物理数学II　Bessel関数

ノートのタイトル　2009/07/01

§5 Bessel関数

§5.1 Bessel関数

母関数
$$\sum_{n=-\infty}^{\infty} t^n J_n(z) = e^{\frac{1}{2}z(t-\frac{1}{t})}$$

Laurent 展開

$$J_n = \frac{1}{2\pi i}\oint dt\, t^{-n-1} e^{\frac{1}{2}z(t-t^{-1})}$$

$t = e^{i\theta}$

$\frac{dt}{t} = i d\theta$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{-in\theta} e^{iz\sin\theta}$$

$$= \frac{1}{\pi}\int_0^{\pi} d\theta \cos(n\theta - z\sin\theta) \quad \cdots (**)$$

解釈　2次元平面波の r 依存性

$x = r\cos\theta$, $y = r\sin\theta$

$$e^{iy} = e^{ir\sin\theta} = \sum_{n\in\mathbb{Z}} e^{in\theta} J_n(r)$$

$$e^{i(k_1 x + k_2 y)} = e^{ikr\sin(\theta+\theta_0)} = \sum_{n\in\mathbb{Z}} e^{in(\theta+\theta_0)} J_n(kr)$$

$$k = \sqrt{k_1^2 + k_2^2}, \quad \theta_0 = \arctan\left(\frac{k_1}{k_2}\right)$$

展開

$$e^{\frac{1}{2}zt} e^{-\frac{z}{2t}} = \sum_{n=0}^{\infty} \frac{z^n t^n}{n!\,2^n} \sum_{m=0}^{\infty} \left(-\frac{1}{2}\right)^m \frac{z^m}{t^m\, m!}$$

t^r の係数

$r>0$, $n-m=r$

$$J_r(z) = \frac{z^r}{r!\,2^r} - \frac{z^{r+2}}{(r+1)!\,2^{r+2}} + \cdots$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{(r+s)!\,s!}\left(\frac{z}{2}\right)^{r+2s}$$

Bessel 関数の性質

① $J_{-n}(z) = (-1)^n J_n(z)$

∵ (＊＊) で $\theta \to \pi - \theta$ とおくと

$$J_n = \frac{1}{\pi}\int_\pi^0 (-d\theta)\ \underbrace{\cos(n\pi - n\theta - z\sin(\pi-\theta))}_{(-1)^n \cos(n\theta + z\sin\theta)}$$

$\sin(\pi-\theta)$ → $\sin\theta$

$$= \frac{1}{\pi}\int_0^\pi d\theta\ (-1)^n \cos(n\theta + z\sin\theta) = (-1)^n J_{-n}(z)\ //$$

② $J_n(x+y) = \sum_{m=-\infty}^{\infty} J_{n-m}(x)\,J_m(y)$

Bessel 関数の加法定理

∵ $e^{\frac{1}{2}(x+y)(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(x+y)$

$$= e^{\frac{1}{2}x(t-\frac{1}{t})}\, e^{\frac{1}{2}y(t-\frac{1}{t})} = \sum_{m,l} t^{m+l} J_m(x)\,J_l(y)$$

$$= \sum_{n=-\infty}^{\infty} t^n \left(\sum_{m=-\infty}^{\infty} J_{n-m}(x)\,J_m(y)\right)$$

③ 漸化式

$$\frac{d}{dz}\left(z^{-n} J_n(z)\right) = -z^{-n} J_{n+1}(z) \quad \cdots\cdots \quad (A)$$

$$\frac{d}{dz}\left(z^{n} J_n(z)\right) = z^{n} J_{n-1}(z) \quad \cdots\cdots \quad (B)$$

☺ (A)の証明

$$\frac{d}{dz}(z^{-n}J_n(z)) = \frac{d}{dz}\left(z^{-n}\oint\frac{dt}{2\pi i}\,t^{-n-1}\,e^{\frac{1}{2}z(t-\frac{1}{t})}\right)$$

$\Leftarrow \zeta = tz$

$$= \frac{d}{dz}\left(\oint\frac{d\zeta}{2\pi i}\,\zeta^{-n-1}\,e^{\frac{1}{2}(\zeta-\frac{z^2}{\zeta})}\right)$$

$$= \oint\frac{d\zeta}{2\pi i}\,\zeta^{-n-1}\left(-\frac{z}{\zeta}\right)e^{\frac{1}{2}(\zeta-\frac{z^2}{\zeta})}$$

$$= -z\cdot\oint\frac{d\zeta}{2\pi i}\,\zeta^{-n-2}\,e^{\frac{1}{2}(\zeta-\frac{z^2}{\zeta})}$$

$\Leftarrow t = \frac{\zeta}{z}$

$$= -z^{-n}\oint\frac{dt}{2\pi i}\,t^{-n-2}\,e^{\frac{1}{2}(z-\frac{z}{t})t}$$

$$= -z^{-n}\,J_{n+1}(z) \quad //$$

(B)の証明も同様 //

④ 微分方程式

$$\left[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + \left(1-\frac{n^2}{z^2}\right)\right]J_n(z) = 0$$

☺

$$(A) \Leftrightarrow z^n\frac{d}{dz}(z^{-n}J_n) = \left(\frac{d}{dz} - \frac{n}{z}\right)J_n = -J_{n+1}$$

$$(B) \Leftrightarrow z^{-n}\frac{d}{dz}(z^nJ_n) = \left(\frac{d}{dz} + \frac{n}{z}\right)J_n = J_{n-1}$$

両側を組み合わせると

$$\left(\frac{d}{dz} + \frac{n+1}{z}\right)\left(\frac{d}{dz} - \frac{n}{z}\right)J_n = -J_n$$

$$\left(\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + \left(1-\frac{n^2}{z^2}\right)\right)J_n = 0 \quad //$$

※ $z=0$ は確定特異点 $\quad p(z) \sim \frac{1}{z}$, $q(z) \sim -\frac{n^2}{z^2}$

決定方程式 $\quad \alpha(\alpha-1) + \alpha - n^2 = 0 \quad \alpha = \pm n$

$\alpha = n > 0 \Rightarrow J_n$: ベッセル $\quad J_n \propto z^n + \cdots$

$\alpha = -n < 0 \Rightarrow N_n$: ノイマン $\quad N_n \propto z^{-n} + \cdots + J_n \ln z + \cdots$

§5.2 波動方程式と Bessel 関数

(D+1)次元波動方程式

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta_D\right)\psi = 0 \qquad \Delta_D = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_D^2}$$

時間変数と変数分離 (Fourier 変換)

$$\psi(t, \vec{x}) = e^{i\omega t}\,\tilde{\psi}_\omega(\vec{x})$$

$$\boxed{\left(\Delta_D + \frac{\omega^2}{c^2}\right)\tilde{\psi}_\omega(\vec{x}) = 0}$$

D次元 Helmholtz eq.

$\left(\frac{\omega^2}{c^2} \equiv k^2 \text{ とおく}\right)$

<u>D=2</u> の場合　　$x = r\cos\theta,\ y = r\sin\theta$

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$\tilde{\psi}_\omega(\vec{x}) = R(r)\,\Theta(\theta)$ とおいて Helmholtz に代入

$$\underbrace{\frac{r^2}{R}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + k^2 R\right)}_{\alpha^2} + \underbrace{\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2}}_{-\alpha^2} = 0$$

Θ）　$\frac{\partial^2\Theta}{\partial\theta^2} + \alpha^2\Theta = 0 \Rightarrow \Theta = Ae^{i\alpha\theta} + Be^{-i\alpha\theta}$

一価性 $\Theta(\theta + 2\pi) = \Theta(\theta)$ を要求　$\alpha = n \in \mathbb{Z}$

R）　$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)R + \left(k^2 - \frac{n^2}{r^2}\right)R = 0$

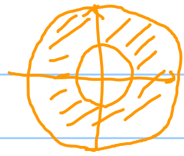
$r = \xi/k$ と変数変換

$$\left(\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} + \left(1 - \frac{n^2}{\xi^2}\right)\right)R = 0$$

: Bessel の DE

$r=0$ で正則な解

$$\Rightarrow \quad R(r) = J_n(\zeta) = J_n(kr)$$



でこのような領域で考えるときは J_n と N_n の組みあわせ

Helmholtz eq. の一般解 $\qquad k = \frac{\omega}{c}$

$$\tilde{\psi}_\omega(\vec{x}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta} J_n(kr)$$

※ 特に $a_n = e^{in\theta_0}$ とおくと平面波になる

$$\sum_{n\in\mathbb{Z}} e^{in(\theta+\theta_0)} J_n(kr) = e^{i(k_x x + k_y y)}$$

$$k_x = k\sin\theta_0, \quad k_y = k\cos\theta_0$$

平面波は確かに $(\Delta_2 + k^2)\psi = 0$ の解

D = 3 $\qquad (r, \theta, \varphi)$: 極座標

$$\Delta_3 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}\hat{\Omega} \qquad \hat{\Omega} Y_{lm} = -l(l+1) Y_{lm}$$

$$\hat{\Omega} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$\tilde{\psi}_\omega(\vec{x}) = R(r) Y_{lm}(\theta, \varphi)$ とおくと

R に対する DE

$$\left(\frac{1}{r}\frac{d^2}{dr^2} r - \frac{l(l+1)}{r^2} + k^2\right) R = 0 \qquad \left(k = \frac{\omega}{c}\right)$$

$\zeta = kr$ とおくと

$$\left(\frac{1}{\zeta}\frac{d^2}{d\zeta^2}\zeta + \left(1 - \frac{l(l+1)}{\zeta^2}\right)\right) R = 0$$

さらに $R(\zeta) = \frac{1}{\sqrt{\zeta}} j(\zeta)$ とおくと

$$\left(\frac{d^2}{d\zeta^2} + \frac{1}{\zeta}\frac{d}{d\zeta} + \left(1 - \frac{(l+\frac{1}{2})^2}{\zeta^2}\right)\right) j(\zeta) = 0$$

これは Bessel の DE で $\nu \to l+\frac{1}{2}$ とおいたもの

$r=0$ で正則な解は

$$R(r) = \sqrt{\frac{\pi}{2kr}} \cdot J_{l+\frac{1}{2}}(kr) \equiv j_l(kr)$$

球 Bessel 関数

$$j_n(x) = \sqrt{\frac{\pi}{2x}} \, J_{n+\frac{1}{2}}(x)$$

n が実数のときの Bessel 関数の定義

$$J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(\nu+s)\, s!} \cdot \left(\frac{x}{2}\right)^{\nu+2s}$$

これは $n \to \nu$ とした Bessel の DE の解

$\nu = n+\frac{1}{2}$ のとき $\Gamma\left(n+\frac{1}{2}+s\right) = \Gamma\left(\frac{1}{2}\right) \dfrac{(2n+2s+1)!}{2^{2n+2s+1}(s+n)!}$

$\sqrt{\pi}$

$$\boxed{j_n(x) = 2^n x^n \cdot \sum_{s=0}^{\infty} \frac{(s+n)!\,(-1)^s}{s!\,(2n+2s+1)!} x^{2s}}$$

$n=0$ のとき $j_0(x) = \displaystyle\sum_{s=0}^{\infty} \frac{\cancel{s!}\,(-1)^s}{\cancel{s!}\,(2s+1)!} x^{2s} = \frac{\sin x}{x}$

$l=0$ のとき DE は

$\dfrac{d}{dr}(rj) + rj = 0$

初等関数で解ける!

球 Bessel に対する Recursion formula

$\dfrac{d}{dx}(x^{-n} f_n) = -x^{-n} f_{n+1}$, $\dfrac{d}{dx}(x^{n+1} f_n) = x^{n+1} f_{n-1}$ を用いると

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)$$

もう一つの解: 球 Neumann 関数

$$n_0(x) = \frac{\cos x}{x}$$

$$n_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right)$$

証明　帰納法で示す。 $n=0$ のときは明らかに成立

$n=m$ のとき

$$\hat{j}_m(x) = (-1)^m x^m \left(\frac{1}{x}\frac{d}{dx}\right)^m \left(\frac{\sin x}{x}\right)$$ が成立したとする。

$n=m+1$ のとき

$$\hat{j}_{m+1}(x) = -x^m \frac{d}{dx}\left(x^{-m} \hat{j}_m\right)$$

$$= (-1)^{m+1} x^m \frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\right)^m \left(\frac{\sin x}{x}\right)$$

$$= (-1)^{m+1} x^{m+1} \left(\frac{1}{x}\frac{d}{dx}\right)^{m+1} \left(\frac{\sin x}{x}\right)$$

Recursion formula の証明

$$L_+^{(n)} := x^n \frac{d}{dx} x^{-n} = \frac{d}{dx} - \frac{n}{x}$$

$$L_-^{(n)} := x^{-n-1} \frac{d}{dx} x^{n+1} = \frac{d}{dx} + \frac{n+1}{x}$$

とおくと 球 Bessel の DE は

$$L_+^{(n-1)} L_-^{(n)} \hat{j}_n = -\hat{j}_n \quad \text{or} \quad L_-^{(n+1)} L_+^{(n)} \hat{j}_n = -\hat{j}_n$$ と書ける。

$$\Delta^{(n)} = L_+^{(n-1)} L_-^{(n)} = L_-^{(n+1)} L_+^{(n)} = \frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{n(n+1)}{x^2}$$ とおく。

$L_+^{(n)} \hat{j}_n \equiv f$ と書くと

$$L_+^{(n)} L_-^{(n+1)} f = L_+^{(n)} L_-^{(n+1)} L_+^{(n)} \hat{j}_n = L_+^{(n)}(-\hat{j}_n) = -f$$

$$\therefore \quad \Delta^{(n+1)} f = -f \quad \Rightarrow \quad f \propto \hat{j}_{n+1}$$

$L_-^{(n)} \hat{j}_n = g$ と書くと

$$\Delta^{(n-1)} g = L_-^{(n)} L_+^{(n-1)} L_-^{(n)} \hat{j}_n = L_-^{(n)} \Delta^{(n)} \hat{j}_n = -L_-^{(n)} \hat{j}_n$$

$$= -g$$

$$\Rightarrow \quad \Delta^{(n-1)} g = -g \quad \Rightarrow \quad g \propto \hat{j}_{n-1}$$

$$\hat{j}_n \sim 2^n x^n \left(\frac{n!}{(2n+1)!} - \frac{(n+1)!}{(2n+3)!} x^2 + O(x^4)\right)$$

$$L_+^{(n)} \hat{j}_n = \left(\frac{d}{dx} - \frac{n}{x}\right)\hat{j}_n = -2^{n+1} x^n \left(\frac{(n+1)!}{(2n+3)!} x^{n+1} + O(x^{n+3})\right) \sim -\hat{j}_{n+1}$$

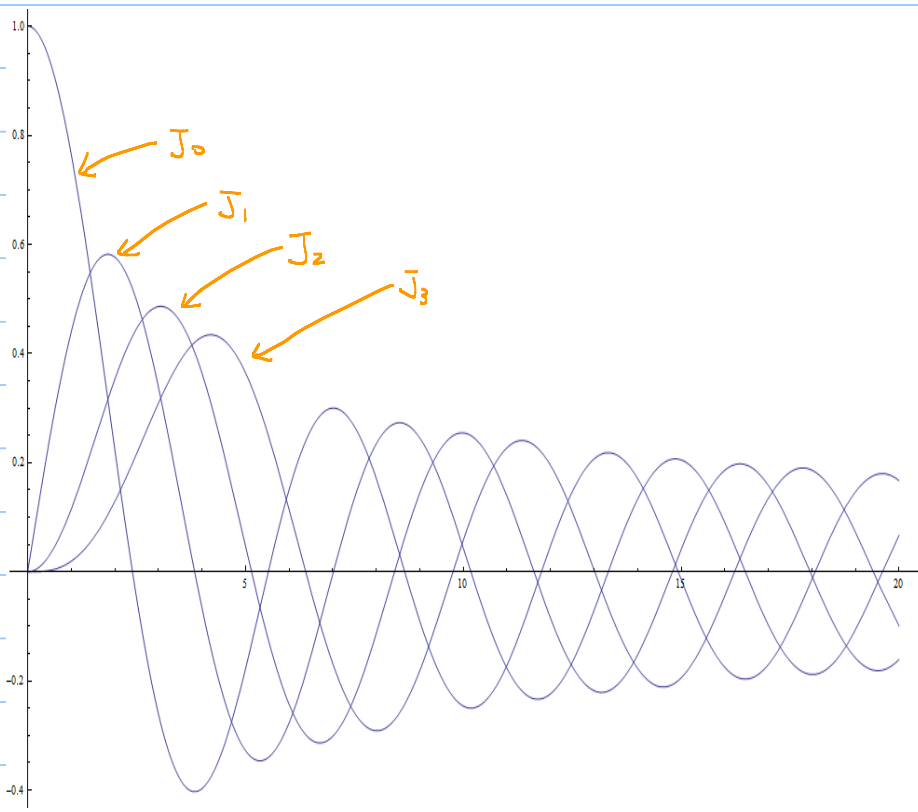
§5.3 Bessel関数の直交性

$$J_n(x) \sim \frac{1}{2^n n!} x^n + O(x^{n+1}) \quad \text{as} \quad x \to 0$$

$$\sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}\left(n + \frac{1}{2}\right)\right) \quad \text{as} \quad x \to \infty$$

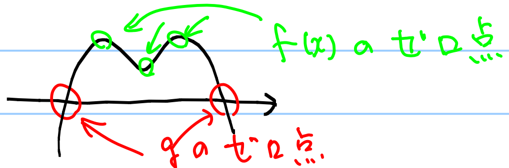
後で導く漸近展開

nが1つ増すたびにphaseが$\frac{\pi}{2}$進む



定理　$J_n(x)$の隣りあうゼロ点の間に$J_{n+1}(x)$は1つだけゼロ点を持つ

Rollの補題：　$f(x) = \frac{dg}{dx}$とすると$g(x)$の隣りあうゼロ点の間に$f(x)$は1つ（以上）ゼロ点を持つ



この補題を $J_n(x)$ の漸化式

$$x^{-n} J_{n+1}(x) = -\frac{d}{dx}\left(x^{-n} J_n(x)\right)$$

$$x^{n+1} J_n(x) = \frac{d}{dx}\left(x^{n+1} J_{n+1}(x)\right)$$

に適用すると定理が成立

直交性

$\alpha_{\nu n}$ を $J_\nu(x)$ の n 番目のゼロ点とすると

$$\int_0^1 J_\nu(\alpha_{\nu n} x) J_\nu(\alpha_{\nu m}) x\,dx = \frac{1}{2}\left(J_{\nu+1}(\alpha_{\nu n})\right)^2 \delta_{n,m}$$

$n \neq m$ の場合

補題 $\hat{H} = -\frac{d^2}{dx^2} - \frac{1}{x}\frac{d}{dx} + \frac{\nu^2}{x^2} = -\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right) + \frac{\nu^2}{x^2}$

$(f, g) = \int_0^1 f(x) g(x) x\,dx$

とおくと $\hat{H}$ は $f(1) = g(1) = 0$ のとき Hermite

$\because$ $(f, \hat{H}g) = \int_0^1 f(x)\left(-\frac{1}{x}\frac{d}{dx}\left(x\frac{dg}{dx}\right) + \frac{\nu^2}{x^2} g(x)\right) x\,dx$

$= \left(-fg'x + gf'x\right)\Big|_0^1$

$+ \int_0^1 \left(-\frac{1}{x}\frac{d}{dx}(xf') + \frac{\nu^2}{x^2} f\right) g\, x\,dx$

$= (\hat{H}f, g)$ //

$\hat{H} J_\nu(\alpha_{\nu n} x) = \alpha_{\nu n}^2 J_\nu(\alpha_{\nu n} x)$, $\quad J_\nu(\alpha_{\nu n} x)\Big|_{x=1} = 0$

なので $n \neq m$ であれば $\alpha_{\nu n}^2 \neq \alpha_{\nu m}^2$ となる. Hermite 性より

$$\underbrace{(\alpha_{\nu n}^2 - \alpha_{\nu m}^2)}_{\neq 0}\ \underbrace{\left(J_\nu(\alpha_{\nu n} x), J_\nu(\alpha_{\nu m} x)\right)}_{= 0} = 0$$

$n = m$ の場合　α, β を任意の実数として

$$(J_\nu(\alpha x), \hat{H} J_\nu(\beta x)) - (\hat{H} J_\nu(\alpha x), J_\nu(\beta x))$$

$$= (\beta^2 - \alpha^2) \int_0^1 J_\nu(\alpha x) J_\nu(\beta x)\, x\, dx$$

$$= \left(- J_\nu(\alpha x) x J_\nu'(\beta x) + J_\nu(\beta x) x J'(\alpha x) \right) \Big|_0^1$$

$\alpha = \alpha_{\nu n}$, $\beta \sim \alpha_{\nu n}$ とすると

$$\lim_{\beta \to \alpha_{\nu n}} \int_0^1 J_\nu(\alpha_{\nu n} x) J_\nu(\beta x)\, x\, dx \qquad \left(= \int_0^1 J_\nu(\alpha_{\nu n} x)^2 x\, dx \right)$$

$$= \lim_{\beta \to \alpha_{\nu n}} \frac{1}{\beta^2 - \alpha_{\nu n}^2} \left(- J_\nu(\alpha_{\nu n} x) x \cancel{\frac{dJ_\nu(\beta x)}{dx}} + \underbrace{J_\nu(\beta x) x \frac{dJ_\nu(\alpha_{\nu n} x)}{dx}}_{\alpha_{\nu n}(\beta - \alpha_{\nu n}) J'(\alpha_{\nu n} x)} \right) \Bigg|_0^1$$

$$= \frac{1}{2\alpha_{\nu n}} \alpha_{\nu n} (J'(\alpha_{\nu n} x))^2 = \frac{1}{2} (J'(\alpha_{\nu n} x))^2$$

一方　$\frac{d}{dx}(x^{-n} J_\nu(x)) = x^{-n} J_{\nu+1}(x)$ より

$$J_\nu'(\alpha_{\nu n}) = J_{\nu+1}(\alpha_{\nu n})$$

$$\therefore \int_0^1 J_\nu(\alpha_{\nu n} x)^2 x\, dx = \frac{1}{2} (J_{\nu+1}(\alpha_{\nu n}))^2 \quad //$$

Fourier-Bessel 級数

$f(x)$: $[0, 1]$ の実関数

$$f(x) = \sum_{n=1}^{\infty} c_{\nu n} J_\nu(\alpha_{\nu 1} x)$$

$$c_{\nu n} = \frac{2}{[J_{\nu+1}(\alpha_{\nu n})]^2} \cdot \int_0^1 f(y) J_\nu(\alpha_{\nu n} y)\, y\, dy$$

$\{ J_\nu(\alpha_{\nu n} x) \}_{n=1,2,\sim}$　は　$[0, 1]$ の関数系の完全系をなす。

Fourier - Bessel 級数の応用

タイコの膜の運動

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)u(x,y,t)=0$$

$$u(x,y,t)\Big|_{x^2+y^2=1}=0$$

解)

$u = u_{n\omega}(r)\, e^{in\theta+i\omega t}$ とおくと

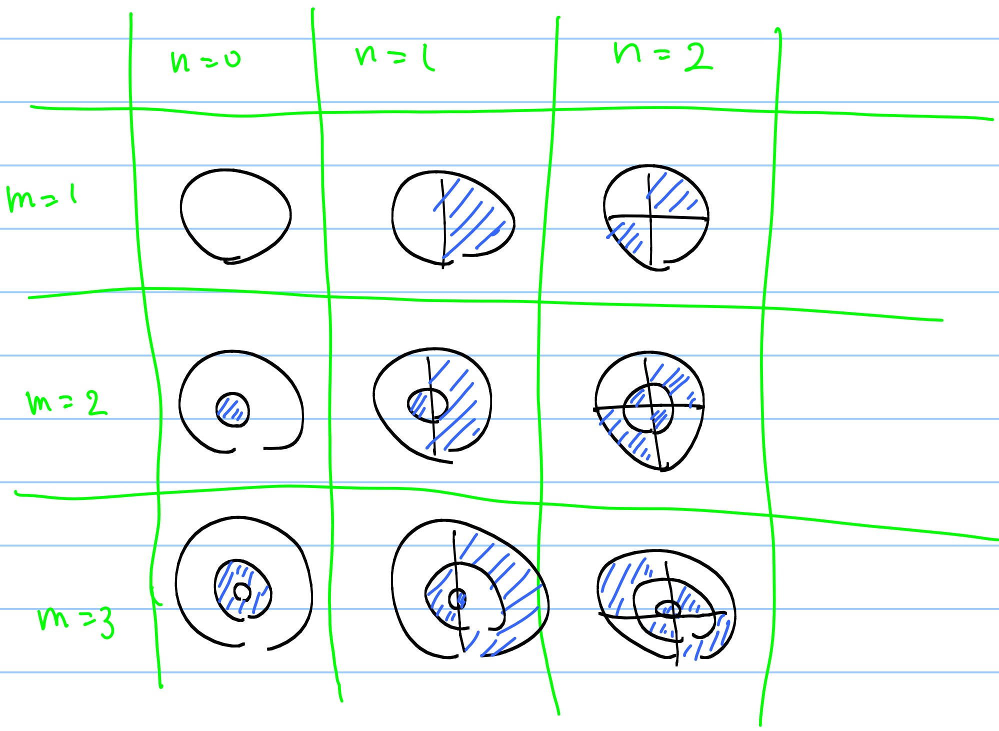
$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}+\omega^2\right)u_{n\omega}(r)=0$$

$r=0$ で正則 → $u_{n\omega}(r)\propto J_n(\omega r)$

境界条件 $u_{n\omega}(r)\Big|_{r=1}=0$ より

$\omega=\alpha_{nm}$ $(m=1,2,3,\ldots)$

$$u(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}A_{nm}\, e^{in\theta+i\alpha_{nm}t}\, J_n(\alpha_{nm}r)$$



§5.4 Bessel 関数の積分表示と漸近形

Bessel の DE

$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+\left(1-\frac{1}{z^2}\right)\right)u(z)=0$$

の解の積分表示は

$$u(z)=z^n\int_C(1+\zeta^2)^{\frac{2n-1}{2}}e^{z\zeta}\,d\zeta$$

ただし $\left.(1+\zeta^2)^{\frac{2n+1}{2}}e^{z\zeta}\right|_{\partial C}=0$

☺ $u(z)=z^n v(z)$ とおいて v に対する DE を求めると

$$\left(\frac{d^2}{dz^2}+\frac{2n+1}{z}\frac{d}{dz}+1\right)v(z)=0$$

$v(z)=\int_C(1+\zeta^2)^{\frac{2n-1}{2}}e^{z\zeta}\,d\zeta$ とおくと

$$\frac{d^2v}{dz^2}=\int_C(1+\zeta^2)^{\frac{2n-1}{2}}\zeta^2e^{z\zeta}\,d\zeta$$

$$\left(\frac{d^2}{dz^2}+1\right)v=\int_C(1+\zeta^2)^{\frac{2n+1}{2}}e^{z\zeta}\,d\zeta$$

$$=\frac{1}{z}\int_C(1+\zeta^2)^{\frac{2n+1}{2}}\frac{\partial}{\partial\zeta}(e^{z\zeta})\,d\zeta$$

$$=\frac{1}{z}\left.(1+\zeta^2)^{\frac{2n+1}{2}}e^{z\zeta}\right|_{\partial C}$$ ← C に対する条件

$$-\frac{2n+1}{z}\cdot\int_C(1+\zeta^2)^{\frac{2n-1}{2}}\zeta e^{z\zeta}\,d\zeta$$

$$=-\frac{2n+1}{z}\frac{d}{dz}v(z) \quad //$$

特に

$$J_n(z) = \frac{z^n}{i\sqrt{\pi}\,\Gamma(n+\frac{1}{2})\,2^n} \int_{-i}^{i} (1+\zeta^2)^{\frac{2n-1}{2}} e^{z\zeta}\, d\zeta \qquad \cdots (*)$$

$\because$ $(1+\zeta^2)^{\frac{2n+1}{2}} e^{z\zeta}\Big|_{\zeta=\pm i} = 0$ なので 積分路はOK.

$$\int_{-i}^{i} (1+\zeta^2)^{\frac{2n-1}{2}} e^{z\zeta}\, d\zeta$$

$$= \int_{-i}^{i} (1+\zeta^2)^{\frac{2n-1}{2}} \sum_{m=0}^{\infty} \frac{(z\zeta)^m}{m!}\, d\zeta$$

m=奇数の場合
積分がゼロになる

$$= 2\sum_{m=0}^{\infty} \frac{z^{2m}}{(2m)!} \int_0^i (1+\zeta^2)^{\frac{2n-1}{2}} \cdot \zeta^{2m}\, d\zeta$$

$$\left(\begin{aligned} &\zeta^2 = -t \\ &d\zeta = \frac{i}{2}\frac{dt}{\sqrt{t}} \end{aligned} \right.$$

$$= i \sum_{m=0}^{\infty} \frac{(-z^2)^m}{(2m)!} \cdot \underbrace{\int_0^1 (1-t)^{\frac{2n-1}{2}} t^{m-\frac{1}{2}}\, dt}_{B(n+\frac{1}{2},\, m+\frac{1}{2}) = \frac{\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(n+m+1)}}$$

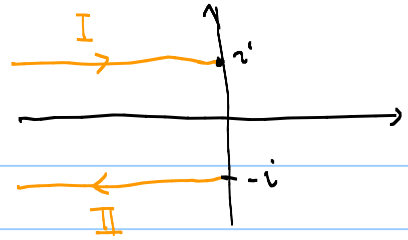
$$\therefore\ (*) = \frac{z^n}{\cancel{i}\sqrt{\pi}\,\cancel{\Gamma(n+\frac{1}{2})}\,2^n} \left(\cancel{i} \sum_{m=0}^{\infty} \frac{(-z^2)^m}{(2m)!} \frac{\cancel{\Gamma(n+\frac{1}{2})}\,\Gamma(m+\frac{1}{2})}{(n+m)!} \right)$$

$$\left(\frac{\Gamma(m+\frac{1}{2})}{(2m)!} = \frac{(m-\frac{1}{2})(m-\frac{3}{2})\cdots\frac{1}{2}\sqrt{\pi}}{(2m)!} = \frac{\sqrt{\pi}}{2^{2m}\, m!} \right)$$

$$= \left(\frac{z}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(n+m)!} \left(\frac{z}{2}\right)^{2m} = J_n(z) \ /\!/$$

その他の Bessel 方程式の解

$$C_n = \frac{z^n}{i\sqrt{\pi}\,\Gamma(n+\frac{1}{2})\,2^n},$$



として

$$J_n(z) = C_n \int_{\mathrm{I}+\mathrm{II}} (1+\zeta^2)^{\frac{2n-1}{2}} e^{z\zeta}\, d\zeta$$

$$N_n(z) = \frac{1}{i} C_n \int_{\mathrm{I}-\mathrm{II}} (\text{same})\, d\zeta$$ Neumann

$$H_n^{(1)}(z) = 2C_n \int_{\mathrm{I}} (\text{same})\, d\zeta$$

$$H_n^{(2)}(z) = 2C_n \int_{\mathrm{II}} (\text{same})\, d\zeta$$

Hankel

Bessel 関数の漸近形 $(z \to \infty)$

$$H_n^{(1)} \sim \sqrt{\frac{2}{\pi z}} \cdot e^{i(z-\frac{\pi}{2}(n+\frac{1}{2}))}$$

$$H_n^{(2)} \sim \sqrt{\frac{2}{\pi z}}\, e^{-i(z-\frac{\pi}{2}(n+\frac{1}{2}))}$$

$$J_n \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{2}(n+\frac{1}{2})\right)$$

$$N_n \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{2}(n+\frac{1}{2})\right)$$

(**)

(∵) I上の積分　$\zeta = i - \frac{t}{z}$ とおくと

$$1+\zeta^2 = 1 + \left(i - \frac{t}{z}\right)^2 = -2i \cdot \frac{t}{z} + \underbrace{\frac{t^2}{z^2}}_{z \to \infty \text{ ではムシできる}}$$

$$\int_{\mathrm{I}} (1+\zeta^2)^{\frac{2n-1}{2}} e^{z\zeta} d\zeta \approx \int_{\infty}^{0} \left(-\frac{dt}{z}\right) \left(-2i\frac{t}{z}\right)^{\frac{2n-1}{2}} e^{z\left(i - \frac{t}{z}\right)}$$

$$= (-2i)^{\frac{2n-1}{2}} \frac{e^{zi}}{z^{n+1/2}} \underbrace{\int_0^{\infty} dt\, e^{-t} t^{n-\frac{1}{2}}}_{\Gamma\left(n+\frac{1}{2}\right)}$$

$$\therefore C_n \int_{\mathrm{I}} (\text{same}) d\zeta = \frac{(-2i)^{n-\frac{1}{2}} z^n e^{iz}}{i\sqrt{\pi}\, \cancel{\Gamma\left(n+\frac{1}{2}\right)}\, 2^n} \cancel{\Gamma\left(n+\frac{1}{2}\right)} \left(1 + O\left(\frac{1}{z}\right)\right)$$

$$= \frac{1}{\sqrt{2\pi z}} e^{i\left(z - \frac{\pi}{2}\left(n+\frac{1}{2}\right)\right)} \left(1 + O\left(\frac{1}{z}\right)\right)$$

同様にして

$$C_n \int_{\mathrm{II}} (\text{same}) d\zeta = \frac{1}{\sqrt{2\pi z}} e^{-i\left(z - \frac{\pi}{2}\left(n+\frac{1}{2}\right)\right)} \left(1 + O\left(\frac{1}{z}\right)\right)$$

これらを組み合わせると (**) が導かれる。