometry in Quantum Mechanics

Let me now explain how the star product (sometimes called the Moyal [29] or Weyl-Moyal product, although it seems to have first appeared explicitly in work of Groenewold [30], see also [31]) arises in quantum mechanics. Quantum Mechanics is inherently noncommutative due to the noncommutative algebra of quantum mechanical operators. Since in general $\hat O_f \hat O_g \neq \hat O_g \hat O_f$, it is natural to ask how this is reflected in the composition law of the Weyl transforms f, g. We can address this question by working out the Weyl transform of the operator $\hat O_f \hat O_g$. We expect to find a noncommutative composition law that reduces to the ordinary commutative product of functions as $\hbar \to 0$.

Using the previous expressions for the Weyl transform we have

$$
\begin{aligned}
\hat{O}_f\hat{O}_g =&\frac{1}{(2\pi)^4}\int d\sigma_1 d\tau_1 d\sigma_2 d\tau_2 U(\tau_1,\sigma_1)U(\tau_2,\sigma_2)\tilde{f}(\tau_1,\sigma_1)\tilde{g}(\tau_2,\sigma_2)\\
=&\frac{1}{(2\pi)^4}\int d\sigma_1 d\tau_1 d\sigma_2 d\tau_2 U(\tau_1+\tau_2,\sigma_1+\sigma_2)e^{\frac{i}{2}(\tau_1\sigma_2-\sigma_1\tau_2)}\tilde{f}(\tau_1,\sigma_1)\tilde{g}(\tau_2,\sigma_2).
\end{aligned}
\tag{1.35}
$$

Now change variables to $\tau_3=\tau_1+\tau_2, \tau_4=(\tau_1-\tau_2)/2$ and $\sigma_3=\sigma_1+\sigma_2, \sigma_4=(\sigma_1-\sigma_2)/2$ to find

$$
\begin{aligned}
\hat{O}_f\hat{O}_g =&\frac{1}{(2\pi)^2}\int d\sigma_3 d\tau_3 U(\tau_3,\sigma_3)\Big[\frac{1}{(2\pi)^2}\int d\sigma_4 d\tau_4 e^{\frac{i}{2}(\sigma_3\tau_4-\tau_3\sigma_4)}\\
&\tilde{f}(\tau_4+\tau_3/2,\sigma_4+\sigma_3/2)\tilde{g}(-\tau_4+\tau_3/2,-\sigma_4+\sigma_3/2)\Big].
\end{aligned}
\tag{1.36}
$$

From (1.6) we see that the quantity in square brackets is just the Fourier transform of $f*g$ for noncommutative $\mathbf{R}^2$ and with $\theta^{12}=1$. Thus we have with this choice understood,

$$
\hat{O}_f\hat{O}_g=\frac{1}{(2\pi)^2}\int d\sigma_3 d\tau_3 U(\tau_3,\sigma_3)\widetilde{f*g}(\tau_3,\sigma_3)=\hat{O}_{f*g}. \tag{1.37}
$$

In other words, the Weyl transform takes operator multiplication into the star product of functions on phase space.

1.5. Noncommutative Field Theory and Solitons

We will now leave the realm of quantum mechanics and use the noncommutative star product which we found there (generalized as in (1.2)) to formulate noncommutative field theories. Instead of viewing the noncommutative coordinates as phase space coordinates as in quantum mechanics, we now will think of them as the spatial coordinates of a noncommutative space. We will here consider only space-space noncommutativity. Including temporal noncommutativity introduces new complications which are only resolved in string theory [32,33,34,35]. We will also assume that we have skew diagonalized θ so that

$$
\theta^{ij}=\begin{pmatrix}
\begin{pmatrix}0&\theta_1\\-\theta_1&0\end{pmatrix} & 0 & \dots & 0\\
0 & \begin{pmatrix}0&\theta_2\\-\theta_2&0\end{pmatrix} & \dots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & \begin{pmatrix}0&\theta_n\\-\theta_n&0\end{pmatrix}
\end{pmatrix} \tag{1.38}
$$

To define noncommutative field theories we consider fields which are functions of time, some set of commutative coordinates y^A and noncommuting coordinates x^i. To begin we consider noncommutative field theories in $2+1$ dimensions involving real scalar fields $\phi(t, x^1, x^2)$ (equivalently, the Weyl transform is a Hermitian operator) with $[x^1, x^2] = i\theta$. Although we will soon want to introduce derivatives and also gauge fields, let's start by considering an action with only potential terms to illustrate one of the new features that arises in the study of solitons in noncommutative field theories. We thus consider the action

$$S = \int dt dx_1 dx_2 V_*(\phi) \tag{1.39}$$

where the $*$ subscript on V indicates that products of fields are evaluated using the star product. We can easily insert the θ dependence into previous formulae by noting that the rescaled coordinates $\tilde{x}^i = x^i/\sqrt{\theta}$ obey $[\tilde{x}^1, \tilde{x}^2] = i$.

Assuming that V is polynomial in ϕ, we can shift ϕ by a constant so that up to an overall constant V has the form

$$V_*(\phi) = \frac{m^2}{2}\phi * \phi + c_1 \phi * \phi * \phi + \cdots \tag{1.40}$$

Using the Weyl correspondence we can also write the action in terms of $\hat{\phi}$, the Weyl transform of ϕ (denoted $\hat{O}_\phi$ previously) as

$$S = 2\pi\theta \int dt \mathrm{Tr}_{\mathcal{H}} V(\hat{\phi}) \tag{1.41}$$

We will use the operator formalism and drop the hats on operators when there is little chance of confusion. Also, in $\mathbf{R}^{2n}$ the prefactor of θ is replaced by $\mathrm{Pf}(\theta^{ij}) = \prod_\alpha \theta_\alpha$.

The equation of motion for this rather trivial action is just the vanishing of the first derivative of V, $V'(\phi) = 0$. Given (1.40) this takes the form

$$V'(\phi) = c\phi(\phi - \lambda_1)\cdots(\phi - \lambda_n) = 0 \tag{1.42}$$

where $\lambda_0 = 0, \lambda_1, \cdots \lambda_n$ are the critical points of V and c is a constant. In commutative field theory the only solution of this equation would be a constant field, $\phi = \lambda_i$ for some i, but in the noncommutative theory we can construct nontrivial solutions. In particular, if

$$\phi = \lambda_i P \tag{1.43}$$

where P is a projection operator, $P^2 = P$, we also get a solution since $V'(\phi) = \lambda_i P \cdots \lambda_i (P-1) = 0$. More generally, if P_i are a set of orthogonal projection operators then $\phi = \sum_i \lambda_i P_i$ is also a solution. Of course since $1 - P$ is a projection operator if P is, $\phi = \lambda_i(1 - P)$ is also a solution. Which of these solutions we choose depends on what we are trying to describe. If $\phi = 0$ is the global minimum of $V(\phi)$ then finite sums of the form $\phi = \sum_i \lambda_i P_i$ with P_i finite rank will be finite energy excitations above the vacuum given by $\phi = 0$. On the other hand, if some $\lambda_i = \lambda_*$ is the global minimum of $V(\phi)$ then a solution $\phi = \lambda_*(1 - P)$ has finite energy. Clearly we also obtain solutions that describe finite energy excitations above (or below) an unstable vacuum, although the physical interpretation of these solutions is less clear.

The existence of simple soliton solutions in noncommutative scalar field theories was first discovered by Gopakumar, Minwalla and Strominger [36][2]. There are two important caveats to keep in mind. First, the expression in terms of projection operators holds only as long as we define the fields so that $V(\phi)$ has no linear term as in (1.40). Second, as pointed out in [38], for special choices of potential there can be soliton solutions which are not of the form (1.43).

To see what these solutions look like we can consider a simple choice of projection operator such as the projection operator onto the harmonic oscillator ground state, $|0\rangle\langle 0|$, and write it as a function on the noncommutative $\mathbf{R}^2$ using the Weyl transform. Using (1.34) this gives

$$P = |0\rangle\langle 0| \to 2e^{-(x_1^2+x_2^2)/\theta}. \tag{1.44}$$

Thus we obtain a localized solution, even in a field theory without kinetic terms [36]! This is possible because the noncommutativity parameter θ introduces a length scale into the problem.

The energy of the solution (1.43) is easily evaluated using $\phi^n = \lambda_i^n P$ to give

$$E = 2\pi\theta \mathrm{Tr}_{\mathcal{H}} V(\phi) = 2\pi\theta V(\lambda_i) \mathrm{Tr}_{\mathcal{H}} P = 2\pi\theta n V(\lambda_i) \tag{1.45}$$

where $n = \mathrm{Tr}_{\mathcal{H}}\, P$ is the rank of the projection operator P.

1.6. The Solution Generating Technique

Before going on to add in derivative terms and couplings to gauge fields, I want to discuss a more formal way to construct this solution which will be useful when we come to more complicated situations. This construction is analogous to solution generating techniques that have been used in a wide variety of other contexts.

[2] Similar constructions appeared earlier in the context of Matrix Theory [37].

The Lagrangian we have used so far,

$$\mathcal{L} = 2\pi\theta \mathrm{Tr}_{\mathcal{H}} V(\phi), \tag{1.46}$$

is invariant under the group of unitary transformations on Hilbert space, $U(\mathcal{H})$[3]. Thus the transformation

$$\phi \to U\phi\overline{U}, \qquad \overline{U}U = U\overline{U} = 1 \tag{1.47}$$

leaves $\mathcal{L}$ invariant and takes solutions of the equation of motion to solutions since

$$\frac{dV}{d\phi} \to U\frac{dV}{d\phi}\overline{U} \tag{1.48}$$

(note that this requires the vanishing of linear terms in V as in (1.40), for further discussion see [40]). Once we add derivatives and gauge fields we will interpret $U(\mathcal{H})$ as a gauge symmetry and solutions related by $U(\mathcal{H})$ transformations as gauge equivalent, but for now we view $U(\mathcal{H})$ as a global symmetry of the simple action (1.46).

In showing that solutions of the equation of motion transform into solutions, that is, that the equations of motion transform covariantly, it was only necessary to use $\overline{U}U = 1$ since this is all that is required to show that powers of ϕ transform covariantly. In a finite dimensional Hilbert space $\overline{U}U = 1$ would imply that $U\overline{U} = 1$, but this is not true in an infinite dimensional Hilbert space. Operators U obeying $\overline{U}U = 1$ are called isometries because they preserve the metric or inner product on Hilbert space:

$$\langle\chi|\psi\rangle \to \langle\chi|\overline{U}U|\psi\rangle = \langle\chi|\psi\rangle. \tag{1.49}$$

If it is also true that $U\overline{U} = 1$ then U is unitary. Thus if we find a non-unitary isometry it will still map solutions to solutions, but these solutions will not be related by the global symmetry (or later the gauge symmetry) of the action. Note that $U\overline{U}U\overline{U} = U\overline{U}$ so $U\overline{U}$ is a projection operator if U is an isometry.

The standard example of a non-unitary isometry is the shift operator

$$S:\ |n\rangle \to |n+1\rangle, \qquad S = \sum_{n=0}^{\infty} |n+1\rangle\langle n|. \tag{1.50}$$

[3] The group $U(\mathcal{H})$ is distinct from $U(\infty)$ defined as the inductive limit of $U(n)$. For example, $U(\mathcal{H})$ is contractible [39] while $\pi_{2n-1}(U(\infty)) = Z$ for n a positive integer.

Clearly $\overline{S}S = 1$ but $S\overline{S} = 1 - P \equiv 1 - |0\rangle\langle 0|$. Note that $\ker S = \{\emptyset\}$ and $\ker \overline{S} = \{|0\rangle\}$ so that the index of S is -1. More generally, $U = S^n$ is a non-unitary isometry with index $-n$ and $U\overline{U} = 1 - P_n$ with $P_n = \sum_{k=0}^{n-1} |k\rangle\langle k|$.

To apply the solution generating technique we start with the trivial constant solution $\phi = \lambda_i I$ with I the identity operator. Transforming this with $U = S^n$ we obtain the new solution $\phi = S^n \lambda_i I \overline{S}^n = \lambda_i(1 - P_n)$. We can thus interpret the solution (1.43) as the result of acting with the solution generating transformation $U = S^n$ on the constant solution $\phi = \lambda_i I$, and if we choose λ_i to be a global minimum of V, then this solution will describe a finite energy excitation above this vacuum.

A slightly more complicated example of this technique is illustrated by starting with the action for a complex noncommutative scalar field ϕ (in operator language we drop the assumption $\phi = \overline{\phi}$). We take

$$\mathcal{L} = 2\pi\theta \mathrm{Tr}_{\mathcal{H}} \left[W(\phi\overline{\phi} - 1) + W(\overline{\phi}\phi - 1) \right] \tag{1.51}$$

and assume that W is stationary at $\overline{\phi}\phi = \phi\overline{\phi} = 1$. In later applications W will be a "mexican hat" potential with a local maximum at $\phi = 0$ and a ring of minima at $|\phi| = 1$.

The Lagrangian $\mathcal{L}$ and the equations of motion are now invariant under $U(\mathcal{H}) \times U(\mathcal{H})$ transformations,

$$\phi \to V\phi\overline{U}, \qquad U, V \in U(\mathcal{H}) \tag{1.52}$$

and the equations of motion are covariant as long as $\overline{U}U = \overline{V}V = 1$. In this case we can generate new solutions from old using independent non-unitary isometries acting on the left and right. Since $\phi = I$ is a solution of the equations of motion, $\phi = V\overline{U}$ will be also if U, V are non-unitary isometries. Note that $\phi = V\overline{U}$ implies that

$$\phi\overline{\phi}\phi = \phi. \tag{1.53}$$

In general, an operator obeying (1.53) is called a partial isometry. Note that (1.53) implies that $\overline{\phi}\phi$ and $\phi\overline{\phi}$ are projection operators. On the subspace of $\mathcal{H}$ where $\overline{\phi}\phi$ has eigenvalue $+1$, ϕ acts as an isometry, hence the name partial isometry.

Choosing $V = S^m$, $U = S^n$, we conclude from the above discussion that $\phi = S^m\overline{S}^n$ is a solution to the equations of motion. Later on we will interpret this solution as m vortices and n anti-vortices or as m $D7$-branes and n $\overline{D7}$-branes following [38,41].

2. Lecture 2

2.1. Derivatives and Gauge Fields in Noncommutative Field Theory

So far we have found non-trivial solutions to trivial noncommutative field theories. To make these theories more realistic we need to understand how to add derivative terms and couplings to gauge fields. For a review of these topics see [42].

The inclusion of derivatives again goes back to the work of Weyl [25]. Returning to the notation of quantum mechanics for a moment, let $f(q,p)$ be a function on phase space and consider the operator corresponding to $\partial f/\partial q$. That is,

$$\hat{O}_{\partial_q f} = \frac{1}{(2\pi)^2}\int d\sigma d\tau dq dp \frac{\partial f}{\partial q} U(\tau,\sigma) e^{(i\tau q + i\sigma p)}. \tag{2.1}$$

Assuming as usual that f has sufficiently fast fall off that we can integrate by parts, we find

$$\hat{O}_{\partial_q f} = \frac{1}{(2\pi)^2}\int d\sigma d\tau dq dp (-i\tau f(q,p)) U(\tau,\sigma) e^{(i\tau q + i\sigma p)}. \tag{2.2}$$

Now note that $U\ \hat{p}\ \overline{U} = \hat{p} + \tau$ implies $[U,\hat{p}] = \tau U$. Applying the same logic to $\hat{O}_{\partial_p f}$ we conclude that

$$\hat{O}_{\partial_q f} = i[\hat{p},\hat{O}_f], \qquad \hat{O}_{\partial_p f} = -i[\hat{q},\hat{O}_f]. \tag{2.3}$$

Weyl then noted that if we define the derivatives of an operator O to be

$$\begin{aligned} \frac{\partial \hat{O}}{\partial q} &= i[\hat{p},\hat{O}] \\ \frac{\partial \hat{O}}{\partial p} &= -i[\hat{q},\hat{O}], \end{aligned} \tag{2.4}$$

then the Heisenberg equations of motion,

$$\begin{aligned} i\frac{\partial \hat{p}}{\partial t} &= [\hat{p},\hat{H}] \\ i\frac{\partial \hat{q}}{\partial t} &= [\hat{q},\hat{H}], \end{aligned} \tag{2.5}$$

take the same form as the classical Hamilton equations,

$$\begin{aligned} \frac{dp}{dt} &= -\frac{\partial H}{\partial q} \\ \frac{dq}{dt} &= \frac{\partial H}{\partial p}. \end{aligned} \tag{2.6}$$

In the noncommutative situation this is easily generalized. With $[x^i, x^j] = i\theta^{ij}$ the natural derivative operator is

$$\partial_i = -i\theta_{ij}[x^j, \ \ ] \equiv -i\theta_{ij}\text{ad } x^j \tag{2.7}$$

with $\theta_{ij} = (\theta^{ij})^{-1}$. This satisfies the properties we expect of a derivative operator, namely $[\partial_i, \partial_j] = 0$, $\partial_i x^j = \delta_i^j$ as well as

$$\begin{aligned} \text{Linearity}: &\quad \partial_i(aO_1 + bO_2) = a\partial_i O_1 + b\partial_i O_2 \\ \text{Leibniz}: &\quad \partial_i(O_1 O_2) = (\partial_i O_1)O_2 + O_1(\partial_i O_2). \end{aligned} \tag{2.8}$$

It is useful to introduce complex coordinates. In two noncommuting dimensions we set $z = (x^1 + ix^2)/\sqrt{2}$ and define $a = z/\sqrt{\theta}$ which then obeys the commutation relation of a annihilation operator, $[a, \overline{a}] = 1$. We then have

$$\begin{aligned} \partial \equiv \partial_z &= -\,\theta^{-1/2}\text{ad } \overline{a} \\ \overline{\partial} \equiv \partial_{\overline{z}} &= \theta^{-1/2}\text{ad } a. \end{aligned} \tag{2.9}$$

In $2n$ dimensions we will block diagonalize θ^{ij} as in (1.38) and introduce complex coordinates z_α, $\alpha = 1, 2 \cdots n$ for each pair of noncommuting coordinates. We then have $\partial_\alpha = -\theta_\alpha^{-1/2}\text{ad } a_\alpha$.

We can now formulate noncommutative field theories that contain kinetic terms as well as potential terms. For example we could consider a scalar field $\phi(t, x_i)$ in $2+1$ dimensions with action

$$S = 2\pi\theta \int dt \text{Tr}_{\mathcal{H}}[\tfrac{1}{2}(\partial_t\phi\partial_t\phi - \partial_i\phi\partial_i\phi) - V(\phi)]. \tag{2.10}$$

Note that the global $U(\mathcal{H})$ symmetry taking $\phi \to U\phi\overline{U}$ is broken by the presence of derivative terms.

We are aiming towards a description of noncommutative theories arising from D-branes and these carry a gauge field. Thus from now on we will focus exclusively on noncommutative theories with gauge fields and we will see that one can naturally gauge the $U(\mathcal{H})$ symmetry which was present in the absence of derivative terms.

To introduce gauge fields we look for a covariant derivative that transforms covariantly under $U(\mathcal{H})$ transformations. Thus if

$$\phi \to U\phi\overline{U}, \tag{2.11}$$

we want

$$D_i\phi = \partial_i\phi - i[A_i,\phi] \to UD_i\phi\overline{U}. \tag{2.12}$$

In complex coordinates we have

$$D_\alpha\phi \equiv D_{z_\alpha}\phi = -\theta_\alpha^{-1/2}[\overline{a}_\alpha + i\theta_\alpha^{1/2}A_\alpha,\phi], \tag{2.13}$$

or defining $C_\alpha = \overline{a}_\alpha + i\theta_\alpha^{1/2}A_\alpha$ we can write

$$D_\alpha\phi = -\theta_\alpha^{-1/2}[C_\alpha,\phi], \qquad \overline{D}_\alpha\phi = \theta_\alpha[\overline{C}_\alpha,\phi]. \tag{2.14}$$

Equation (2.12) then requires that C_α transform as $C_\alpha \to UC_\alpha\overline{U}$, or that

$$A_\alpha \to UA_\alpha\overline{U} - i\theta_\alpha^{1/2}U[\overline{a}_\alpha,\overline{U}]. \tag{2.15}$$

We can introduce a covariant field strength in the usual way as

$$F_\alpha \equiv iF_{z_\alpha,\overline{z}_\alpha} = i(\partial_\alpha\overline{A}_\alpha - \overline{\partial}_\alpha A_\alpha - i[A_\alpha,\overline{A}_\alpha]) = \theta_\alpha^{-1}\left([C_\alpha,\overline{C}_\alpha]+1\right). \tag{2.16}$$

With these ingredients we can construct an action for the noncommutative fields (ϕ, A_μ) with $U(\mathcal{H})$ gauge symmetry:

$$S = 2\pi \mathrm{Pf}(\theta^{ij})\mathrm{Tr}_{\mathcal{H}}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}D^\mu\phi D_\mu\phi - V(\phi)\right). \tag{2.17}$$

Let me make a few comments about this action and its generalizations:

1. We have implicitly chosen ϕ to transform in the adjoint representation of $U(\mathcal{H})$. There are other possibilities, for example we could consider the fundamental representation as well with $\phi \to U\phi$. I have focused on the adjoint representation with D-brane applications in mind.
2. We have constructed what might be called noncommutative $U(1)$ gauge theory. This can be easily generalized to noncommutative $U(N)$ gauge theory by tensoring with $N \times N$ matrices and generalizing operator products (or star products) to include matrix multiplication. This generalization will be used below in sec 2.4.
3. In the limit $\theta_\alpha \to \infty$ it is easy to see by rescaling coordinates that the first two terms in (2.17) involving the gauge field strength and covariant derivative terms vanish relative to the potential term [36]. Thus in this limit the previous noncommutative soliton becomes a solution to the theory which includes derivatives and gauge fields.

2.2. Noncommutative Solitons at Finite θ

At first glance it would seem substantially more complicated to find solutions at finite values of θ when gauge fields and derivatives are included, and this is true if one tries to analyze the equations of motion directly. Results along these lines can be found in [43,44,45,46,47,48,49,50]. The solution generating technique provides a unified way of understanding the structure of many of these solutions as well as of generating new kinds of solutions [51,52,53,54,55] since it depends only on the symmetry structure of the Lagrangian and not on its detailed form. I will illustrate this for a choice of potential which will arise later in string theory applications of this formalism.

Consider a noncommutative theory in $2+1$ dimensions with action given by (2.17). Furthermore take the potential $V(\phi-\phi_*)$ to have a local minimum at $\phi=\phi_*$ and a local maximum at $\phi=0$. Finally, adjust the constant term in V so that at the local minimum we have $V(0)=0$.

As before, we start with the "vacuum" with $\phi=\phi_* I$ and $C=\overline{a}$ (so that the gauge fields A_μ vanish). This is clearly a solution to the equations of motion, and as before, the equations of motion transform covariantly under transformations U which are "almost" gauge in that $\overline{U}U=1$ but $U\overline{U}\neq 1$. Under this transformation we have

$$\begin{aligned} &\phi \to U\phi_* I\overline{U} = \phi_* U\overline{U} \\ &C \to U\overline{a}\overline{U}. \end{aligned} \tag{2.18}$$

Choosing U to be the n^{th} power of the shift operator, $U=S^n$ then gives the solution

$$\begin{aligned} \phi =&\phi_*(I-P_n) \\ C =&S^n\overline{a}\overline{S}^n \\ \overline{C} =&S^n a\overline{S}^n \end{aligned} \tag{2.19}$$

Computing the field strength we find

$$F=\frac{1}{\theta}\left([C,\overline{C}]+1\right)=\frac{1}{\theta}\left(S^n[\overline{a},a]\overline{S}^n+1\right)=\frac{P_n}{\theta} \tag{2.20}$$

and the energy of the solution is

$$E=2\pi\theta\mathrm{Tr}\left(\frac{1}{2}F^2+\frac{1}{\theta}[C,\phi][\overline{C},\phi]+V(\phi-\phi_*)\right)=2\pi\theta n\left(\frac{1}{2\theta^2}+V(-\phi_*)\right). \tag{2.21}$$

Three comments concerning this solution:

1. The energy depends on the "flux" n and the value of V at the local maximum $\phi = 0$ but is independent of the detailed form of V. This would not be the case for a soliton in a commutative theory.
2. If we take $\phi_* = 0$, or equivalently just drop the tachyon field, then this solution reduces to the "fluxon" solutions discussed in [45,49,50]
3. This construction can be trivially extended to noncommutative $\mathbf{R}^{2n}$ and again the energy diverges as $\theta_\alpha \to 0$. This in accord with the lack of finite energy solutions in commutative gauge theory with the action (2.17) for $n \geq 2$. In string theory we will find noncommutative solitons with energy which remains finite as $\theta_\alpha \to 0$ by modifying the action (2.17).

2.3. Vortices and Their Analogs in Commutative Field Theory

Before moving onto string theory I would like to discuss one generalization of noncommutative solitons involving noncommutative versions of vortices and their generalizations. Let me first recall some aspects of these solitons in commutative field theories.

Consider a commutative field theory with scalar (Higgs) fields and gauge fields in $2n+1$ dimensions. We will look for soliton solutions which are localized in $\mathbf{R}^{2n}$ (equivalently we could look for p-brane solutions in higher dimensions by taking $\mathbf{R}^{2n}$ to be the space transverse to the p-brane). As the radial coordinate $r \to \infty$ in $\mathbf{R}^{2n}$ the scalar fields must approach their values in vacuum in order to have finite energy. Thus the scalar fields define a map from the sphere at infinity, S^{2n-1}, into $\mathcal{M}$, the vacuum manifold of the theory. If

$$\pi_{2n-1}(\mathcal{M}) = \mathrm{Z} \tag{2.22}$$

then we can construct topologically non-trivial field configurations.

To be concrete take the gauge group to be $U(N)_1 \times U(N)_2$ with a Higgs field ϕ transforming in the bifundamental representation, $\phi \sim (N, \overline{N})$ so that $\phi \to V\phi\overline{U}$ with $U \in U(N)_1$, $V \in U(N)_2$ under gauge transformations. This theory arises in the low-energy description of N $Dp - \overline{Dp}$-branes. For a suitable choice of potential the Higgs vacuum expectation value will break $U(N)_1 \times U(N)_2$ down to the diagonal subgroup and the vacuum manifold will be

$$\mathcal{M} = \frac{U(N)_1 \times U(N)_2}{U(N)} \sim U(N). \tag{2.23}$$

This theory supports topologically non-trivial field configurations in $\mathbf{R}^{2n}$ since $\pi_{2n-1}(U(N)) = Z$ for sufficiently large N. For $N = n = 1$ this solution is the well known vortex solution of the Abelian Higgs model. For the vortex solution the winding of ϕ around the S^1 at infinity requires that derivatives of ϕ scale at large r like $\partial\phi \sim 1/r$. Finite energy then requires that $D\phi$ vanish at large r faster than $1/r$, which in turn requires that the gauge field have an angular component scaling like $1/r$. Hence the gauge field strength scales like $1/r^2$. In $\mathbf{R}^2$ this is consistent with finite energy, but for $n > 1$ the gauge field contribution to the energy scales like

$$\int d^{2n}xF^2 \sim \int r^{2n-1}F^2 dr \sim \int r^{2n-3}dr, \tag{2.24}$$

and diverges at large r unless $n < 2$. Thus in higher dimensions the field configurations with non-trivial topology will have divergent energy with a conventional action for the gauge field.

As we will see, this problem with divergent energy is solved in noncommutative gauge theory and also apparently in string theory. With this application in mind it is useful to find an explicit construction of a topologically non-trivial gauge field. This can be done using an elegant construction due to Atiyah, Bott and Shapiro [56]. The construction uses the gamma matrices of the transverse rotation group to construct explicit generators of $\pi_{2n-1}(U(N))$.

The $Spin(2n)$ rotation group of $\mathbf{R}^{2n}$ has two irreducible spinor representations, $S_\pm$, of dimension 2^{n-1}. Let Γ_i be the gamma matrices mapping S_+ to S_-,

$$\Gamma_i : \; S_+ \to S_-. \tag{2.25}$$

The usual Dirac gamma matrices γ_i can be constructed in terms of the Γ_i and their Hermitian conjugates $\overline{\Gamma}_i$ as

$$\gamma_i = \begin{pmatrix} 0 & \Gamma_i \\ \overline{\Gamma}_i & 0 \end{pmatrix}. \tag{2.26}$$

The result of ABS is that $\Gamma_i x^i/|x|$ is a generator of $\pi_{2n-1}(U(2^{n-1}))$ where x^i are Cartesian coordinates on $\mathbf{R}^{2n}$.

To take advantage of this construction we can consider $U(N) \times U(N)$ gauge theory in $\mathbf{R}^{2n}$ as before but with the special choice $N = 2^{n-1}$. A Higgs field in the bifundamental representation with non-trivial topology at infinity is then constructed as

$$\phi = f(|x|)\frac{\Gamma_i x^i}{|x|} \tag{2.27}$$

with $f(|x|)$ approaching a constant as $|x| \to \infty$.

2.4. The Noncommutative ABS Construction

We can now generalize this construction to the noncommutative situation following [57]. We first generalize the vortex construction above with $N = n = 1$. To do this we start with the action for a complex scalar ϕ given in (1.51) and generalize it to include derivatives and gauge fields. We can gauge the $U(\mathcal{H}) \times U(\mathcal{H})$ symmetry under which ϕ transforms as $\phi \to V\phi\overline{U}$ by introducing gauge fields A^+_μ, A^-_μ and covariant derivatives

$$\begin{aligned} D_\mu\phi =&\partial_\mu\phi - i(A^+_\mu\phi - \phi A^-_\mu) \\ D_\mu\overline{\phi} =&\partial_\mu\overline{\phi} + i(\overline{\phi}A^+_\mu - A^-_\mu\overline{\phi}). \end{aligned} \tag{2.28}$$

We then consider the action for $A^\pm_\mu, \phi$ given by

$$\begin{aligned} S = 2\pi\theta \int dt \mathrm{Tr}_{\mathcal{H}} \Big[-\frac{1}{4}(F^+_{\mu\nu})^2 - \frac{1}{4}(F^-_{\mu\nu})^2 + \frac{1}{4}(D^\mu\phi D_\mu\overline{\phi} + D^\mu\overline{\phi}D_\mu\phi) \\ - W(\phi\overline{\phi} - 1) - W(\phi\overline{\phi} - 1)\Big]. \end{aligned} \tag{2.29}$$

Working on noncommutative $\mathbf{R}^2$ we can generate a solution starting with the vacuum configuration $\phi = 1, C^- = \overline{a}, C^+ = \overline{a}$. Acting with $U = S^n, V = S^m$ generates the solution

$$\begin{aligned} \phi &= S^m\overline{S}^n \\ C^- &= S^n\overline{a}\overline{S}^n \\ C^+ &= S^m\overline{a}\overline{S}^m. \end{aligned} \tag{2.30}$$

Computing the field strengths we find $F^- = P_n/\theta$, $F^+ = P_m/\theta$, which implies that the fluxes in the two gauge fields are given by

$$\begin{aligned} \int d^2xF^+ =&2\pi\theta\mathrm{Tr}_{\mathcal{H}}F^+ = 2\pi m \\ \int d^2xF^- =&2\pi\theta\mathrm{Tr}_{\mathcal{H}}F^- = 2\pi n. \end{aligned} \tag{2.31}$$

In the commutative limit we have the symmetry breaking pattern $U(1)_+ \times U(1)_- \to U(1)_{\mathrm{diag}}$. The broken $U(1)_{\mathrm{rel}}$ has generator $Q_{\mathrm{rel}} = Q_+ - Q_-$ with $Q_\pm$ the generators of $U(1)_\pm$. The $U(1)_{\mathrm{rel}}$ flux carried by the vortex is thus $2\pi(m-n)$ showing that we should interpret the solution as m vortices and n anti-vortices.

To generalize this construction to $\mathbf{R}^{2n}$ we follow the ABS construction and consider $U(N)$ noncommutative gauge theory with $N = 2^{n-1}$ in R^{2n}. In terms of operators this means that ϕ is a map from $\mathcal{H} \otimes S^+$ into $\mathcal{H} \otimes S^-$:

$$\phi : \ \mathcal{H} \otimes S^+ \to \mathcal{H} \otimes S^- \ . \tag{2.32}$$

We would like to generalize the ABS construction by finding a solution of noncommutative gauge theory with the above field content which in the commutative limit reduces to $\phi = f(|x|)\Gamma^i x_i$.

To implement the solution generating technique we need an analog of the shift operator, that is an operator from $\mathcal{H} \otimes S^+$ to $\mathcal{H} \otimes S^-$ which is a non-unitary isometry with index -1. It is not hard to see that

$$\overline{\mathcal{S}} = \frac{1}{\sqrt{\Gamma_j x^j \overline{\Gamma}_k x^k}} \Gamma_i x^i \tag{2.33}$$

does the trick. Note that $\overline{\mathcal{S}}\mathcal{S} = 1$. It is also fairly straightforward to see that dim ker $\overline{\mathcal{S}} = 1$ and that

$$\mathcal{S}\overline{\mathcal{S}} = 1 - P_{\ker\,\overline{\mathcal{S}}} \tag{2.34}$$

where $P_{\ker\,\overline{\mathcal{S}}}$ is the projection operator onto the kernel of $\overline{\mathcal{S}}$. The relevant calculation is

$$\Gamma_i x^i \overline{\Gamma}_j x^j = \sum_{\alpha=1}^{n} 2\theta_\alpha (N_\alpha + 1/2) - i\Sigma_{ij}\theta^{ij} \tag{2.35}$$

where $\alpha = 1, \cdots n$ labels the n complex coordinates z_α, θ^{ij} is assumed to be block diagonalized as in (1.38),

$$\Sigma_{ij} = \frac{1}{4}\left(\Gamma_i \overline{\Gamma}_j - \Gamma_j \overline{\Gamma}_i\right), \tag{2.36}$$

are generators in the S_- spinor representation of $SO(2n)$ and $N_\alpha = \overline{a}_\alpha a_\alpha$ are number operators. Equation (2.35) implies that the kernel of $\overline{\mathcal{S}}$ consists of the oscillator ground state times the lowest weight spinor of $SO(2n)$.

Since $\mathcal{S}$ is a non-unitary isometry, the previous argument goes through and allows us to construct a solution in the noncommutative theory using the solution generating technique with $U = \overline{\mathcal{S}}$ and $V = 1$:

$$\begin{aligned} \phi &= \mathcal{S} \\ C_\alpha^+ &= \overline{a}_\alpha \\ C_\alpha^- &= \overline{\mathcal{S}}\overline{a}_\alpha \mathcal{S}. \end{aligned} \tag{2.37}$$

Computing the field strength as before one finds $F^+ = 0$ and $F^- = P_{\ker\,\overline{\phi}}/\mathrm{Pf}(\theta^{ij})$.

As mentioned earlier, the solution has finite energy at finite θ^{ij}. Also as expected, the energy diverges as $\theta^{ij} \to 0$. Note that this solution is trivially generalized to a multi ABS/anti-ABS configuration by taking $U = \overline{\mathcal{S}}^n$ and $V = \overline{\mathcal{S}}^m$.

To summarize what we have done so far, we have found that one can construct exact soliton solutions in a variety of noncommutative Yang-Mills Higgs theories using the solution generating technique. The scalar (Higgs) field are constructed in terms of projection operators and partial isometries.

3. Lecture 3

3.1. String Theory and Noncommutative Geometry

The main point of these lectures is the connection between noncommutative solitons and D-branes in string theory, so it is time to explain what all of this has to do with string theory and tachyon condensation. Over the last few years it has been realized that noncommutative geometry arises as a limit of string theory when one considers D-branes in a background B field [58]. It would take too long to go through the full details here, so I will just make a few comments and refer to the literature for details.

A standard example of noncommutative geometry arises when we consider a particle of charge q moving in a plane (with coordinates x, y) in the presence of a magnetic field transverse to the plane ($\vec{B} = B\hat{z}$). If we parallel transport the particle around a closed loop in the plane then we know that the wave function of the particle acquires a phase proportional to the magnetic flux through the loop. As an example of such a loop consider a square with sides of length a. If $T_x(a)$, $T_y(a)$ are the translation operators by a in the x, y directions then we have

$$T_x(a)T_y(a)T_x^{-1}(a)T_y^{-1}(a) = e^{2\pi i qBa^2}. \tag{3.1}$$

Thus we have a projective representation of the translation group on the plane. As in quantum mechanics where the plane was two-dimensional phase space, this can be used to define a noncommutative product of functions on the plane.

Indeed, the Lagrangian for a charged particle of mass m in a constant magnetic field is

$$\mathcal{L} = \frac{m}{2}\left(\frac{dx}{dt}^2 + \frac{dy}{dt}^2\right) + qBx\frac{dy}{dt}. \tag{3.2}$$

As is well known, the spectrum consists of infinitely degenerate Landau levels with the energy spacing between levels given by qB/m. As $m \to 0$ this gap becomes infinite and physics is restricted to the lowest Landau level. In this limit the Lagrangian reduces to the second term in (3.2) and the momentum canonically conjugate to y becomes

$$p_y = \frac{\partial \mathcal{L}}{\partial(dy/dt)} = qBx. \tag{3.3}$$

Thus the canonical commutation relations imply

$$[x, y] = iq/B, \tag{3.4}$$

showing that the coordinates on the plane do not commute. This treatment was rather naive, but a more complete treatment leads to a similar conclusion [59].

Similarly, in string theory a constant antisymmetric two-form potential $B_{\mu\nu}$ couples to the string via

$$\int_\Sigma B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu = \int_{\partial\Sigma} B_{\mu\nu} X^\mu \partial_t X^\nu \tag{3.5}$$

where Σ is the string world-sheet with boundary $\partial\Sigma$ and ∂_t is the derivative tangential to the boundary. Since with constant B the interaction is a surface term, it does not change the dynamics unless the string-world sheet has a boundary, that is unless there are D-branes present. In the presence of D-branes, the interaction (3.5) modifies the boundary conditions obeyed by the string coordinates X and hence changes the propagator. Seiberg and Witten [60] showed, using previous results on the quantization of open strings [61,62,63] and building on previous work [64,65,66,67] that the effect of B could be summarized as follows.

There is a two-form field Φ whose value depends on how the theory is defined (that is, how it is regularized). The action for the spacetime fields associated to the different modes of the string depends on the combination $\hat{F}_{\mu\nu} + \Phi_{\mu\nu}$ where $\hat{F}$ is the gauge field. For further details of the role played by $\Phi_{\mu\nu}$ see [68]. Assuming that B has only spatial components and making the special choice $\Phi_{ij} = -B_{ij}$, the changes to the effective action for the open string modes due to the presence of B are:

1. In the action use the open string metric G_{ij} defined in terms of the background closed string metric g_{ij} by

$$G_{ij} = -(2\pi\alpha')^2 \left(B \frac{1}{g} B \right)_{ij} \tag{3.6}$$

2. Change the string coupling constant from its value g_s when $B = 0$ to

$$G_s = g_s \det(2\pi\alpha' B g^{-1})^{1/2} \tag{3.7}$$

3. Use star products to multiply fields with

$$\theta^{ij} = \left(\frac{1}{B} \right)^{ij}. \tag{3.8}$$

In the rest of this lecture we will consider various unstable or non-BPS D-brane configurations in the bosonic string and type II superstring in the presence of a background B field. The effective field theory of the tachyon and gauge field degrees of freedom of the open strings on the D-brane will be a noncommutative field theory determined by the above procedure in terms of the effective field theory at $B = 0$. We will see that the noncommutative solitons studied in the previous lectures can be identified with lower-dimensional D-branes which arise from tachyon condensation [69,38]. We will carry out the analysis using an effective field theory approach, although for large B the analysis can also be done directly in string field theory [41].

3.2. Bosonic D-branes as Noncommutative Solitons

The first example we will consider is the bosonic string. The bosonic string contains unstable Dp-branes for all p, for simplicity we start by considering a space-filling $D25$-brane. The open string field theory describing the excitations of the $D25$-brane involves an infinite number of ordinary component fields, the tachyon ϕ, a massless gauge field A_μ, and an infinite tower of massive states. In string field theory these fields are organized into a string field Φ with a gauge transformation law which when expressed in terms of the component fields is complicated and differs from the usual gauge transformation laws when interactions are included. Nonetheless, it should be the case that if we integrate out all fields other than ϕ, A_μ to obtain an effective action, $S_{\rm eff}(\phi, A_\mu)$, then this effective action is invariant under standard gauge transformations since this is the only consistent way to decouple the negative norm states of A_μ.

In the commutative theory the tachyon field is a singlet under $U(1)$ gauge transformations, but an analysis of disk diagrams shows that in the noncommutative theory the tachyon is in the adjoint representation of the noncommutative $U(1)$ gauge theory [70]. Also, while higher derivative terms in the effective action are not known precisely, it is known that the leading terms for constant tachyon and gauge field strength have the Born-Infeld form [71,72,73].

Thus integrating out the massive string degrees of freedom leads to an effective action of the form (assuming for now that $B_{ij} = 0$)

$$S_{\rm eff} = \frac{c}{g_s} \int d^{26}x \left[-V(\phi - 1)\sqrt{-\det(g + 2\pi\alpha' F)} + \cdots + \frac{1}{2}\sqrt{g} f(\phi - 1)\partial^\mu \phi \partial_\mu \phi + \cdots \right] \tag{3.9}$$

Here $c = T_{25} g_s$ is independent of g_s with T_{25} the $D25$-brane tension. The zero point and scale of the potential $V(\phi - 1)$ have been chosen so that there is a local maximum at $\phi = 0$ with $V(-1) = 1$ which represents the unstable $D25$-brane configuration, and a local minimum at $\phi = 1$ with $V(0) = 0$ which is supposed to represent the closed string vacuum according to the conjecture of Sen [74,75]. Finally, the ellipsis in (3.9) indicate higher derivative terms.

We now turn on a background B field. The simplest choice (which is easily generalized) is to turn on a B field along a $\mathbf{R}^2$ in $\mathbf{R}^{25,1}$, say $B_{24,25} = b$. I will choose $b < 0$ for convenience in later expressions. Following the prescription described above (3.6), (3.7) we need the open string metric, coupling constant and noncommutativity parameter which are

$$\begin{aligned} G_{\mu\nu} &= \mathrm{diag}(1, -1, -1, \cdot - 1, -(2\pi\alpha' b)^2, -(2\pi\alpha' b)^2) \\ G_s &= g_s(2\pi\alpha'|b|) \\ \theta^{24,25} &\equiv \theta = 1/|b|. \end{aligned} \tag{3.10}$$

Using the operator formalism in the noncommuting directions the action then becomes

$$S = \frac{2\pi\theta c}{G_s}\int d^{24}x \mathcal{L}_{\mathrm{nc}} \tag{3.11}$$

with

$$\mathcal{L}_{\mathrm{nc}} = \mathrm{Tr}_{\mathcal{H}}\left[-V(\phi-1)\sqrt{det(G_{\mu\nu}2\pi\alpha'(F+\Phi)_{\mu\nu})} + \frac{1}{2}\sqrt{G}f(\phi-1)D^\mu\phi D_\mu\phi + \cdots\right] \tag{3.12}$$

where $F_{24,25} + \Phi_{24,25} = -iF_{z,\overline{z}} + 1/\theta = -[C, \overline{C}]/\theta$.

We can now use the solution generating technique to construct a solution starting from the vacuum configuration[4]

$$\phi = 1, \qquad C = \overline{a}, \qquad A_\mu = 0, \qquad \mu = 0, 1, \cdots 23. \tag{3.13}$$

Although we do not know the form of the infinite number of higher derivative terms in the effective action, we do know that they transform covariantly under gauge transformations, and this is sufficient to construct a solution using the solution generating technique. Thus the transformation

$$\begin{aligned} \phi &\to U\phi\overline{U} \\ C &\to UC\overline{U} \\ A_\mu &\to UA_\mu\overline{U} \end{aligned} \tag{3.14}$$

[4] Other choices of vacuum have been discussed in [76,68] and it was argued in [77] that this redundancy in the choice of vacuum reflects a bad choice of coordinates in field space. In any event, the solution generating technique works best if we start from something rather than nothing.

generates a new solution from the vacuum as long as $\overline{U}U = 1$ and $U\overline{U} = 1 - P_n$. Choosing $U = S^n$ then gives the solution

$$\begin{aligned} \phi &= S^n \overline{S}^n = (1 - P_n) \\ C &= S^n \overline{a} \overline{S}^n \\ A_\mu &= 0. \end{aligned} \tag{3.15}$$

Although we know this is a solution which is localized in $\mathbf{R}^2$ and hence represents a 23-brane, it might seem difficult to calculate the tension and spectrum of fluctuations about the solution without detailed knowledge of the higher derivative terms in (3.9). Luckily, this is not the case and we can identify this solution with the $D23$-brane of the bosonic string by computing the tension and spectrum of low-lying fluctutations and showing that they agree with those of a $D23$-brane.

To compute the tension note that $D\phi = \overline{D}\phi = 0$ since $D\phi$ vanishes in vacuum and transforms covariantly under the solution generating transformation. Similarly, $DF = \overline{D}F = 0$. Since derivatives of ϕ and F must appear as covariant derivatives in the action, this shows that any term containing derivatives of ϕ or F does not contribute to the tension. This leaves the potential term and terms involving F^2 and higher powers of F (i.e. those appearing in the expansion of the DBI action).

Terms involving the gauge field strength also do not contribute to the tension as a result of the vanishing of $V(\phi - 1)$ in the closed string vacuum. Specifically, using $V(0) = 0$ we find

$$V(\phi - 1)[C, \overline{C}]^2 = V(-P_n)(1 - P_n) = V(-1)P_n(1 - P_n) = 0. \tag{3.16}$$

Thus the $V(\phi - 1)(F + \Phi)^2$ term does not contribute to the tension. Similarly, the higher powers of $(F + \Phi)$ in an expansion of (3.12) are also multiplied by $V(\phi - 1)$ and so do not contribute.

We are then left with the potential contribution to the tension. Using

$$V(\phi - 1) = V(-P_n) = V(-1)P_n = P_n \tag{3.17}$$

and

$$\frac{\sqrt{G}\theta}{G_s} = \frac{2\pi\alpha'}{g_s} \tag{3.18}$$

we find that the action evaluated for the solution (3.15) is

$$S = (2\pi)^2 \alpha' n \frac{c}{g_s} \int d^{24}x. \tag{3.19}$$

Thus the tension of the solution is

$$T_{23} = \frac{(2\pi)^2 \alpha' n c}{g_s} = (2\pi)^2 \alpha' n T_{25}^D = n T_{23}^D, \tag{3.20}$$

with T_p^D the tension of a Dp-brane of the bosonic string as give in [78].

Let me make a few comments concerning this solution and the identification with a $D23$-brane:

1. In addition to the tension matching that of a $D23$-brane, it is not hard to show that the tachyon and gauge fields on the $D23$-brane arise from a collective coordinate analysis of fluctuations about the solution. For example, the solution preserves a $U(n)$ subgroup of $U(\mathcal{H})$ acting in the image of P_n and as in the usual Higgs mechanism this leads to massless $U(n)$ gauge fields. This non-Abelian gauge symmetry would be very difficult to see in a description of D-branes as solitons in a commutative theory. An analysis of the tachyon field fluctuations leads to tachyons in the adjoint of $U(n)$ on the $D23$-brane [38]. One also finds massive fluctuations, but their mass cannot be computed reliably without knowledge of the higher derivative terms in (3.9).
2. The discussion presented here (following [79]) improves on the discussion in [38] by not requiring that one take the limit $\alpha' B \to \infty$. In particular, the construction of D-branes as noncommutative solitons holds for finite noncommutativity and finite coupling constant.
3. The solution we have found has a non-zero gauge field. This is to be contrasted with the constructions of D-branes as lumps in truncated string field theory [80] or in background independent string field theory where the gauge field vanishes. There is no obvious contradiction since the transformation between the gauge field variables here and those used in these other two treatments is undoubtedly subtle and complicated. Still, it would be nice to understand this better.
4. The construction can easily be extended to all Dp-branes with p odd by turning on a B field in the even-dimensional transverse space to the Dp-brane and repeating the above analysis with the obvious modifications.

3.3. Type II D-branes as Noncommutative Solitons

These ideas can also be applied to construct D-branes in type II string theory starting from unstable brane configurations. There are two general classes of unstable brane configurations of interest. The first consists of non-BPS Dp-branes for p odd in IIA theory or p even in IIB theory. The second is the $Dp-\overline{Dp}$ system of BPS branes and anti-branes for p even in IIA and p odd in IIB. The first case is very similar to the analysis presented above for the bosonic string [5]. Because of this I will focus on the second case.

Consider a space-filling $D9-\overline{D9}$ system in IIB string theory. The low-lying excitations consist of two gauge fields $A_\mu^\pm$ coming from open strings which begin and end on the same D-brane and a complex tachyon ϕ from open strings which begin on one D-brane and end on the other (the tachyon is complex because of the two orientations of these open strings). This system is closely related to the field theory studied at the beginning of sec 3.4. The action of $A_\mu^\pm, \phi$ in type II string theory has not been as well studied as in the bosonic string, so we will have to proceed with a bit more guesswork that in the previous analysis.

The action is constrained by the presence of a Z_2 symmetry, denoted by $(-1)^{F_L}$, which exchanges the $D9$ and $\overline{D9}$-branes and takes

$$\begin{aligned} \phi &\leftrightarrow \overline{\phi}, \\ A^+ &\leftrightarrow A^- \ . \end{aligned} \tag{3.21}$$

Sen's conjectures regarding tachyon condensation and explicit calculations referred to earlier in truncated open string field theory and BSFT suggest a potential of the form $V(\phi\overline{\phi}-1)+V(\overline{\phi}\phi-1)$ with $V(0)=1$ at the local maximum and a ring of minima at $|\phi|=1$ with $V(0)=0$.

In the noncommutative theory F^- and F^+ transform differently under the two $U(1)$ factors, but one can form linear combinations by noting that F^- and $\overline{\phi}F^+\phi$ transform in the same way. So an acceptable gauge kinetic term is

$$\begin{aligned} \mathcal{L}_{gauge} = {} & h_+(\overline{\phi}\phi-1)\left\{F^-_{\mu\nu}+\Phi_{\mu\nu}+\overline{\phi}(F^+_{\mu\nu}+\Phi_{\mu\nu})\phi\right\}^2 \\ & + h_-(\overline{\phi}\phi-1)\left\{F^-_{\mu\nu}+\Phi_{\mu\nu}-\overline{\phi}(F^+_{\mu\nu}+\Phi_{\mu\nu})\phi\right\}^2 \\ & + \ \left\{\phi\leftrightarrow\overline{\phi}, \quad A^+\leftrightarrow A^-\right\}. \end{aligned} \tag{3.22}$$

The last line is included to enforce symmetry under $(-)^{F_L}$. A similar expression holds for terms including higher powers of the gauge fields. We expect that the kinetic term for the "center of mass" $U(1)$ should vanish in the closed string vacuum which implies $h_+(0)=0$.

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[5] Except for the subtle issue of how to interpret the new tensionless brane solutions found in [69,38]. In [81] it is proposed that these are gauge equivalent to the vacuum solution.

It is less clear whether $h_-(0) = 0$ or not. We will see that the solution has the correct tension whether or not $h_-(0) = 0$.

Tachyon kinetic terms appear as in (2.29), but now multiplied by functions $f(\overline{\phi}\phi - 1)$ and symmetrized. Symmetry under $(-)^{F_L}$ implies that the potential is of the form

$$V(\phi\overline{\phi} - 1) + V(\overline{\phi}\phi - 1) \ . \tag{3.23}$$

In the notation of (2.29) we take $V = W$.

Now we use our solution generating transformation to construct exact solitons representing BPS D-branes. As before, we will explicitly consider the codimension two case; starting with a spacefilling $D9 - \overline{D9}$ system of IIB this will produce BPS $D7$-branes. As in (2.30), the solution we generate starting from the closed string vacuum is

$$\begin{aligned} \phi &= S^m \overline{S}^n \ , \\ C^- &= S^n \overline{a} \overline{S}^n \ , \\ C^+ &= S^m \overline{a} \overline{S}^m \ , \\ A_\mu^+ &= A_\mu^- = 0 \ , \quad \mu = 0 \ldots 7 \ . \end{aligned} \tag{3.24}$$

We will see that this solution represents m $D7$-branes coincident with n $\overline{D7}$-branes.

We now work out the energy of this solution. As in the bosonic case, covariant derivatives of the tachyon and gauge field strengths vanish before, and thus after the solution generating transformation, and so do not contribute to the energy. It is less trivial to verify that the gauge field terms (3.22) do not contribute. We need to compute

$$h_+(\overline{\phi}\phi - 1)\left\{[C^-, \overline{C^-}] + \overline{\phi}[C^+, \overline{C^+}]\phi\right\}^2 + h_-(\overline{\phi}\phi - 1)\left\{[C^-, \overline{C^-}] - \overline{\phi}[C^+, \overline{C^+}]\phi\right\}^2 . \tag{3.25}$$

For the solution,

$$\begin{aligned} \phi\overline{\phi} &= I - P_m \ , \\ \overline{\phi}\phi &= I - P_n \ , \\ [C^-, \overline{C^-}] &= -(I - P_n) \ , \\ \overline{\phi}[C^+, \overline{C^+}]\phi &= -(I - P_n) \ , \end{aligned} \tag{3.26}$$

the first term in (3.25) vanishes since $h_+(\overline{\phi}\phi - 1) = h_+(-P_n) = h_+(-1)P_n$, which is orthogonal to $I - P_n$; here we used that $h_+(0) = 0$. The second term in (3.25) vanishes without a similar assumption about h_-. So, as in the bosonic theory, the only contribution to the energy comes from the potential term, which we find to be

$$V(\phi\overline{\phi}-1)+V(\overline{\phi}\phi-1)=V(-P_m)+V(-P_n)=V(-1)(P_m+P_n). \tag{3.27}$$

Repeating the computation leading to (3.20) in the bosonic case now gives the tension

$$T_{nm}=\frac{(2\pi)^2\alpha'(n+m)c}{g_s}T_9^D=(n+m)T_7^D. \tag{3.28}$$

as expected for m $D7$-branes plus n $\overline{D7}$-branes.

Using the ABS construction of solitons in section 3.4, it is straightforward to generalize the above discussion to codimension $2p$ solitons representing coincident $D(9-2p)$ and $\overline{D(9-2p)}$ branes. It is also possible to extend the previous discussion of the fluctuation spectrum to show that the expected low-lying spectrum of excitations ($U(n)\times U(m)$ gauge fields and tachyons in the (n,m)) arise on the lower-dimensional D-branes.

4. Lecture 4

4.1. Noncommutative Solitons and K-theory

In the previous lectures we have developed an approach to D-branes that describes them as solitons in noncommutative field theory. We have seen that the tension and the low-lying spectrum of excitations are both correctly obtained in this picture. It is natural to ask whether other aspects of D-branes can be understood from this point of view or whether new points of view on D-brane physics are suggested by the noncommutative approach. Since $B=0$ is a special value while $B\neq 0$ is generic, one might argue that the noncommutative description is more general, and hence the proper formulation of D-branes should naturally involve concepts of noncommutative geometry.

Until recently field theorists have been able to rely on the basic tools of algebraic topology (that is homotopy, homology and cohomology) to compute the charges of solitons. Roughly speaking, these tools (or functors) provide a natural way to associate an Abelian group (representing the charges) to a topological space. When it comes to D-branes new tools are needed because D-branes naturally carry gauge fields and thus one needs a way to associate an Abelian group to topological spaces equipped with vector bundles. This leads to the identification of D-brane charge with K-theory [82,83,84]. The noncommutative description of D-branes as solitons makes this fact manifest, but also suggests some interesting connections to some more exotic aspects of K-theory and noncommutative geometry [85,86,57]. The correct tool for studying D-brane charge in noncommutative field theory might be called noncommutative algebraic topology and should associate Abelian groups to C^* operator algebras. This leads to the K-theory of C^* algebras. For reasons of both time and competence I will only discuss some elementary aspects of this theory. A brief summary of some of the relevant mathematics can be found in [87]. For more details see [88,89,90].

To start with let me a give a low-brow summary of a construction which is central to K theory. An Abelian semi-group is a set $S = a, b, c, \cdots$ with a binary Abelian composition law $\circ$

$$a \circ b = b \circ a = c \in S \ . \tag{4.1}$$

We do not assume the existence of an inverse or an identity, hence this is a semi-group rather than a group. Associated to S there is a Grothendieck-given Abelian group, $G(S)$, which consists of ordered pairs of elements of S, (a, b) with the identification

$$(a, b) \sim (a \circ c, b \circ c), \qquad c \in S \ . \tag{4.2}$$

A group multiplication law, $+$, can be defined for these ordered pairs by the rule

$$(a, b) + (c, d) \equiv (a \circ c, b \circ d) \ . \tag{4.3}$$

It is simple to check that this defines an Abelian group with identity $\mathrm{id} = (a, a)$ for any $a \in S$ and with inverse $(a, b)^{-1} = (b, a)$:

$$\begin{aligned} (a, a) + (c, d) &= (a \circ c, a \circ d) = (c \circ a, d \circ a) = (c, d) \\ (a, b) + (b, a) &= (a \circ b, b \circ a) = (a \circ b, a \circ b) = \mathrm{id} \ . \end{aligned} \tag{4.4}$$

We now consider some examples of this construction.

1. Take the set S to consist of the set of natural numbers, $S = N = 1, 2, 3, \cdots$ and the composition law $\circ$ to be addition. Then it is easy to check that $G(S)$ is isomorphic to the Abelian group formed by the set of all integers with group multiplication being addition of integers and $n^{-1} = -n$. The isomorphism identifies the ordered pair (n, m) with the integer $n - m$.
2. Take the set S to consist of vector bundles over a manifold X. The composition law $\circ$ is $\oplus$, the direct sum of vector bundles. Thus we identify the pair of vector bundles (E, F) with $(E \oplus G, F \oplus G)$ for an arbitrary vector bundle G. The K group of X (or

$K_0(X)$) is the Grothendieck group $G(S)$. In [91] the above equivalence relation was interpreted as the creation of brane-antibrane pairs following the work of Sen.

3. Let $S = [X, \mathcal{F}]$ be the set of homotopy classes of maps from a compact manifold X into the space of Fredholm operators on Hilbert space [6]. It is proved in [92] that one can define a composition law such that once again $G(S) = K(X)$. If X is a single point then the set of homotopy classes of maps from X into $\mathcal{F}$ has one element for each disconnected component of $\mathcal{F}$. Since the disconnected components are labelled by the index, this shows that $K(\text{point}) = Z$.
4. Let S be the set of Murray von-Neumann equivalence classes of projection operators in a C^* algebra $\mathcal{A}$. Recall that a C^* algebra can be thought of as a self-adjoint subalgebra of the algebra of bounded operators on Hilbert space, $B(\mathcal{H})$. Two projection operators p, q are Murray-von Neumann equivalent if $p = \overline{v}v$, $q = \overline{v}v$ for some partial isometry v. In general the sum of two projection operators is not a projection operator, so to define a composition law one must "stabilize" by considering infinite dimensional matrices with entries in $\mathcal{A}$. One can then move projection operators down the diagonal to make them orthogonal and define a binary composition law on S. The Abelian group $G(S)$ constructed this way is called the K group of $\mathcal{A}$, $K(\mathcal{A})$. As an example, the space of continuous functions on a manifold X is a commutative C^* algebra, $C(X)$ with a norm defined by $||f|| = \sup_{x \in X} f(x)$. It is a standard result [88] that $K(C(X)) = K(X)$.

The second definition of the K-theory group of a manifold X has been discussed extensively in connection with D-branes. The third and fourth definitions on the other hand involve concepts that we have already encountered in the description of D-branes as noncommutative solitons [86,57]. For example, consider tachyon condensation on a $D9 - \overline{D9}$ system in IIB string theory to make a $D7$-brane as described in the previous lecture. In order to have finite action the tachyon field ϕ must be Fredholm and we saw that the induced $D7$-brane charge was just the index of ϕ. Let X denote the $D7$-brane world volume and consider X-dependent tachyon configurations. Finite action configurations of this sort give us a map

$$T : X \to \mathcal{F} \tag{4.5}$$

of X into Fredholm operators. These configurations will be classified by the homotopy class of the map (4.5), which according to the third definition is just $K(X)$. We could also

[6] Recall that an operator O is Fredholm if its image is closed and its kernel and cokernel are both finite dimensional

consider such a construction in IIA theory starting with an unstable $D9$-brane. In this case the tachyon field is real, and so we should consider maps of X (the world volume of a lower-dimensional D-brane constructed as a noncommutative soliton) into the space of self-adjoint Fredholm operators, $\mathcal{F}^{sa}$. The homotopy classes of maps $[X, \mathcal{F}^{sa}]$ provides a model of the K-group $K^1(X)$ [93], an identification which was utilized in the proposal of [85].

4.2. D-brane Charge and Toeplitz Operators

There is a deeper aspect of the connection between noncommutative tachyon condensation and K-theory which I would like to outline briefly. To begin with, let me consider the related problem of understanding the connection between the topological charge and the index of Fredholm operators as it appears in the constructions described in the previous lecture. For example, in the noncommutative set-up the $D7$-brane charge is the index of the tachyon operator ϕ. On the other hand, in the commutative theory the $D7$-brane charge is given by the winding number of the classical tachyon field,

$$Q_{D7} \sim \frac{1}{2\pi|\phi(\infty)|^2} \int_{S^1_\infty} \phi d\phi \sim \frac{1}{2\pi} \int_{R^2} F \ . \tag{4.6}$$

What is the connection between these two facts? We need an operator analog of the winding number. To construct this we consider a model for the shift operator S which appears in the construction of the $D7$-brane solution where S acts on the Hilbert space of states on a circle (which we think of physically as an S^1 encircling the $D7$-brane).

So consider $\mathcal{H} = L^2(S^1)$ with orthonormal basis

$$\psi_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}}, \qquad n \in Z \ . \tag{4.7}$$

A function in the C^* algebra of functions on S^1, $f \in C(S^1)$ determines an operator M_f acting on $\mathcal{H}$ in the obvious way:

$$M_f \psi(\theta) = f(\theta)\psi(\theta) \ . \tag{4.8}$$

An operator which shifts $\psi_n \to \psi_{n+1}$ has index 0 since now n runs over all integers. To find an analog of the shift operator we had in the harmonic oscillator basis for $L^2(R)$ we define a subspace $\mathcal{H}_+$ of $\mathcal{H}$ by

$$\mathcal{H}_+ = \mathrm{span}\{\psi_n(\theta), \ n \geq 0\} \tag{4.9}$$

and let P be the projection operator onto $\mathcal{H}_+$. We can think of H_+ as the space of boundary values of holomorphic functions on $\mathbf{R}^2$.

We now define a Toeplitz operator on $\mathcal{H}_+$ which generalizes the shift operator by multiplying by a general $f(\theta)$ and then projecting back to $\mathcal{H}_+$:

$$T_f = PM_f : \ \mathcal{H}_+ \to \mathcal{H}_+ \ . \tag{4.10}$$

We can see that this generalizes the shift operator by noting that the Toeplitz operator constructed from $f_l(\theta) = e^{il\theta}$ for $l > 0$ acts as the l^{th} power of the shift operator:

$$T_{f_l} : \ \psi_n(\theta) \to \psi_{n+l}(\theta) \ . \tag{4.11}$$

An index theorem for Toeplitz operators [94] says that

$$\text{ind } T_f = \frac{1}{2\pi} \int \overline{f} df \tag{4.12}$$

thus providing the desired connection between the index and winding number. This argument and the index theorem can be extended to include the noncommutative ABS configurations defined in $\mathbf{R}^{2n}$ [57].

4.3. BDF and all that

The formalism described above provides one of the simplest examples of a general structure analyzed by Brown, Douglas and Fillmore (BDF) [95] which may ultimately be of some importance in understanding D-branes. To explain this, note that

$$T_f T_g - T_{fg} \in \mathcal{K}(\mathcal{H}) \tag{4.13}$$

where $\mathcal{K}(\mathcal{H})$ is the space of compact operators. For example,

$$T_{f_l} T_{f_l^*} - T_1 = P_l \tag{4.14}$$

where P_l is the projection operator onto the space spanned by $\psi_0, \psi_1, \cdots \psi_{l-1}$.

The Toeplitz operators form a C^* algebra, which we denote by $\mathcal{T}$, which maps to the C^* algebra $C(S^1)$ of continuous functions on S^1 by $T_f \to f$. Furthermore, since $T_f T_g - T_{fg}$ is compact and $fg - gf = 0$ in $C(S^1)$ (since $C(S^1)$ is commutative), the kernel of this map consists of the compact operators. We have therefore deduced the existence of a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T} \to C(S^1) \to 0. \tag{4.15}$$

Exact sequences of this form play an important role in the study of operator algebras and their invariants. In particular, BDF classified the "extensions of $C(X)$ by $\mathcal{K}$" meaning the possible C^* algebras $\mathcal{A}$ such that one has a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{A} \to C(X) \to 0 \ . \tag{4.16}$$

This classification is based on an invariant of such extensions called the Busby invariant and an associated Abelian group, $\mathbf{Ext}(C(X), \mathcal{K})$. Without going into detail, it is possible to motivate the definitions of these objects by using the Toeplitz algebra as an example.

For solitons in commutative field theory the topology is captured by the gauge invariant "winding" of the fields at infinity. We have seen that the Weyl transform maps bounded operators to functions which can be non-vanishing at infinity. Compact operators on the other hand map to functions vanishing at infinity. Thus we expect the "topology" of a noncommutative field theory to be independent of a change by compact operators, that is the topology should depend only on the quotient of bounded operators by compact operators, $Q(\mathcal{H}) = B(\mathcal{H})/\mathcal{K}(\mathcal{H})$, otherwise known as the Calkin algebra.

The Busby invariant is a map from $C(X)$ into $Q(\mathcal{H})$ defined for $f \in C(X)$ by choosing a $T_f \in \mathcal{A}$ which maps to f and defining $\tau(f) = \pi(T_f)$ where $\pi : B(\mathcal{H}) \to Q(\mathcal{H})$ is the projection. Two configurations should have the same topology if they are gauge equivalent, in operator language this is known as "strong equivalence". Two extensions are strongly equivalent if the Busby invariants are related by $\tau_2(f) = \pi(U)\tau_1(f)\pi(U)^*$. The set of strong equivalence classes of extensions of $C(X)$ by $\mathcal{K}$ is denoted $\mathbf{Ext}(C(X), \mathcal{K})$. It is possible to show that one can define a sum of extensions that turns $\mathbf{Ext}(C(X), \mathcal{K})$ into an Abelian group which we can identify with the D-brane charge. These strong equivalence classes of extensions can be used to define a variant of K-theory sometimes known as K-homology. This and other considerations discussed in [96,57] suggest that D-brane charge should really be associated to K-homology. Some other aspects of K-theory such as Bott periodicity also find a natural setting in the language of noncommutative solitons.

In these lectures I have only described D-branes as noncommutative solitons in $\mathbf{R}^{2n}$. It is obviously interesting to extend these constructions to more complicated spaces. See [97,98] and references therein for the extension to tori and orbifolds.

One motivation for the study of tachyon condensation in open string theory is the hope that open string theory might provide an alternate starting point for a fundamental formulation of string theory. If this is correct and amenable to practical analysis, then closed fundamental strings and NS-branes must also make an appearance. Closed strings of course appear in perturbation theory about the unstable vacuum, but finding them in the stable vacuum in terms of open string fields is problematic. Closed strings and NS-branes do not have the correct tension to be constructed as solitons of open string field theory, at least naively. There have been attempts to describe closed fundamental strings as electric flux tubes in the tachyon condensed vacuum [99,100,38,101,102,103,104] but we seem far from a definitive picture. The description of NS-branes seems even more difficult since they are naturally viewed as solitons of closed string field theory [105,106,107,108]. A preliminary attempt has been made to describe D-branes as noncommutative solitons in the presence of NS-branes by using twisted $U(\mathcal{H})/U(1)$ bundles [57]. It would be interesting to pursue these ideas further.

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